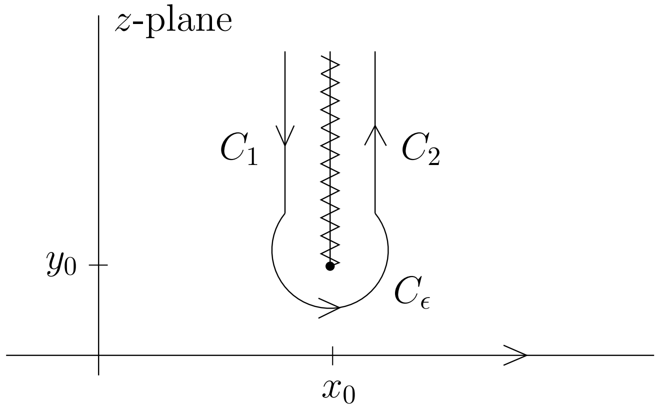
5. transforms

$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}dx$: Fourier transform , $k \to \infty$

We could integrate by parts, but instead we deform the path of integration.

case 1 : $f(z)$ has a branch point of order $\gamma > -1$ at $z_0 = x_0 + iy_0$, $y_0 > 0$



$f(z) = (z - z_0)^\gamma \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for $|z - z_0| < \delta$, assume $-\frac{3\pi}{2} < \arg(z - z_0) < \frac{\pi}{2}$

$|e^{ikz}| = e^{-ky}$: deform path into $y > 0$, assume $\int_{C_R}$ vanishes , $\int_{-\infty}^{\infty} = \int_{C_1} + \int_{C_\epsilon} + \int_{C_2}$

$z \in C_\epsilon \;\Rightarrow\; z = \epsilon e^{i\theta}$

$\int_{C_\epsilon} f(z)e^{ikz}dz = \int_{-3\pi/2}^{\pi/2} f(z_0 + \epsilon e^{i\theta})e^{ik(z_0+\epsilon e^{i\theta})}\epsilon i e^{i\theta}d\theta \sim O(\epsilon^{\gamma+1}) \to 0$ as $\epsilon \to 0$

$z \in C_1 \;\Rightarrow\; z = z_0 + se^{-3\pi i/2}$, $z \in C_2 \;\Rightarrow\; z = z_0 + se^{\pi i/2}$

$\left(\int_{C_1} + \int_{C_2}\right) f(z)e^{ikz}dz \;=\; \int_0^{\infty} (f(z_0 + se^{\pi i/2}) - f(z_0 + se^{-3\pi i/2}))e^{ik(z_0+is)}i ds$

$\hat{f}(k) = e^{ikz_0} i \int_0^{\infty} (f(z_0 + se^{\pi i/2}) - f(z_0 + se^{-3\pi i/2}))e^{-ks}ds$: Watson's lemma

domain of integration : $(0, \infty) \to (0, s_\delta) \to (0, \infty)$

$f(z_0 + se^{\pi i/2}) = (se^{\pi i/2})^\gamma \sum_{n=0}^{\infty} a_n (si)^n$, $f(z_0 + se^{-3\pi i/2}) = (se^{-3\pi i/2})^\gamma \sum_{n=0}^{\infty} a_n (si)^n$

$$\hat{f}(k) \sim e^{ikz_0} i (e^{\gamma\pi i/2} - e^{-3\gamma\pi i/2}) \sum_{n=0}^{\infty} a_n i^n \int_0^{\infty} e^{-ks} s^{\gamma+n} ds$$

$$= \; e^{i(kz_0 - \gamma\pi/2)} \cdot \frac{e^{\gamma\pi i} - e^{-\gamma\pi i}}{-i} \cdot \sum_{n=0}^{\infty} a_n i^n \frac{\Gamma(\gamma + n + 1)}{k^{\gamma+n+1}}$$

$$\hat{f}(k) \sim -2\sin\gamma\pi \cdot a_0 \Gamma(\gamma + 1) e^{i(kx_0 - \gamma\pi/2)} \frac{e^{-ky_0}}{k^{\gamma+1}} = O(k^{-(\gamma+1)}e^{-ky_0}) \text{ as } k \to \infty$$

ex 1

$f_1(x) = (x^2+\delta^2)^{-1/2} \Rightarrow$ branch point : $z_0 = i\delta$, $\delta > 0$, $\gamma = -\frac{1}{2}$

