6.1 differential equations

$w'' + pw' + qw = 0\ ,\ w = w(z)\ , p = p(z)\ , q = q(z)$

convergent series , $w(z) = \dots$ for $|z - z_0| < R$

asymptotic expansions , $w(z) \sim \dots$ as $z \to \infty$

ex

$-\nabla^2\phi = \phi$ in 2D

$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} = -\phi$, look for $\phi(r,\theta) = R(r)T(\theta)$

$R''T + \frac{1}{r}R'T + \frac{1}{r^2}RT'' = -RT \Rightarrow \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{T''}{T} = -1$

$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2 = -\frac{T''}{T} = c \Rightarrow T'' + cT = 0\ ,\ \text{PBC} \Rightarrow c = n^2\ ,\ T(\theta) = e^{\pm in\theta}$

$r^2R'' + rR' + (r^2 - n^2)R = 0$: Bessel equation

$w'' + pw' + qw = 0$: eliminate pw'

$w = uv$

$w' = uv' + u'v$

$w'' = uv'' + 2u'v' + u''v$

$uv'' + 2u'v' + u''v + p(uv' + u'v) + quv = 0$

$uv'' + (2u' + pu)v' + (u'' + pu' + qu)v = 0$

set $2u' + pu = 0 \Rightarrow u' = -\frac{1}{2}pu \Rightarrow \frac{du}{u} = -\frac{1}{2}pdz \Rightarrow \log u = -\frac{1}{2}\int^z p$

$u = e^{-\frac{1}{2}\int^z p}$

$u' = -\frac{1}{2}pu$

$u'' = -\frac{1}{2}(pu' + p'u) = -\frac{1}{2}(p \cdot -\frac{1}{2}pu + p'u) = \frac{1}{4}p^2u - \frac{1}{2}p'u$

$uv'' + (\frac{1}{4}p^2u - \frac{1}{2}p'u - \frac{1}{2}p^2u + qu)v = 0$

summary : $w = uv\ ,\ v'' + (q - \frac{1}{4}p^2 - \frac{1}{2}p')v = 0\ ,\ u = e^{-\frac{1}{2}\int^z p}$

Hence we consider $w'' + fw = 0$.

ex : $z^2 w'' + z w' + (z^2 - n^2) w = 0$: Bessel equation

$w'' + \frac{1}{z} w' + (1 - \frac{n^2}{z^2}) w = 0 \ , \ p = \frac{1}{z} \ , \ q = 1 - \frac{n^2}{z^2}$

$u = e^{-\frac{1}{2}\int^z p} = e^{-\frac{1}{2}\int \frac{1}{z}} = e^{-\frac{1}{2}\log z} = z^{-1/2}$

$w = z^{-1/2} v$

$w' = z^{-1/2} v' - \frac{1}{2} z^{-3/2} v = z^{-1/2}(v' - \frac{1}{2} z^{-1} v)$

$w'' = z^{-1/2} v'' - z^{-3/2} v' + \frac{3}{4} z^{-5/2} v = z^{-1/2}(v'' - z^{-1} v' + \frac{3}{4} z^{-2} v)$: Liebniz rule

cancel $z^{-1/2} \Rightarrow v'' - \cancel{z^{-1} v'} + \frac{3}{4} z^{-2} v + \cancel{z^{-1} v'} - \frac{1}{2} z^{-2} v + (1 - \frac{n^2}{z^2}) v = 0$

$v'' + \left(1 - \frac{(n^2 - \frac{1}{4})}{z^2}\right) v = 0 \ , \ q - \frac{1}{4} p^2 - \frac{1}{2} p' = 1 - \frac{n^2}{z^2} - \frac{1}{4z^2} + \frac{1}{2z^2} = 1 - \frac{(n^2 - \frac{1}{4})}{z^2}$ ok

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$w'' + f w = 0$, assume $f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$ as $z \to \infty$, $a_0 \neq 0$

ex : $f(z) = 1 - \frac{(n^2 - \frac{1}{4})}{z^2}$: Bessel equation , $a_0 = 1$, $a_1 = 0$, $a_2 = -(n^2 - \frac{1}{4})$

method 0

$w \sim \alpha_0 + \frac{\alpha_1}{z} + \frac{\alpha_2}{z^2} + \cdots + \frac{\alpha_n}{z^n} + \cdots$ as $z \to \infty$, need to solve for α_0 , α_1 , α_2 , ...

$$w' \sim -\frac{\alpha_1}{z^2} - \frac{2\alpha_2}{z^3} - \cdots - \frac{n\alpha_n}{z^{n+1}} - \cdots$$

$$w'' \sim \frac{2\alpha_1}{z^3} + \frac{6\alpha_2}{z^4} + \cdots + \frac{n(n+1)\alpha_n}{z^{n+2}} - \cdots$$

$$f w \sim \left(a_0 + \frac{a_1}{z} + \cdots\right)\left(\alpha_0 + \frac{\alpha_1}{z} + \cdots\right)$$

$$\sim a_0\alpha_0 + \frac{1}{z}(a_0\alpha_1 + a_1\alpha_0) + \frac{1}{z^2}(a_0\alpha_2 + a_1\alpha_1 + a_2\alpha_0)$$

$$+ \cdots + \frac{1}{z^n}(a_0\alpha_n + \cdots + a_n\alpha_0) + \cdots$$

$w'' + f w = 0$

z^0 : $a_0\alpha_0 = 0 \Rightarrow \alpha_0 = 0$

z^{-1} : $a_0\alpha_1 + a_1\cancel{\alpha_0} = 0 \Rightarrow \alpha_1 = 0$

z^{-2} : $a_0\alpha_2 + a_1\cancel{\alpha_1} + a_2\cancel{\alpha_0} = 0 \Rightarrow \alpha_2 = 0$

z^{-3} : $2\cancel{\alpha_1} + a_0\alpha_3 + a_1\cancel{\alpha_2} + a_2\cancel{\alpha_1} + a_3\cancel{\alpha_0} = 0 \Rightarrow \alpha_3 = 0$

z^{-n} : $(n-2)(n-1)\cancel{\alpha_{n-2}} + a_0\alpha_n + \cdots + a_n\cancel{\alpha_0} = 0 \Rightarrow \alpha_n = 0$, $n \geq 4$

This method fails to find a nonzero solution.

