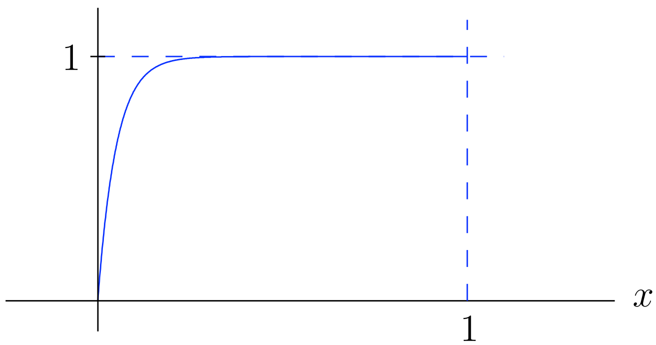
7.1 singular perturbation problems

ex 1 : $\epsilon u'' + u' = 0$, $0 \le x \le 1$, $u(0) = 0$, $u(1) = 1$, $\epsilon \to 0^+$

We could use the method of chapter 6, $u(x,\epsilon) \sim \exp(g_0(\epsilon)\psi_0(x) + \cdots)$, but we want to develop a method that can also handle nonlinear problems.

solution : $u(x,\epsilon) = \dfrac{1}{1 - e^{-1/\epsilon}}\left(1 - e^{-x/\epsilon}\right)$, check ...



This looks like a velocity profile for viscous fluid flow past a solid boundary; in that case $\epsilon = Re^{-1}$, where Re is the Reynolds number.

1. $u(x,\epsilon)$ has a boundary layer of width $O(\epsilon)$ near $x = 0$.
2. $\lim_{\epsilon\to 0} u(x,\epsilon) = \begin{cases} 0 , & x = 0 \\ 1 , & 0 < x \le 1 \end{cases}$: non-uniform convergence as $\epsilon \to 0$
3. $\lim_{x\to 0}\lim_{\epsilon\to 0} u(x,\epsilon) = 1$, $\lim_{\epsilon\to 0}\lim_{x\to 0} u(x,\epsilon) = 0$: limits do not commute
4. $u(x,\epsilon)$ has an essential singularity at $\epsilon = 0$.

goal : Find an approximation which is uniformly valid for $0 \le x \le 1$ as $\epsilon \to 0$.

regular perturbation series

$u(x,\epsilon) \sim u_0(x) + \epsilon u_1(x) + \cdots$ as $\epsilon \to 0$

$\epsilon(u_0'' + \epsilon u_1'' + \cdots) + u_0' + \epsilon u_1' + \cdots \sim 0$

order ϵ^0 : $u_0' = 0 \Rightarrow u_0(x) = \text{cnst}$: cannot satisfy both BC

case 1 : $u_0(x) = 1$

Since $u_0(0) \ne 0$, $u_0(x)$ is not valid near $x = 0$.

order ϵ^1 : $u_0'' + u_1' = 0 \Rightarrow u_1(x) = \text{cnst}$, $u_1(1) = 0 \Rightarrow u_1(x) = 0$: no progress

We implicitly assumed that $\epsilon u'' = O(\epsilon)$, which is not true near $x = 0$;
to approximate the solution near $x = 0$, we consider a rescaled problem.

$s = \dfrac{x}{\phi(\epsilon)}$, $v(s) = u(x)$, $u' = \dfrac{v'}{\phi(\epsilon)}$, $u'' = \dfrac{v''}{\phi(\epsilon)^2}$

$\epsilon u'' + u' = 0 \Rightarrow \epsilon \dfrac{v''}{\phi(\epsilon)^2} + \dfrac{v'}{\phi(\epsilon)} = 0$

choose $\phi(\epsilon) = \epsilon$, $s = \dfrac{x}{\epsilon}$

then $v'' + v' = 0$, $v(0) = 0 \Rightarrow v(s) = c\,(1 - e^{-s})$

$v(s)$: valid for x near 0 , inner solution

$u_0(x)$: valid for x away from 0 , outer solution

note : The terminology comes from aerodynamics.

matching

$\lim_{s\to\infty} v(s) = \lim_{x\to 0} u_0(x) \Rightarrow c = 1$

outer limit of inner solution = inner limit of outer solution

uniform asymptotic approximation

$\bar{u}(x) = v(s) + u_0(x) - c = 1 - e^{-x/\epsilon}$

Then $u(x) = \bar{u}(x) + O(\epsilon)$ as $\epsilon \to 0$, uniformly for $0 \le x \le 1$,
i.e. there exist $K > 0, \epsilon_0 > 0$ st $0 < \epsilon < \epsilon_0 \Rightarrow \max_{0\le x\le 1} |u(x,\epsilon) - \bar{u}(x,\epsilon)| < K\epsilon$.

case 2 : $u_0(x) = 0$

Since $u_0(1) \neq 1$, $u_0(x)$ is not valid near $x = 1$. As before, higher order terms in the regular perturbation series are no help, so consider a rescaled problem near $x = 1$.