$f_1(z) = (z+i\delta)^{-1/2}(z-i\delta)^{-1/2} \Rightarrow a_0 = (2i\delta)^{-1/2} = \dfrac{e^{-\pi i/4}}{(2\delta)^{1/2}}$

$\hat{f}_1(k) \sim -2\sin(-\frac{\pi}{2}) \cdot \dfrac{e^{-\pi i/4}}{(2\delta)^{1/2}}\,\Gamma(\frac{1}{2})\,e^{\pi i/4}\,\dfrac{e^{-k\delta}}{k^{1/2}} = \left(\dfrac{2\pi}{\delta k}\right)^{1/2} e^{-\delta k}$ as $k \to \infty$

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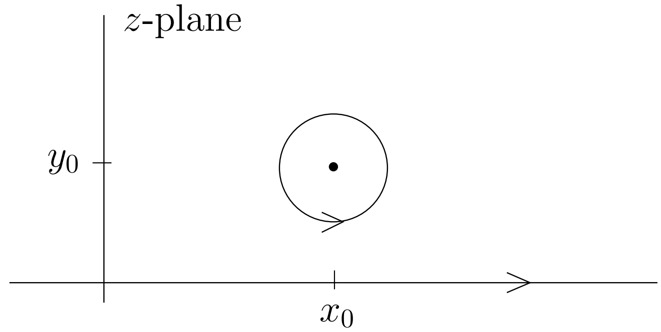
case 2 : $f(z)$ has a pole at $z_0 = x_0 + iy_0$, $y_0 > 0$



deform path into a circle around the pole

$\hat{f}(k) = 2\pi i\,\mathrm{Res}(f(z)e^{ikz}; z = z_0) = 2\pi i a_{-1} e^{ikx_0} e^{-ky_0}$ for all $k > 0$

$= O(e^{-ky_0})$ as $k \to \infty$: compare to case 1

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ex 2

$f_2(x) = (x^2+\delta^2)^{-1} \Rightarrow$ pole : $z_0 = i\delta$

$f_2(z) = (z+i\delta)^{-1}(z-i\delta)^{-1} \Rightarrow a_{-1} = (2i\delta)^{-1}$

$\hat{f}_2(k) = 2\pi i(2i\delta)^{-1}e^{-k\delta} = \dfrac{\pi}{\delta}\,e^{-\delta k}$ for all $k > 0$

note

$f_1(x)$ is smoother than $f_2(x) \Leftrightarrow \hat{f}_1(k)$ decays faster than $\hat{f}_2(k)$ as $k \to \infty$

$\hat{f}_2(k)$ ········ " ······· $\hat{f}_1(k) \Leftrightarrow f_2(x)$ ········ " ········ $f_1(x)$ as $x \to \infty$

---

ex 3

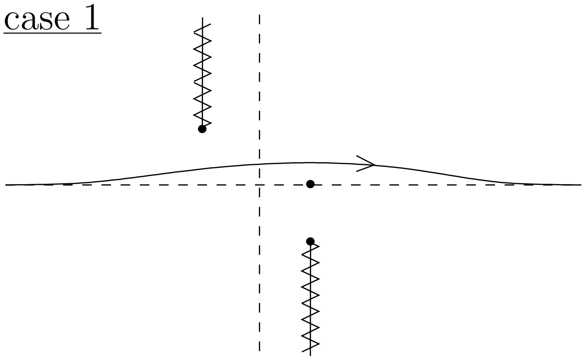
$\hat{f}(k) = \int_C (z-1)^{-1}(z^2+2i)^{-1/2}e^{ikz}dz$, C goes from $-\infty$ to ∞

$f(z) = (z-1)^{-1}(z^2+2i)^{-1/2}$

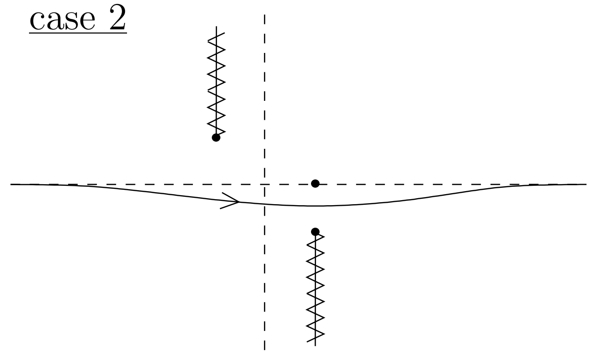
pole : $z = 1$, branch points : $z = \pm(-2i)^{1/2} = \pm\sqrt{2}\,e^{-\pi i/4} = \pm(1-i)$, $\gamma = -\frac{1}{2}$

There are several options for C; the choice depends on the application.