<u>method 1</u>

$$w \sim e^{\lambda z} z^{\sigma}\Big(\alpha_0 + \frac{\alpha_1}{z} + \frac{\alpha_2}{z^2} + \cdots + \frac{\alpha_n}{z^n} + \cdots\Big) \text{ as } z \to \infty$$

$$w' \sim e^{\lambda z} z^{\sigma}\Big(-\frac{\alpha_1}{z^2} - \frac{2\alpha_2}{z^3} - \cdots - \frac{n\alpha_n}{z^{n+1}} - \cdots\Big)$$

$$+\Big(e^{\lambda z}\sigma z^{\sigma-1} + \lambda e^{\lambda z} z^{\sigma}\Big)\Big(\alpha_0 + \frac{\alpha_1}{z} + \frac{\alpha_2}{z^2} + \cdots + \frac{\alpha_n}{z^n} + \cdots\Big)$$

$$= e^{\lambda z} z^{\sigma}\Big(\alpha_0\lambda + \frac{1}{z}(\alpha_1\lambda + \alpha_0\sigma) + \frac{1}{z^2}(\alpha_2\lambda + \alpha_1(\sigma-1)) + \cdots$$

$$+\frac{1}{z^n}(\alpha_n\lambda + \alpha_{n-1}(\sigma-(n-1))) + \cdots\Big) : \text{ same form as } w$$

$$w'' \sim e^{\lambda z} z^{\sigma}\Big(\alpha_0\lambda^2 + \frac{1}{z}(\alpha_1\lambda^2 + 2\alpha_0\lambda\sigma) + \frac{1}{z^2}(\alpha_2\lambda^2 + 2\alpha_1\lambda(\sigma-1) + \alpha_0\sigma(\sigma-1))$$

$$+ \cdots + \frac{1}{z^n}(\alpha_n\lambda^2 + 2\alpha_{n-1}\lambda(\sigma-(n-1))$$

$$+ \alpha_{n-2}(\sigma-(n-1))(\sigma-(n-2))) + \cdots\Big)$$

$$fw \sim \Big(a_0 + \frac{a_1}{z} + \cdots\Big)\cdot e^{\lambda z} z^{\sigma}\Big(\alpha_0 + \frac{\alpha_1}{z} + \cdots\Big)$$

recall : $w'' + fw = 0$

$$z^0 : \alpha_0\lambda^2 + a_0\alpha_0 = 0 \Rightarrow \alpha_0(\lambda^2 + a_0) = 0 \Rightarrow \lambda = \pm i a_0^{1/2}$$

$$z^{-1} : \cancel{\alpha_1\lambda^2} + 2\alpha_0\lambda\sigma + \cancel{a_0\alpha_1} + a_1\alpha_0 = 0 \Rightarrow \alpha_0(2\lambda\sigma + a_1) = 0 \Rightarrow \sigma = -\frac{a_1}{2\lambda}$$

$$z^{-2} : \cancel{\alpha_2\lambda^2} + 2\alpha_1\lambda(\cancel{\cancel{\sigma}} - 1) + \alpha_0\sigma(\sigma-1) + \cancel{a_0\alpha_2} + \cancel{\cancel{a_1\alpha_1}} + a_2\alpha_0 = 0$$

$$\Rightarrow \alpha_1 = \frac{\alpha_0(\sigma(\sigma-1) + a_2)}{2\lambda}$$

$$z^{-n} : \cancel{\alpha_n\lambda^2} + 2\alpha_{n-1}\lambda(\cancel{\cancel{\sigma}} - (n-1)) + \alpha_{n-2}(\sigma-(n-1))(\sigma-(n-2))$$

$$+ \cancel{a_0\alpha_n} + \cancel{\cancel{a_1\alpha_{n-1}}} + a_2\alpha_{n-2} + a_3\alpha_{n-3} + \cdots + a_n\alpha_0 = 0$$

$$\Rightarrow \alpha_{n-1} = \frac{\alpha_{n-2}((\sigma-(n-1))(\sigma-(n-2)) + a_2) + a_3\alpha_{n-3} + \cdots + a_n\alpha_0}{2\lambda(n-1)}$$

check $n = 2$ <u>ok</u>

note

1. set $s_n(z) = e^{\lambda z} z^\sigma\Big(\alpha_0 + \dfrac{\alpha_1}{z} + \cdots + \dfrac{\alpha_n}{z^n}\Big)$, then $s_n'' + f s_n = O\Big(\dfrac{e^{\lambda\sigma} z^\sigma}{z^{n+2}}\Big)$ as $z \to \infty$

2. $w(z) \sim e^{\lambda z} z^\sigma = z^{\pm i a_1/2a_0^{1/2}} e^{\pm i a_0^{1/2} z}$ as $z \to \infty$

3. If $a_0 = 0, a_1 \neq 0$, then $w(z) \sim z^{1/4} e^{\pm 2 i a_1^{1/2} z^{1/2}}$ as $z \to \infty$. pf ...

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ex

$x^2 w'' + x w' + (x^2 - n^2) w = 0$: Bessel equation

recall : $w = x^{-1/2} v \Rightarrow v'' + \Big(1 - \frac{(n^2 - \frac14)}{x^2}\Big) v = 0$

$f(x) = 1 - \frac{(n^2 - \frac14)}{x^2}$

$a_0 = 1\,,\ a_1 = 0\,,\ a_2 = -(n^2 - \frac14)\,,\ a_3 = \cdots = 0$

$\lambda = \pm i a_0^{1/2} = \pm i\ ,\ \sigma = -\dfrac{a_1}{2\lambda} = 0$

$e^{\lambda x} x^\sigma = e^{\pm i x}$

choose $\alpha_0 = 1$

$\alpha_1 = \dfrac{\alpha_0(\sigma(\sigma - 1) + a_2)}{2\lambda} = \dfrac{-(n^2 - \frac14)}{\pm 2i} = \pm \dfrac{i(n^2 - \frac14)}{2}$

$v(x) \sim A e^{ix}\Big(1 + \frac{i(n^2 - \frac14)}{2x} + \cdots\Big) + B e^{-ix}\Big(1 - \frac{i(n^2 - \frac14)}{2x} + \cdots\Big)$

set $A = (2\pi)^{-1/2} e^{-i\frac{\pi}{2}(n + \frac12)}$, $B = \bar{A}$

$w(x) \sim \big(\frac{2}{\pi x}\big)^{1/2} \cos(x - \frac{\pi n}{2} - \frac{\pi}{4}) + O(x^{-3/2})$ as $x \to \infty$

recall : $J_n(\lambda) = \frac{1}{\pi}\int_0^\pi \cos(nt - \lambda \sin t)\,dt \sim \big(\frac{2}{\pi\lambda}\big)^{1/2} \cos(\lambda - \frac{\pi n}{2} - \frac{\pi}{4})$ as $\lambda \to \infty$