$s = \dfrac{1-x}{\phi(\epsilon)}$, $v(s) = u(x)$, $u' = -\dfrac{v'}{\phi(\epsilon)}$, $u'' = \dfrac{v''}{\phi(\epsilon)^2} \Rightarrow \dfrac{\epsilon v''}{\phi(\epsilon)^2} - \dfrac{v'}{\phi(\epsilon)} = 0$

choose $\phi(\epsilon) = \epsilon$, $s = \dfrac{1-x}{\epsilon}$

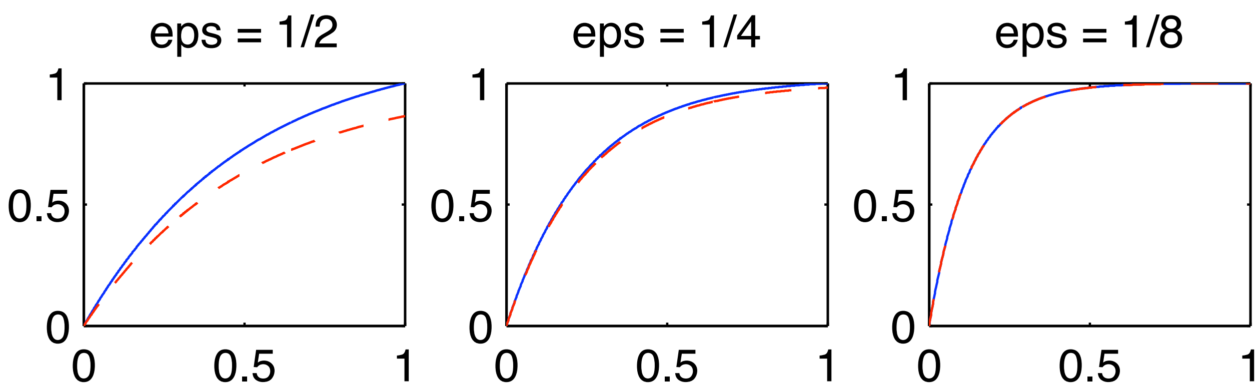
then $v'' - v' = 0$, $v(0) = 1 \Rightarrow v(s) = 1 + c(1 - e^{s})$

matching $\Rightarrow \lim_{s\to\infty} v(s) = \lim_{x\to 1} u_0(x)$: impossible ex 1

$\epsilon u'' + u' = 0$, $u(0) = 0$, $u(1) = 1$

$u(x) = \dfrac{1}{1 - e^{-1/\epsilon}} \left(1 - e^{-x/\epsilon}\right)$: exact solution , solid

$\bar{u}(x) = 1 - e^{-x/\epsilon}$: uniform asymptotic approximation to $O(\epsilon)$, dashed



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ex 2

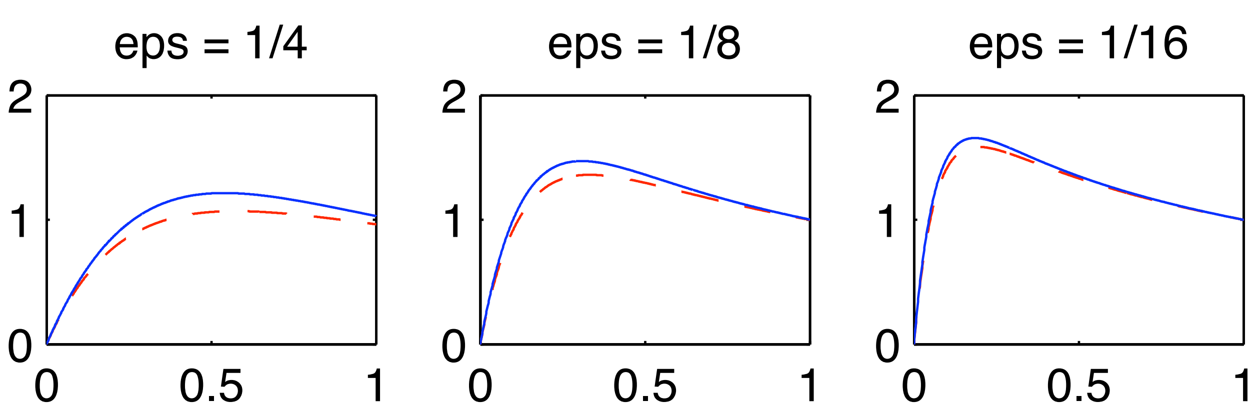
$\epsilon u'' + (1+x)u' + u = 0$, $u(0) = 0$, $u(1) = 1$