<u>case 1</u>



<u>case 2</u>



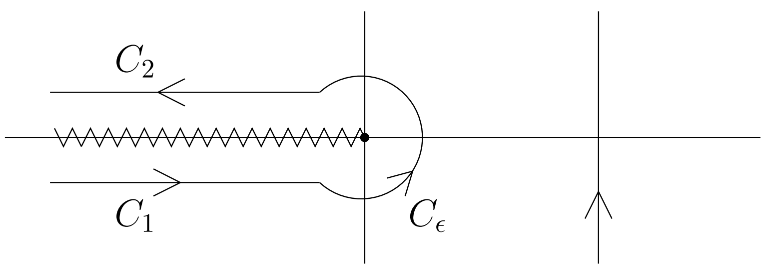
$\widehat{f}_1(k) = O(k^{-1/2}e^{-k})$ as $k \to \infty$

$\widehat{f}_2(k) \sim 2\pi i \operatorname{Res}(f(z)e^{ikz}; z = 1) = 2\pi i(1 + 2i)^{-1/2}e^{ik} = O(1)$ as $k \to \infty$

<u>ex 4</u> : 98/4(ii)

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{tz}z^{-1/2}e^{-z^{1/2}}dz\,,\ \alpha > 0\ :\ \text{inverse Laplace transform}\ ,\ t \to \infty$$

$f(z) = z^{-1/2}e^{-z^{1/2}}$: branch point at $z_0 = 0$, $-\pi < \arg z < \pi$



$z = re^{i\theta} \Rightarrow z^{1/2} = r^{1/2}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})$, $|e^{tz}z^{-1/2}e^{-z^{1/2}}| = e^{tx}r^{-1/2}e^{-r^{1/2}\cos(\theta/2)}$

$-\infty < x \le \alpha$, $\cos\frac{\theta}{2} \ge 0 \Rightarrow$ deform path into $C_1 + C_\epsilon + C_2$

$z \in C_\epsilon \Rightarrow z = \epsilon e^{i\theta}$

$$\left|\int_{C_\epsilon} e^{tz}z^{-1/2}e^{-z^{1/2}}dz\right| \le \int_{-\pi}^{\pi} e^{\epsilon t\cos\theta}\epsilon^{-1/2}e^{-\epsilon^{1/2}\cos(\theta/2)}\epsilon d\theta = O(\epsilon^{1/2}) \to 0 \text{ as } \epsilon \to 0$$

$z \in C_1 \Rightarrow z = -x$, $x > 0 \Rightarrow z^{1/2} = -ix^{1/2} \Rightarrow z^{-1/2} = ix^{-1/2}$

$z \in C_2 \Rightarrow z = -x$, $x > 0 \Rightarrow z^{1/2} = ix^{1/2} \Rightarrow z^{-1/2} = -ix^{-1/2}$

$$\int_{C_1} e^{tz}z^{-1/2}e^{-z^{1/2}}dz = \int_{\infty}^{0} e^{-tx}ix^{-1/2}e^{ix^{1/2}} \cdot -dx$$

$$\int_{C_2} e^{tz}z^{-1/2}e^{-z^{1/2}}dz = \int_{0}^{\infty} e^{-tx} \cdot -ix^{-1/2}e^{-ix^{1/2}} \cdot -dx$$

$$\int_{C_1+C_2} = 2i\int_0^\infty e^{-tx}x^{-1/2}\cos(x^{1/2})dx \sim 2i\frac{\Gamma(1/2)}{t^{1/2}} \text{ as } t \to \infty : \text{Watson's lemma}$$

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{tz}z^{-1/2}e^{-z^{1/2}}dz \sim \left(\frac{1}{\pi t}\right)^{1/2} \text{ as } t \to \infty\ ,\ \text{why decay? } e^{tz} = e^{tx} \cdot e^{ity} \ldots$$