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note

$w \sim e^{\lambda z} z^\sigma (\alpha_0 + \frac{\alpha_1}{z} + \cdots) = \exp(\lambda z + \sigma \log z + \log(\alpha_0 + \frac{\alpha_1}{z} + \cdots))$

$\sim \exp(\lambda z + \sigma \log z + \log \alpha_0 + \log(1 + \frac{\alpha_1}{\alpha_0 z} + \cdots))$

$\sim \exp(\lambda z + \sigma \log z + \log \alpha_0 + \frac{\alpha_1}{\alpha_0 z} + \cdots)$

<u>method 2</u> : "more fundamental and general"

$w \sim \exp(\phi_0 + \phi_1 + \cdots)$

$\phi_0(z)\,,\ \phi_1(z)\,,\ \ldots$ is an asymptotic sequence as $z \to \infty$

$w'' + fw = 0\ ,\ \ f \sim a_0 + \dfrac{a_1}{z} + \cdots\ ,\ \ a_0 \neq 0$

$w' \sim (\phi_0' + \phi_1' + \cdots)\exp(\phi_0 + \phi_1 + \cdots)$

$w'' \sim (\phi_0'' + \phi_1'' + \cdots + (\phi_0' + \phi_1' + \cdots)^2)\exp(\phi_0 + \phi_1 + \cdots)$

$\phi_0'' + \phi_1'' + \cdots + (\phi_0' + \phi_1' + \cdots)^2 + f \sim 0$

$\phi_0'' + \phi_1'' + \cdots + (\phi_0')^2 + 2\phi_0'\phi_1' + (\phi_1')^2 + 2\phi_0'\phi_2' + \cdots + a_0 + \dfrac{a_1}{z} + \dfrac{a_2}{z^2} + \cdots \sim 0$

need to balance terms at each order

<u>order z^0</u>

try $\phi_0'' + a_0 \sim 0\ \Rightarrow\ \phi_0'' \sim -a_0\ \Rightarrow\ \phi_0' \sim -a_0 z\ \Rightarrow\ (\phi_0')^2 \sim a_0^2 z^2 \ldots$: fails

try $\phi_0'' + (\phi_0')^2 + a_0 \sim 0$

try $\phi_0 = O(z^\beta)$, where $\beta > 0$

then $\phi_0'' = O(z^{\beta-2})\ ,\ (\phi_0')^2 = O(z^{2\beta-2})\ \Rightarrow\ \phi_0'' = o((\phi_0')^2)$ as $z \to \infty$

$\Rightarrow\ (\phi_0')^2 + a_0 \sim 0\ \Rightarrow\ (\phi_0')^2 \sim -a_0\ \Rightarrow\ \phi_0' \sim \pm i a_0^{1/2} = \lambda$

set $\phi_0 = \lambda z + \log \alpha_0$, note : $\beta = 1\,,\ \phi_0'' = 0$

<u>order z^{-1}</u>

$2\phi_0'\phi_1' + \dfrac{a_1}{z} \sim 0\ \Rightarrow\ \phi_1' \sim -\dfrac{a_1}{2\lambda z} = \dfrac{\sigma}{z}\ ,\ \sigma = -\dfrac{a_1}{2\lambda}\ ,\ \phi_1 = \sigma \log z$

<u>order z^{-2}</u>

$\phi_1'' + (\phi_1')^2 + 2\phi_0'\phi_2' + \dfrac{a_2}{z^2} \sim 0$

$-\dfrac{\sigma}{z^2} + \dfrac{\sigma^2}{z^2} + 2\lambda\phi_2' + \dfrac{a_2}{z^2} \sim 0$

$\phi_2' \sim -\dfrac{(\sigma^2 - \sigma + a_2)}{2\lambda z^2}\ \Rightarrow\ \phi_2 = \dfrac{\alpha_1}{\alpha_0 z}\ ,\ \dfrac{\alpha_1}{\alpha_0} = \dfrac{\sigma^2 - \sigma + a_2}{2\lambda}$

$\Rightarrow\ \phi_0 + \phi_1 + \phi_2 \cdots = \lambda z + \log \alpha_0 + \sigma \log z + \dfrac{\alpha_1}{\alpha_0 z} + \cdots$

$w \sim \exp(\phi_0 + \phi_1 + \cdots) = \exp(\lambda z + \log \alpha_0 + \sigma \log z + \frac{\alpha_1}{\alpha_0 z} + \cdots)$: same as before

<u>ex</u>

$w'' - z^2 w = 0$: parabolic cylinder functions

Our previous assumption that $f \sim a_0 + \frac{a_1}{z} + \cdots$ doesn't apply, but we can still try $w \sim \exp(\phi_0 + \phi_1 + \cdots)$.

$$\phi_0'' + \phi_1'' + \cdots (\phi_0' + \phi_1' + \cdots)^2 \sim z^2$$

$$\cancel{\phi_0''} + \cancel{\phi_1''} + \cancel{\phi_2''} + \cdots + (\cancel{\phi_0'})^2 + 2\cancel{\phi_0'}\phi_1' + (\cancel{\phi_1'})^2 + 2\cancel{\phi_0'}\phi_2' + 2\cancel{\phi_0'}\phi_3' + 2\cancel{\phi_1'}\phi_2'$$
$$+ (\cancel{\phi_2'})^2 + 2\cancel{\phi_0'}\phi_4' + 2\cancel{\phi_1'}\phi_3' + \cdots \sim \cancel{z^2}$$

<u>order z^2</u>

try $\phi_0'' \sim z^2 \Rightarrow \phi_0' \sim \frac{1}{3}z^3 \Rightarrow (\phi_0')^2 \sim \frac{1}{9}z^6 \ldots$: fails

try $(\phi_0')^2 \sim z^2 \Rightarrow \phi_0' \sim \pm z \Rightarrow \phi_0'' \sim \pm 1$, $\phi_0 = \pm\frac{1}{2}z^2$

<u>order z</u>

$2\phi_0'\phi_1' \sim 0 \Rightarrow \pm 2z\phi_1' \sim 0$, set $\phi_1 = 0$

<u>order 1</u>

$\phi_0'' + 2\phi_0'\phi_2' \sim 0 \Rightarrow \pm 1 + 2(\pm z)\phi_2' \sim 0 \Rightarrow \phi_2' \sim -\frac{1}{2z}$, $\phi_2 = -\frac{1}{2}\log z$

<u>order z^{-1}</u>

$2\phi_0'\phi_3' \sim 0$, set $\phi_3 = 0$

<u>order z^{-2}</u>

$\phi_2'' + (\phi_2')^2 + 2\phi_0'\phi_4' \sim 0 \Rightarrow \frac{1}{2z^2} + \frac{1}{4z^2} + 2(\pm z)\phi_4' \sim 0 \Rightarrow \phi_4' \sim \mp\frac{3}{8z^3}$, $\phi_4 = \pm\frac{3}{16z^2}$