$u(x, \epsilon) = e^{(4-(1+x)^2)/2\epsilon} \int_0^x e^{(1+t)^2/2\epsilon} dt \Big/ \int_0^1 e^{(1+t)^2/2\epsilon} dt$: exact solution

$\bar{u}_0(x), \bar{u}_1(x)$: uniform asymptotic approximations to order $O(\epsilon), O(\epsilon^2)$

$\bar{u}_0(x) = \dfrac{2}{1+x} - 2e^{-x/\epsilon}$, dashed

$\bar{u}_1(x) = \dfrac{2}{1+x} + \epsilon\left(\dfrac{2}{(1+x)^3} - \dfrac{1}{2(1+x)}\right) + \dfrac{e^{-x/\epsilon}}{\epsilon}\left(x^2 - 2\epsilon - \tfrac{3}{2}\epsilon^2\right)$, solid



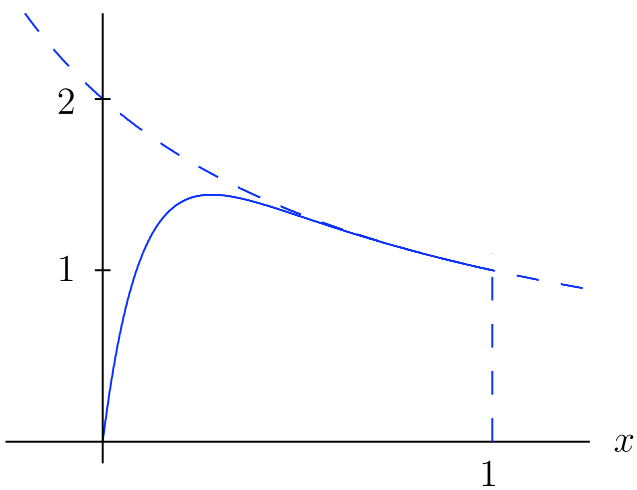
<u>ex 2</u> : $\epsilon u'' + (1+x)u' + u = 0$, $u(0) = 0$, $u(1) = 1$

solution : $u(x,\epsilon) = e^{(4-(1+x)^2)/2\epsilon} \int_0^x e^{(1+t)^2/2\epsilon} dt \Big/ \int_0^1 e^{(1+t)^2/2\epsilon} dt$, check . . .

regular perturbation series : $u(x,\epsilon) \sim u_0(x) + \epsilon u_1(x) + \cdots$

order ϵ^0 : $(1+x)u_0' + u_0 = 0 \Rightarrow ((1+x)u_0)' = 0 \Rightarrow u_0(x) = \dfrac{c}{1+x}$

choose $u_0(1) = 1 \Rightarrow u_0(x) = \dfrac{2}{1+x}$



order ϵ^1 : $u_0'' + (1+x)u_1' + u_1 = 0 \Rightarrow ((1+x)u_1)' = -u_0''$

$$\Rightarrow (1+x)u_1 = -u_0' + c = \frac{2}{(1+x)^2} + c$$

$u_1(1) = 0 \Rightarrow c = -\frac{1}{2} \Rightarrow u_1(x) = \dfrac{2}{(1+x)^3} - \dfrac{1}{2(1+x)}$

outer solution : $u_0(x) + \epsilon u_1(x) = \dfrac{2}{1+x} + \epsilon\left(\dfrac{2}{(1+x)^3} - \dfrac{1}{2(1+x)}\right)$

rescaling : $s = \dfrac{x}{\phi(\epsilon)}$, $v(s) = u(x)$, $u' = \dfrac{v'}{\phi(\epsilon)}$, $u'' = \dfrac{v''}{\phi(\epsilon)^2}$