<u>summary</u>

$\phi_0 + \phi_1 + \phi_2 + \cdots = \pm\frac{1}{2}z^2 - \frac{1}{2}\log z \pm \frac{3}{16z^2} + \cdots$

$w \sim e^{\pm\frac{1}{2}z^2} z^{-1/2}(1 \pm \frac{3}{16z^2} + \cdots)$ as $z \to \infty$

$w \sim e^{\lambda z^2} z^{\sigma}(\alpha_0 + \frac{\alpha_1}{z} + \cdots)$ as $z \to \infty$

<u>plan</u>

6.1 : $w'' + f(z)w = 0$, $z \to \infty$

6.2 : $w'' + f(x,\lambda)w = 0$, $\lambda \to \infty$, $f(x,\lambda) \neq 0$: WKB method

6.3 : $w'' + f(x,\lambda)w = 0$, $\lambda \to \infty$, $f(x_0,\lambda) = 0$: turning points

6.2 WKB method : Wentzel, Kramers, Brillouin

$w'' + f(x,\lambda)w = 0\ ,\ \lambda \to \infty\ ,\ f(x,\lambda) \neq 0$

assume $f(x,\lambda) \sim \lambda^2\phi_0(x) + \lambda\phi_1(x) + \phi_2(x) + \cdots$ as $\lambda \to \infty$

$w(x,\lambda) \sim e^{\lambda W(x)}U(x)\Big(1 + \frac{V_1(x)}{\lambda} + \frac{V_2(x)}{\lambda^2} + \cdots\Big)$: WKB (hw6)

$w(x,\lambda) \sim \exp(g_0(\lambda)\psi_0(x) + g_1(\lambda)\psi_1(x) + \cdots)$: Murray

we expect $\{g_n(\lambda)\}$ to be an asymptotic sequence as $\lambda \to \infty$

$$g_0(\lambda)\psi_0''(x) + g_1(\lambda)\psi_1''(x) + \cdots + (g_0(\lambda)\psi_0'(x) + g_1(\lambda)\psi_1'(x) + \cdots)^2 + \lambda^2\phi_0(x) + \lambda\phi_1(x) + \phi_2(x) + \cdots \sim 0$$

$$g_0\psi_0'' + g_1\psi_1'' + \cdots + g_0^2(\psi_0')^2 + 2g_0g_1\psi_0'\psi_1' + g_1^2(\psi_1')^2 + 2g_0g_2\psi_0'\psi_2' + \cdots + \lambda^2\phi_0 + \lambda\phi_1 + \phi_2 + \cdots \sim 0$$

order λ^2

$g_0^2(\psi_0')^2 + \lambda^2\phi_0 \sim 0 \Rightarrow g_0(\lambda) = \lambda\ ,\ \psi_0' = \pm i\sqrt{\phi_0} \Rightarrow \psi_0(x) = \pm i\int^x\sqrt{\phi_0}$

order λ

$g_0\psi_0'' + 2g_0g_1\psi_0'\psi_1' + \lambda\phi_1 \sim 0$

$g_1(\lambda) = 1\ ,\ \psi_1' = -\frac{(\psi_0'' + \phi_1)}{2\psi_0'} \Rightarrow \psi_1 = -\frac{1}{2}\log\psi_0' - \int^x\frac{\phi_1}{2\psi_0'}$

$\psi_1(x) = -\frac{1}{2}\log(\pm i) - \frac{1}{2}\log\sqrt{\phi_0(x)} \pm i\int^x\frac{\phi_1}{2\sqrt{\phi_0}}$

order 1

$g_1\psi_1'' + g_1^2(\psi_1')^2 + 2g_0g_2\psi_0'\psi_2' + \phi_2 \sim 0$

$g_2(\lambda) = \lambda^{-1}\ ,\ \psi_2' = -\frac{(\psi_1'' + (\psi_1')^2 + \phi_2)}{2\psi_0'} = \cdots$

summary

$g_0\psi_0 + g_1\psi_1 + \cdots = \pm i\lambda\int^x\sqrt{\phi_0} + \log\phi_0^{-\frac{1}{4}}(x) \pm i\int^x\frac{\phi_1}{2\sqrt{\phi_0}} + O(\lambda^{-1})$

$w(x,\lambda) \sim \phi_0^{-\frac{1}{4}}(x)\exp\Big(\pm i\int^x\Big(\lambda\sqrt{\phi_0} + \frac{\phi_1}{2\sqrt{\phi_0}}\Big) + O(\lambda^{-1})\Big)$ as $\lambda \to \infty$

special case : $w'' + \lambda^2\phi_0(x)w = 0$: Liouville equation

$w(x,\lambda) \sim \phi_0^{-\frac{1}{4}}(x)\exp\Big(\pm i\lambda\int^x\sqrt{\phi_0} + O(\lambda^{-1})\Big)$ as $\lambda \to \infty$

<u>note</u>

1. The sign of $\phi_0(x)$ determines the form of $w(x,\lambda)$ (oscillatory/exponential).
2. Unlike the approximations in section 6.1, which were valid for $x \to \infty$, the WKB approximation can be used to satisfy IC/BC at finite values of x.
3. The expansion fails if $\phi_0(x) = 0$, but the idea still works.

<u>ex</u>: $w'' + (\lambda + x)w = 0\ ,\ \lambda \to \infty\ ,\ x \geq 0\ ,\ w(0,\lambda) = a\ ,\ w'(0,\lambda) = b$

$f(x,\lambda) = \lambda + x \ \Rightarrow\ \phi_0(x) = 0\ ,\ \phi_1(x) = 1\ ,\ \phi_2(x) = x$

recall : $w''(\lambda) - \lambda w(\lambda) = 0\ :\ \mathrm{Ai}(\lambda)\ ,\ \mathrm{Bi}(\lambda) \to \mathrm{Ai}(-(\lambda+x))\ ,\ \mathrm{Bi}(-(\lambda+x))$

<u>case 1</u> : $x = O(1)$ as $\lambda \to \infty$

$$w(x,\lambda) \sim \exp(g_0(\lambda)\psi_0(x) + g_1(\lambda)\psi_1(x) + \cdots)$$

$$g_0\psi_0'' + g_1\psi_1'' + g_2\psi_2'' + g_3\psi_3'' + \cdots + g_0^2(\psi_0')^2 + 2g_0g_1\psi_0'\psi_1' + g_1^2(\psi_1')^2 + 2g_0g_2\psi_0'\psi_2'$$
$$+ 2g_0g_3\psi_0'\psi_3' + 2g_1g_2\psi_1'\psi_2' + g_2^2(\psi_2')^2 + 2g_0g_4\psi_0'\psi_4' + 2g_1g_3\psi_1'\psi_3' + \cdots + \lambda + x \sim 0$$

<u>order λ</u>

$$g_0^2(\psi_0')^2 + \lambda \sim 0 \Rightarrow g_0(\lambda) = \lambda^{1/2}\ ,\ \psi_0' = \pm i\ ,\ \psi_0(x) = \pm ix$$

<u>order $\lambda^{1/2}$</u>

$$2g_0g_1\psi_0'\psi_1' \sim 0 \Rightarrow g_1(\lambda) = 1\ ,\ \psi_1' = 0\ ,\ \psi_1(x) = 0$$

<u>order 1</u>

$$2g_0g_2\psi_0'\psi_2' + x \sim 0 \Rightarrow g_2(\lambda) = \lambda^{-1/2}\ ,\ \psi_2' = \frac{-x}{2\psi_0'} = \pm\frac{ix}{2}\ ,\ \psi_2(x) = \pm\frac{ix^2}{4}$$

<u>order $\lambda^{-1/2}$</u>

$$g_2\psi_2'' + 2g_0g_3\psi_0'\psi_3' \sim 0 \Rightarrow g_3(\lambda) = \lambda^{-1}\ ,\ \psi_3' = \frac{-\psi_2''}{2\psi_0'} = -\frac{1}{4}\ ,\ \psi_3(x) = -\frac{x}{4}$$

<u>order λ^{-1}</u>

$$g_2^2(\psi_2')^2 + 2g_0g_4\psi_0'\psi_4' \sim 0 \Rightarrow g_4(\lambda) = \lambda^{-3/2},\ \psi_4' = \frac{-(\psi_2')^2}{2\psi_0'} = \mp\frac{ix^2}{8}\ ,\ \psi_4(x) = \mp\frac{ix^3}{24}$$

<u>summary</u>

$$g_0\psi_0 + g_1\psi_1 + \cdots = \pm i\lambda^{1/2}x \pm \frac{ix^2}{4\lambda^{1/2}} - \frac{x}{4\lambda} + O\left(\frac{ix^3}{\lambda^{3/2}}\right)$$

$$w(x,\lambda) \sim e^{-x/4\lambda}\exp\left[\pm ix\left(\lambda^{1/2} + \frac{x}{4\lambda^{1/2}} + O\left(\frac{x^2}{\lambda^{3/2}}\right)\right)\right]$$

$$\sim \left(1 - \frac{x}{4\lambda} + \cdots\right)\exp\left[\pm ix\left(\lambda^{1/2} + \frac{x}{4\lambda^{1/2}} + \cdots\right)\right] \text{ as } \lambda \to \infty\ ,\ x = O(1)$$

The solution satisfying the IC at $x = 0$ has the following approximation.

$$w(x,\lambda) \sim \left(1 - \frac{x}{4\lambda}\right)\left(a\cos x\left[\lambda^{1/2} + \frac{x}{4\lambda^{1/2}}\right] + \lambda^{-1/2}\left(b + \frac{a}{4\lambda}\right)\sin x\left[\lambda^{1/2} + \frac{x}{4\lambda^{1/2}}\right]\right)$$

$$\text{as } \lambda \to \infty\,,\ x = O(1)$$

$$w(0,\lambda) \sim \left(1 - \frac{x}{4\lambda}\right)\left(a\Big(1 + O(x^2)\Big) + \lambda^{-1/2}\left(b + \frac{a}{4\lambda}\right)\left(x\left[\lambda^{1/2} + \frac{x}{4\lambda^{1/2}}\right] + O(x^3)\right)\right)$$

$$= a + bx + O(x^2) \quad \underline{\text{ok}}$$

---

case 2 : $x = O(\lambda)$ as $\lambda \to \infty$, the previous result is not valid in this case

$w'' + (\lambda + x)w = 0$, set $y = \dfrac{x}{\lambda}$, $W(y,\lambda) = w(x,\lambda) \Rightarrow w' = \dfrac{dw}{dx} = \dfrac{dW}{dy}\dfrac{1}{\lambda} = \dfrac{1}{\lambda}W'$

$$\frac{1}{\lambda^2}W'' + (\lambda + \lambda y)W = 0 \Rightarrow W'' + \lambda^3(1+y)W = 0$$

recall : $W'' + \tilde{\lambda}^2\phi_0(y)W = 0 \Rightarrow W(y,\tilde{\lambda}) \sim \phi_0^{-\frac{1}{4}}(y)\exp\left(\pm i\tilde{\lambda}\int^y\sqrt{\phi_0} + O\big(\tilde{\lambda}^{-1}\big)\right)$

set $\tilde{\lambda}^2 = \lambda^3$, $\phi_0(y) = 1 + y$

$$W(y,\lambda) \sim (1+y)^{-1/4}\exp\left(\pm i\lambda^{3/2}\int^y\sqrt{1+\tilde{y}}\,d\tilde{y} + O\big(\lambda^{-3/2}\big)\right) \text{ as } \lambda \to \infty\,,\ y = O(1)$$

return to original problem

$$w(x,\lambda) \sim \left(1 + \frac{x}{\lambda}\right)^{-1/4}\exp\left[\pm\frac{2i}{3}\lambda^{3/2}\left(1 + \frac{x}{\lambda}\right)^{3/2} + O\big(\lambda^{-3/2}\big)\right] \text{ as } \lambda \to \infty\,,\ x = O(\lambda)$$

The solution satisfying the IC at $x = 0$ has the following approximation.

$$w(x,\lambda) \sim \left(1 + \frac{x}{\lambda}\right)^{-1/4}\left(a\cos\left[\frac{2}{3}\lambda^{3/2}\left(\left(1 + \frac{x}{\lambda}\right)^{3/2} - 1\right)\right]\right.$$

$$\left. + \lambda^{-1/2}\left(b + \frac{a}{4\lambda}\right)\sin\left[\frac{2}{3}\lambda^{3/2}\left(\left(1 + \frac{x}{\lambda}\right)^{3/2} - 1\right)\right]\right) \text{ as } \lambda \to \infty\,,\ x = O(\lambda)$$

note : The approximation is <u>uniformly valid</u> for $0 \le x \le \lambda$.