$\epsilon u'' + (1+x)u' + u = 0 \Rightarrow \epsilon \dfrac{v''}{\phi(\epsilon)^2} + (1+\phi(\epsilon)s)\dfrac{v'}{\phi(\epsilon)} + v = 0$

choose $\dfrac{\epsilon}{\phi(\epsilon)^2} = \dfrac{1}{\phi(\epsilon)} \Rightarrow \phi(\epsilon) = \epsilon$, note : $\dfrac{\epsilon}{\phi(\epsilon)^2} = 1 \Rightarrow \phi(\epsilon) = \sqrt{\epsilon}$: fails

$v'' + (1+\epsilon s)v' + \epsilon v = 0$, $v(0) = 0$

regular perturbation series in rescaled variable

$v(s,\epsilon) = v_0(s) + \epsilon v_1(s) + \cdots$

order ϵ^0 : $v_0'' + v_0' = 0\ ,\ v_0(0) = 0\ \Rightarrow\ v_0(s) = c(1 - e^{-s})$

matching at ϵ^0 : $\lim_{s\to\infty} v_0(s) = \lim_{x\to 0} u_0(x)\ \Rightarrow\ c = 2\ \Rightarrow\ v_0(s) = 2(1 - e^{-s})$

<u>uniform approximation to $O(\epsilon)$</u>

$$\bar{u}_0(x) = u_0(x) + v_0(s) - c = \frac{2}{1+x} - 2e^{-x/\epsilon}$$

order ϵ^1 : $v_1'' + v_1' + sv_0' + v_0 = 0\ ,\ v_1(0) = 0$

$$v_1'' + v_1' = -sv_0' - v_0 = -(sv_0)'$$

$$v_1' + v_1 = -sv_0 + c = -2s(1 - e^{-s}) + c$$

$$(e^s v_1)' = -2s(e^s - 1) + ce^s = 2s(1 - e^s) + ce^s$$

$$e^s v_1(s) = 2\int_0^s t(1 - e^t)dt + c(e^s - 1) = 2\left(\frac{s^2}{2} - e^s(s-1) - 1\right) + c(e^s - 1)$$

$$v_1(s) = e^{-s}(s^2 - 2) + 2(1 - s) + c(1 - e^{-s})$$

matching at ϵ^1 : $\lim_{s\to\infty} v_1(s) = \lim_{x\to 0} u_1(x)$: fails

<u>Van Dyke matching principle</u>

match inner and outer expansions to $O(\epsilon)$

recall : $s = \dfrac{x}{\epsilon}\ ,\ x = \epsilon s$

$$\begin{aligned} v_0(s) + \epsilon v_1(s) &= 2(1 - e^{-s}) + \epsilon(e^{-s}(s^2 - 2) + 2(1 - s) + c(1 - e^{-s})) \\ &\sim 2 + \epsilon(2(1 - s) + c) = 2(1 - \epsilon s) + \epsilon(2 + c) \text{ as } s \to \infty \end{aligned}$$

$$\begin{aligned} u_0(x) + \epsilon u_1(x) &= \frac{2}{1+x} + \epsilon\left(\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right) \\ &\sim\ 2(1 - x) + \epsilon(2 - \tfrac{1}{2}) \text{ as } x \to 0\ \Rightarrow\ c = -\tfrac{1}{2} \end{aligned}$$

<u>uniform approximation to order $O(\epsilon^2)$</u>

$$\begin{aligned} \bar{u}_1(x) &= u_0(x) + \epsilon u_1(x) + v_0(s) + \epsilon v_1(s) - (2(1 - x) + \tfrac{3}{2}\epsilon) \\ &= \frac{2}{1+x} + \epsilon\left(\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right) + 2(\cancel{1} - e^{-x/\epsilon}) \\ &\quad + \epsilon\left(e^{-x/\epsilon}\left(\frac{x^2}{\epsilon^2} - 2\right) + \cancel{2\left(1 - \frac{x}{\epsilon}\right)} - \cancel{\tfrac{1}{2}(1} - e^{-x/\epsilon})\right) - (\cancel{2(1 - x)} + \cancel{\tfrac{3}{2}\epsilon}) \\ &= \frac{2}{1+x} + \epsilon\left(\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right) + \frac{e^{-x/\epsilon}}{\epsilon}\ (x^2 - 2\epsilon - \tfrac{3}{2}\epsilon^2) \end{aligned}$$