Consider $x \to 0$.

$$\left(1 + \frac{x}{\lambda}\right)^{-1/4} = 1 - \frac{x}{4\lambda} + \cdots$$

$$\frac{2}{3}\lambda^{3/2}\left(\left(1 + \frac{x}{\lambda}\right)^{3/2} - 1\right) = \frac{2}{3}\lambda^{3/2}\left(1 + \frac{3}{2}\frac{x}{\lambda} + \frac{3}{8}\frac{x^2}{\lambda^2} + \cdots - 1\right) = x\left(\lambda^{1/2} + \frac{x}{4\lambda^{1/2}} + \cdots\right)$$

Hence case 2 reduces to case 1 for $x \to 0$.

ex

$w'' + (1+\epsilon t)w = 0\ ,\ \epsilon \to 0\ ,\ w(0,\epsilon) = c\ ,\ w'(0,\epsilon) = d$

This equation describes an oscillation with slowly varying frequency.

case 1 : $t = O(1)$

$w(t,\epsilon) \sim F_0(t) + \epsilon F_1(t) + \epsilon^2 F_2(t) + \cdots$: regular perturbation series

$F_0'' + \epsilon F_1'' + \cdots + (1+\epsilon t)(F_0 + \epsilon F_1 + \cdots) \sim 0$

order 1

$F_0'' + F_0 = 0\ ,\ F_0(0) = c\ ,\ F_0'(0) = d\ \Rightarrow\ F_0(t) = c\cos t + d\sin t$

order ϵ

$F_1'' + F_1 + tF_0 = 0\ \Rightarrow\ F_1'' + F_1 = -t(c\cos t + d\sin t)\,,\ F_1(0) = 0\,,\ F_1'(0) = 0$

$F_1(t) = f(t)\cos t + g(t)\sin t\,,\ f(0) = 0\,,\ f'(0) + g(0) = 0$

ODE theory ensures that $f(t)$ and $g(t)$ are quadratic polynomials. (pf ...)

$F_1 = f\cos t + g\sin t$

$F_1' = (f' + g)\cos t + (-f + g')\sin t$

$F_1'' = (f'' + 2g' - f)\cos t + (g'' - 2f' - g)\sin t$

$F_1'' + F_1 = (f'' + 2g')\cos t + (g'' - 2f')\sin t$

$f'' + 2g' = -ct\ \Rightarrow\ \cancel{f'''} + 2g'' = -c\ \Rightarrow\ g'' = -\frac{1}{2}c$

$g'' - 2f' = -dt\ \Rightarrow\ f' = \frac{1}{2}dt - \frac{1}{4}c\ \Rightarrow\ f = \frac{1}{4}dt^2 - \frac{1}{4}ct\ \Rightarrow f'(0) = -\frac{1}{4}c$

$g' = -\frac{1}{2}ct - \frac{1}{4}d\ \Rightarrow\ g = -\frac{1}{4}ct^2 - \frac{1}{4}dt + \frac{1}{4}c$

$F_1(t) = \frac{1}{4}(dt^2 - ct)\cos t - \frac{1}{4}(ct^2 + dt - c)\sin t$

summary

$$w(t,\epsilon) \sim \Big(c + \frac{\epsilon}{4}\Big(dt^2 - ct\Big)\Big)\cos t + \Big(d - \frac{\epsilon}{4}\Big(ct^2 + dt - c\Big)\Big)\sin t + O(\epsilon^2)$$

as $\epsilon \to 0\,,\ t = O(1)$

Note that the approximation is not valid for $t = O(\epsilon^{-1})$, because then the terms containing ϵt are no longer $O(\epsilon)$; these terms (e.g. $\epsilon t\cos t, \ldots$) are called secular terms; they are unbounded as $t \to \infty$. Let us derive an approximation that is uniformly valid for $0 \le t \le O(\epsilon^{-1})$.

<u>case 2</u> : $t = O(\epsilon^{-1})$

set $s = \epsilon t$, $\lambda = \dfrac{1}{\epsilon}$, $W(s,\lambda) = w(t,\epsilon) \Rightarrow w' = \dfrac{dw}{dt} = \dfrac{dW}{ds}\dfrac{1}{\lambda} = \dfrac{1}{\lambda}W'$

$w'' + (1+\epsilon t)w = 0 \Rightarrow \dfrac{1}{\lambda^2}W'' + (1+s)W = 0 \Rightarrow W'' + \lambda^2(1+s)W = 0$

recall : $W'' + \lambda^2\phi_0(x)W = 0 \Rightarrow W(x,\lambda) \sim \phi_0^{-\frac{1}{4}}(x)\exp\Big(\pm i\lambda\int^x\sqrt{\phi_0} + O\big(\lambda^{-1}\big)\Big)$

$W(s,\lambda) \sim (1+s)^{-1/4}\exp\Big(\pm i\lambda\int^s\sqrt{1+\tilde{s}}\,d\tilde{s} + O\big(\lambda^{-1}\big)\Big)$ as $\lambda \to \infty$, $s = O(1)$

$w(t,\epsilon) \sim (1+\epsilon t)^{-1/4}\exp\left(\pm\dfrac{i}{\epsilon}\dfrac{2}{3}(1+\epsilon t)^{3/2} + O(\epsilon)\right)$ as $\epsilon \to 0$, $t = O(\epsilon^{-1})$

note : this approximation has no secular terms

The solution satisfying the IC at $t = 0$ has the following approximation.

$$w(t,\epsilon) \sim (1+\epsilon t)^{-1/4}\left(c\cos\frac{2}{3\epsilon}((1+\epsilon t)^{3/2}-1) + \left(d + \frac{\epsilon c}{4}\right)\sin\frac{2}{3\epsilon}((1+\epsilon t)^{3/2}-1)\right)$$

as $\epsilon \to 0$, $t = O(\epsilon^{-1})$

Consider $t \to 0$, omit terms of order $O(\epsilon^2)$.

$$(1+\epsilon t)^{-1/4} = 1 - \frac{\epsilon}{4}t + \cdots$$

$$\frac{2}{3\epsilon}((1+\epsilon t)^{3/2}-1) = \frac{2}{3\epsilon}\left(1 + \frac{3}{2}\epsilon t + \frac{3}{8}(\epsilon t)^2 + \cdots - 1\right) = t + \frac{\epsilon}{4}t^2 + \cdots$$

$$\cos\frac{2}{3\epsilon}((1+\epsilon t)^{3/2}-1) \sim \cos\left(t + \frac{\epsilon}{4}t^2 + \cdots\right)$$

$$\sim \cos t\cos\left(\frac{\epsilon}{4}t^2 + \cdots\right) - \sin t\sin\left(\frac{\epsilon}{4}t^2 + \cdots\right) \sim \cos t + \cdots - \frac{\epsilon}{4}t^2\sin t + \cdots$$

$$\sin\frac{2}{3\epsilon}((1+\epsilon t)^{3/2}-1) \sim \sin\left(t + \frac{\epsilon}{4}t^2 + \cdots\right)$$

$$\sim \sin t\cos\left(\frac{\epsilon}{4}t^2 + \cdots\right) + \cos t\sin\left(\frac{\epsilon}{4}t^2 + \cdots\right) \sim \sin t + \cdots + \frac{\epsilon}{4}t^2\cos t + \cdots$$

$$w(t,\epsilon) \sim \left(1 - \frac{\epsilon}{4}t\right)\left(c\left(\cos t - \frac{\epsilon}{4}t^2\sin t\right) + \left(d + \frac{\epsilon c}{4}\right)\left(\sin t + \frac{\epsilon}{4}t^2\cos t\right)\right)$$

$$\sim \left(c + \frac{\epsilon}{4}\Big(dt^2 - ct\Big)\right)\cos t + \left(d - \frac{\epsilon}{4}\Big(ct^2 + dt - c\Big)\right)\sin t + O(\epsilon^2)$$

Hence case 2 reduces to case 1 for $t \to 0$.

ex: $u_{tt} = c^2(x)u_{xx}$: wave equation in non-uniform medium in 1D

$u(x,t) = v(x,\omega)e^{i\omega t}$, ω : frequency

$v'' + \frac{\omega^2}{c^2(x)}v = 0$, $\omega \to \infty$: high-frequency waves

$$v(x,\omega) \sim \sqrt{c(x)}\exp\Big(\pm i\omega \int^x \frac{d\tilde{x}}{c(\tilde{x})}\Big)$$

$$u(x,t) \sim \sqrt{c(x)}\exp\Big(i\omega\Big(t \pm \int^x \frac{d\tilde{x}}{c(\tilde{x})}\Big)\Big) \text{ as } \omega \to \infty$$

note : amplitude is proportional to square root of wave speed

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ex: $u_{tt} = c^2(u_{xx} + u_{yy})$: wave equation in uniform medium in 2D

$u(x,y,t) = v(x,y,\omega)e^{i\omega t} \Rightarrow v_{xx} + v_{yy} + \lambda^2 v = 0$, $\lambda = \frac{\omega}{c}$: Helmholtz equation

$v(x,\omega) \sim \exp(ig_0(\lambda)\psi_0(x,y) + \cdots)$

$g_0(\psi_{0xx} + \psi_{0yy}) + \cdots + (g_0\psi_{0x} + \cdots)^2 + (g_0\psi_{0y} + \cdots)^2 \sim \lambda^2$

order λ^2

$g_0^2(\psi_{0x}^2 + \psi_{0y}^2) \sim \lambda^2 \Rightarrow g_0(\lambda) = \lambda$, $\psi_{0x}^2 + \psi_{0y}^2 = 1$: eikonal equation

$u(x,y,t) \sim \exp\Big(i\frac{\omega}{c}\Big(\psi_0(x,y) + ct\Big)\Big)$ as $\omega \to \infty$: geometrical optics

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ex: $i\hbar\psi_t = -\frac{\hbar^2}{2m}\psi_{xx} + V(x)\psi$: Schrödinger equation

$\hbar$: Planck constant , m : particle mass , $V(x)$: potential

$\psi(x,t)$: wavefunction , $|\psi(x,t)|^2$: particle pdf

$\psi(x,t) = u(x)e^{-i\omega t} \Rightarrow \hbar\omega u = -\frac{\hbar^2}{2m}u'' + V(x)u$, $\hbar\omega = E$: energy , $\frac{\hbar^2}{2m} = \frac{1}{\lambda^2}$

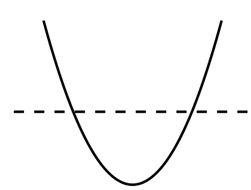
$u'' + \lambda^2(E - V(x))u = 0$: eigenvalue problem , $\lambda \to \infty$

$$u(x) \sim (E - V(x))^{-\frac{1}{4}}\exp\Big(\pm i\lambda\int^x \sqrt{E - V(\tilde{x})}\,d\tilde{x}\Big) : \begin{cases} V(x) < E : \text{oscillatory} \\ V(x) > E : \text{exponential} \end{cases}$$

example : $V(x) \sim x^2$: harmonic oscillator

classical mechanics : particle is confined to $V(x) < E$

quantum mechanics : tunneling



def : A point x_0 at which $V(x_0) = E$ is called a turning point.

<u>6.3 turning points</u>

consider $w'' + \lambda^2 \phi(x) w = 0\,,\ \lambda \to \infty$

recall : $w(x,\lambda) \sim \phi^{-\frac{1}{4}}(x) \exp\Big(\pm i\lambda \int^x \sqrt{\phi}\,\Big)$ as $\lambda \to \infty$: WKB approximation

$\phi(x) > 0\ :\ w(x,\lambda) \sim w_+(x,\lambda) = \phi^{-\frac{1}{4}}(x)\Big(c\cos\Big(\lambda\int^x\sqrt{\phi}\,\Big) + d\sin\Big(\lambda\int^x\sqrt{\phi}\,\Big)\Big)$

$\phi(x) < 0\ :\ w(x,\lambda) \sim w_-(x,\lambda) = \Big|\phi^{-\frac{1}{4}}(x)\Big|\Big(A\exp\Big(\lambda\int^x\sqrt{|\phi|}\,\Big) + B\exp\Big(-\lambda\int^x\sqrt{|\phi|}\,\Big)\Big)$

These expressions are valid for $\phi(x) \neq 0$ in the limit $\lambda \to \infty$.

Now assume that $\phi(x) \sim \nu^2 x$ as $x \to 0\,,\ \nu > 0\,,\ x_0 = 0$: turning point.

<u>goals</u>

1. For a given solution $w(x,\lambda)$, find the relation between c, d and A, B.
2. Find an approximation of $w(x,\lambda)$ valid in a neighborhood of $x = 0$.

"There is a transition solution valid near $x = 0$."

$x \to 0 \;\Rightarrow\; \phi(x) \sim \nu^2 x \;\Rightarrow\; w_0'' + \lambda^2\nu^2 x w_0 = 0$: a form of Airy's equation

<u>plan</u>

$x > 0 :\ \lim\limits_{x\to 0^+} w_+(x,\lambda) \sim \lim\limits_{\lambda\to\infty} w_0(x,\lambda)$

$x < 0 :\ \lim\limits_{x\to 0^-} w_-(x,\lambda) \sim \lim\limits_{\lambda\to\infty} w_0(x,\lambda)$

inner limit of outer solution $\sim$ outer limit of inner solution

"This is a splendid example of an asymptotic matching process."