ex 3 : $\epsilon u'' + uu' - u = 0$, $u(0) = A$, $u(1) = B$ (Kevorkian and Cole, p. 56)

outer solution

$u(x, \epsilon) = u_0(x) + \epsilon u_1(x) + \cdots$

$\epsilon(u_0'' + \epsilon u_1'' + \cdots) + (u_0 + \epsilon u_1 + \cdots)(u_0' + \epsilon u_1' + \cdots) - (u_0 + \epsilon u_1 + \cdots) \sim 0$

order ϵ^0 : $u_0 u' - u_0 = 0 \Rightarrow u_0(u_0' - 1) = 0 \Rightarrow u_0 = 0$ or $u_0' = 1$

There are 3 possible outer solutions.

$u_0(x) = 0$, $u_0(x) = x + A$, $u_0(x) = x + B - 1$

inner solution

$s = \dfrac{x - x_0}{\phi(\epsilon)}$, $v(s) = \psi(\epsilon)u(x)$: we scale u because the equation is nonlinear

$$u' = \frac{v'}{\psi\phi} \ , \ u'' = \frac{v''}{\psi\phi^2}$$

$$\epsilon u'' + uu' - u = 0 \ \Rightarrow \ \frac{\epsilon v''}{\psi\phi^2} + \frac{v}{\psi} \cdot \frac{v'}{\psi\phi} - \frac{v}{\psi} = 0$$

$$\frac{\epsilon v''}{\phi^2} + \frac{vv'}{\psi\phi} - v = 0$$

case a

$$\frac{\epsilon}{\phi^2} = \frac{1}{\psi\phi} = \frac{1}{\epsilon} \ \Rightarrow \ \phi(\epsilon) = \epsilon \ , \ \psi(\epsilon) = 1 \ \Rightarrow \ s = \frac{x - x_0}{\epsilon} \ , \ u(x) = v(s)$$

$$v'' + vv' - \epsilon v = 0 \ \Rightarrow \ v_0'' + v_0 v_0' = 0$$

case b

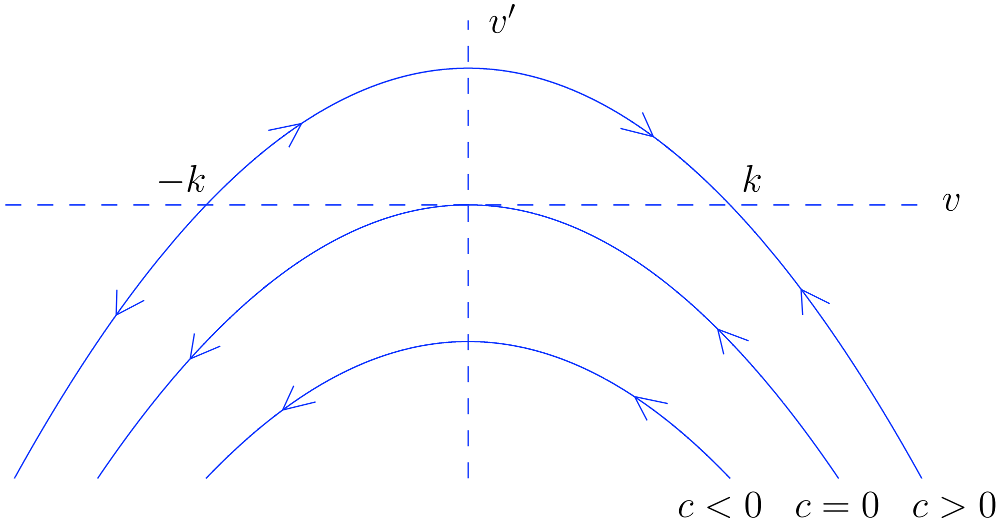
$$\frac{\epsilon}{\phi^2} = \frac{1}{\psi\phi} = 1 \ \Rightarrow \ \phi(\epsilon) = \sqrt{\epsilon} \ , \ \psi(\epsilon) = \frac{1}{\sqrt{\epsilon}} \ \Rightarrow \ s = \frac{x - x_0}{\sqrt{\epsilon}} \ , \ u(x) = \sqrt{\epsilon} v(s)$$

$$v'' + vv' - v = 0 \ \Rightarrow \ v_0'' + v_0 v_0' - v_0 = 0$$

case c

$$\frac{\epsilon}{\phi^2} = 1 \ \Rightarrow \ \phi(\epsilon) = \sqrt{\epsilon} \ , \ \text{choose } \psi(\epsilon) = \frac{1}{\epsilon} \ \Rightarrow \ s = \frac{x - x_0}{\sqrt{\epsilon}} \ , \ u(x) = \epsilon v(s)$$

$v'' + \sqrt{\epsilon} vv' - v = 0 \ \Rightarrow \ v_0'' - v_0 = 0$: will not need this case consider case a : $v_0'' + v_0 v_0' = 0 \;\Rightarrow\; v'' + (\frac{1}{2}v^2)' = 0 \;\Rightarrow\; v' + \frac{1}{2}v^2 = c$



$c > 0$, $c = \frac{1}{2}k^2 \;\Rightarrow\; v' = \frac{1}{2}(k^2 - v^2)$, $k > 0$

$$\frac{dv}{ds} = \tfrac{1}{2}(k^2 - v^2) \;\Rightarrow\; \frac{dv}{k^2 - v^2} = \frac{ds}{2}$$

$$\frac{1}{k^2 - v^2} = \frac{1}{2k}\left(\frac{1}{k+v} + \frac{1}{k-v}\right) \;\Rightarrow\; \frac{dv}{k+v} + \frac{dv}{k-v} = kds$$

<u>case a1</u> : $-k < v < k$

$$\ln(k+v) - \ln(k-v) = \ln\left(\frac{k+v}{k-v}\right) = k(s+s_0) \;\Rightarrow\; \frac{k+v}{k-v} = e^{k(s+s_0)}$$

$$\Rightarrow\; v(1 + e^{k(s+s_0)}) = k(e^{k(s+s_0)} - 1) \;\Rightarrow\; v = k\left(\frac{e^{k(s+s_0)} - 1}{e^{k(s+s_0)} + 1}\right) = k\tanh\tfrac{1}{2}k(s+s_0)$$

<u>case a2</u> : $v > k$

$$\frac{dv}{k+v} - \frac{dv}{v-k} = kds$$

$$\ln(k+v) - \ln(v-k) = \ln\left(\frac{v+k}{v-k}\right) = k(s+s_0) \;\Rightarrow\; \frac{v+k}{v-k} = e^{k(s+s_0)}$$

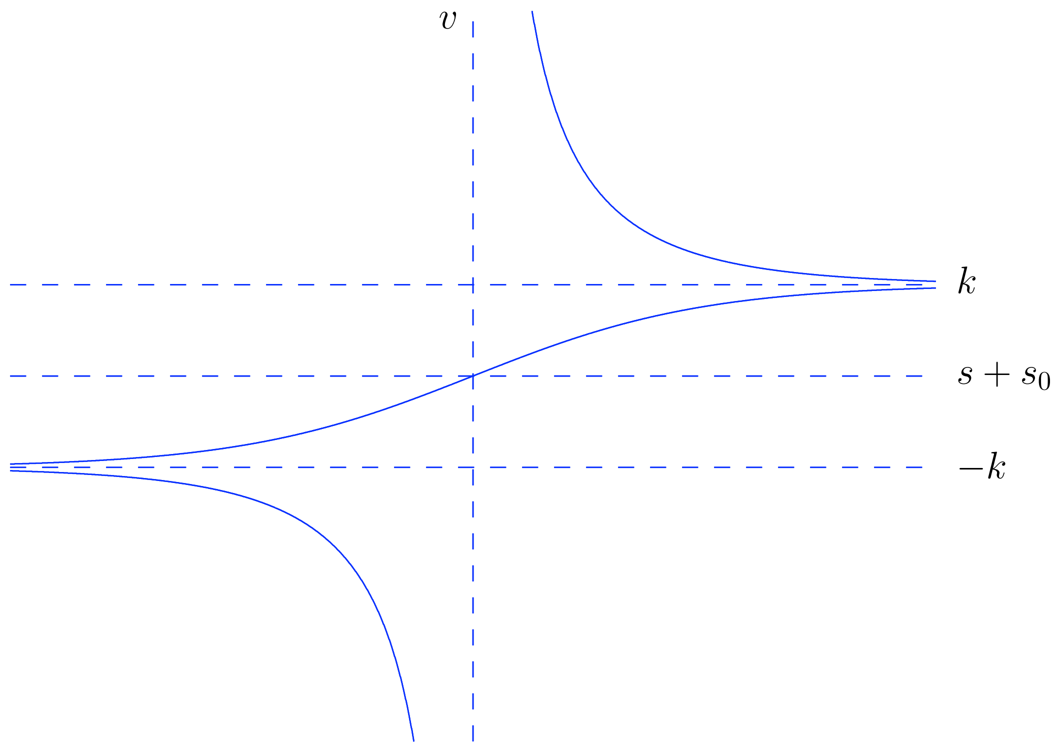
$$\Rightarrow\; v(1 - e^{k(s+s_0)}) = -k(e^{k(s+s_0)} + 1) \;\Rightarrow\; v = k\left(\frac{e^{k(s+s_0)} + 1}{e^{k(s+s_0)} - 1}\right) = k\coth\tfrac{1}{2}k(s+s_0)$$