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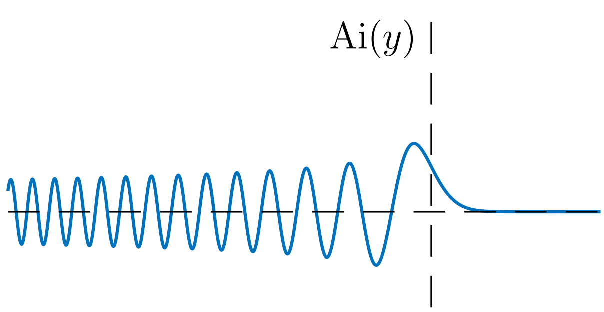
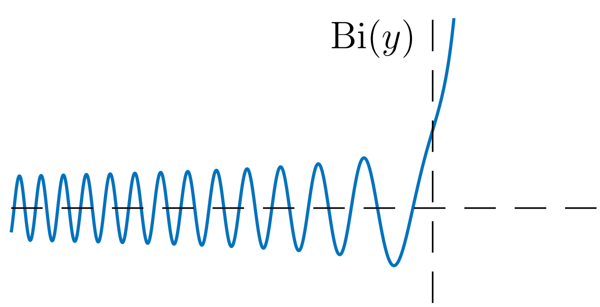
recall : $w''(y) - y w(y) = 0$: Airy equation , "of fundamental importance"

$$\mathrm{Ai}(y) = \frac{1}{2\pi i}\int_{C_1} e^{yz - \frac{1}{3}z^3} dz \sim \begin{cases} \frac{1}{2\sqrt{\pi}}\, y^{-\frac{1}{4}} \exp\big(-\frac{2}{3}y^{3/2}\big) & \text{as } y \to \infty \\ \frac{1}{\sqrt{\pi}}\, |y|^{-\frac{1}{4}} \sin\big(\frac{2}{3}|y|^{3/2} + \frac{\pi}{4}\big) & \text{as } y \to -\infty \end{cases}$$

$$\mathrm{Bi}(y) = \frac{1}{2\pi}\Big(\int_{C_2} - \int_{C_3}\Big) e^{yz - \frac{1}{3}z^3} dz \sim \begin{cases} \frac{1}{\sqrt{\pi}}\, y^{-\frac{1}{4}} \exp\big(\frac{2}{3}y^{3/2}\big) & \text{as } y \to \infty \\ \frac{1}{\sqrt{\pi}}\, |y|^{-\frac{1}{4}} \cos\big(\frac{2}{3}|y|^{3/2} + \frac{\pi}{4}\big) & \text{as } y \to -\infty \end{cases}$$





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return to equation for w_0

set $y = -(\lambda\nu)^{2/3}x$, $w_0(x,\lambda) = w(y)$

$w_0'' = w'' \cdot (\lambda\nu)^{4/3} = -(\lambda\nu)^{2/3}x \cdot w \cdot (\lambda\nu)^{4/3} = -\lambda^2\nu^2 x w = -\lambda^2\nu^2 x w_0$ <u>ok</u>

$w_0(x,\lambda) = a\,\mathrm{Ai}(-(\lambda\nu)^{2/3}x) + b\,\mathrm{Bi}(-(\lambda\nu)^{2/3}x)$

1. outer limit of inner solution , use approximations of $\mathrm{Ai}(y), \mathrm{Bi}(y)$ as $y \to \pm\infty$

$$w_0(x,\lambda) \sim \begin{cases} x > 0 \,:\, \frac{1}{\sqrt{\pi}}(\lambda\nu)^{-\frac{1}{6}}x^{-\frac{1}{4}}\Big(a\sin\big(\frac{2}{3}\lambda\nu x^{3/2} + \frac{\pi}{4}\big) + b\cos\big(\frac{2}{3}\lambda\nu x^{3/2} + \frac{\pi}{4}\big)\Big) \\ x < 0 \,:\, \frac{1}{\sqrt{\pi}}(\lambda\nu)^{-\frac{1}{6}}|x|^{-\frac{1}{4}}\Big(\frac{1}{2}a\exp\big(-\frac{2}{3}\lambda\nu|x|^{3/2}\big) + b\exp\big(\frac{2}{3}\lambda\nu|x|^{3/2}\big)\Big) \end{cases}$$

2. inner limit of outer solutions , let $x \to 0$ in WKB approximations

$$x > 0 \,:\, w_+(x,\lambda) \sim \nu^{-\frac{1}{2}}x^{-\frac{1}{4}}\Big(c\cos\big(\tfrac{2}{3}\lambda\nu x^{3/2}\big) + d\sin\big(\tfrac{2}{3}\lambda\nu x^{3/2}\big)\Big)$$

$$x < 0 \,:\, w_-(x,\lambda) \sim \nu^{-\frac{1}{2}}|x|^{-\frac{1}{4}}\Big(A\exp\big(\tfrac{2}{3}\lambda\nu|x|^{3/2}\big) + B\exp\big(-\tfrac{2}{3}\lambda\nu|x|^{3/2}\big)\Big)$$

now apply asymptotic matching to obtain <u>connection formulas</u>

$$\left.\begin{array}{l} x > 0 \,:\, c = \frac{1}{\sqrt{2\pi}}\lambda^{-\frac{1}{6}}\nu^{\frac{1}{3}}(a+b) \ ,\ d = \frac{1}{\sqrt{2\pi}}\lambda^{-\frac{1}{6}}\nu^{\frac{1}{3}}(a-b) \\ x < 0 \,:\, A = \frac{1}{\sqrt{\pi}}\lambda^{-\frac{1}{6}}\nu^{\frac{1}{3}}b \ ,\ B = \frac{1}{2\sqrt{\pi}}\lambda^{-\frac{1}{6}}\nu^{\frac{1}{3}}a \end{array}\right\} \Rightarrow \begin{cases} c = \frac{1}{\sqrt{2}}(A + 2B) \\ d = \frac{1}{\sqrt{2}}(2B - A) \end{cases}$$

The goals have been achieved.

1. Given A, B, we can determine c, d, and vice versa.

2. Given either A, B or c, d, we can determine a, b, and then $w_0(x,\lambda)$ is the required asymptotic approximation of $w(x,\lambda)$ which is uniformly valid in a neighborhood of the turning point.

<u>ex</u> : 137/1

$w'' + \lambda^2(x^2 - 1)w = 0$, $0 \le x \le 2$, $w(0) = 0$, $w'(0) = 1$

turning point : $x_0 = 1$, numerical solution by trapezoid method