<u>case a3</u> : $v < -k$, write $w = -v$, so $w > k$

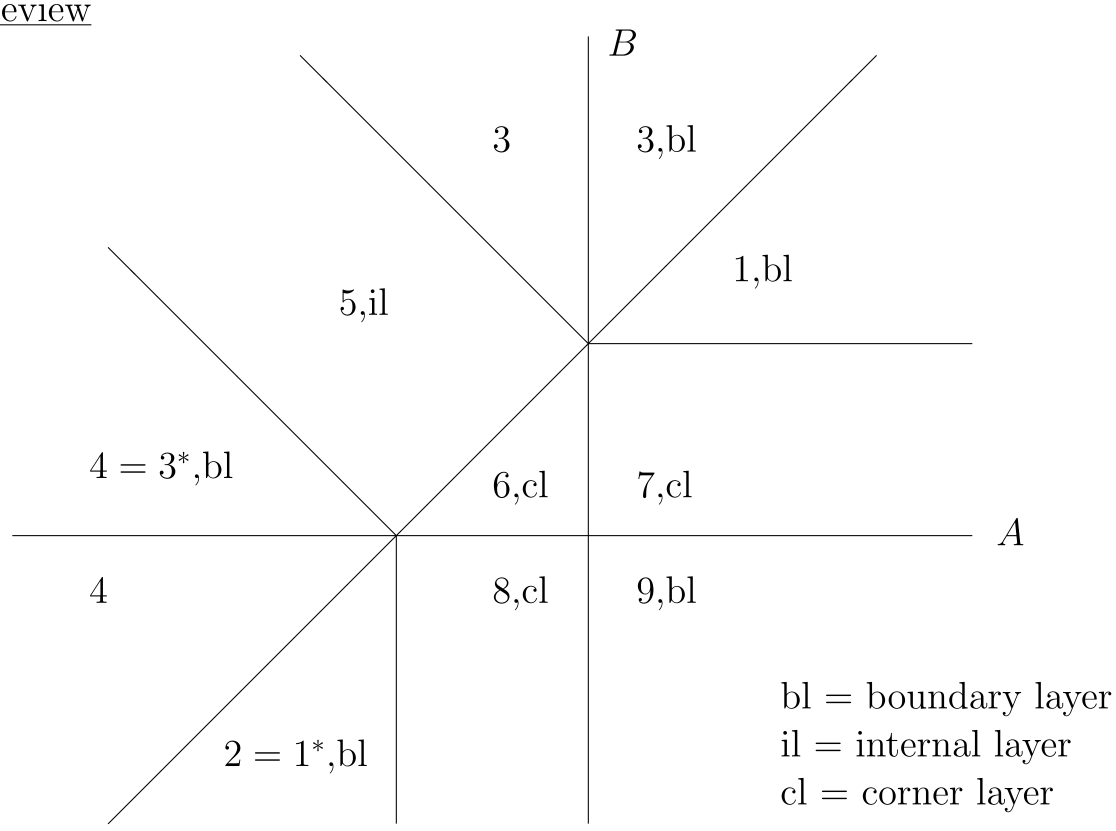
$$\frac{-dw}{k-w} + \frac{-dw}{k+w} = \frac{dw}{w-k} - \frac{dw}{w+k} = kds \;\Rightarrow\; \cdots \;\Rightarrow\; v = k\coth\tfrac{1}{2}k(s+s_0)$$

<u>summary</u>

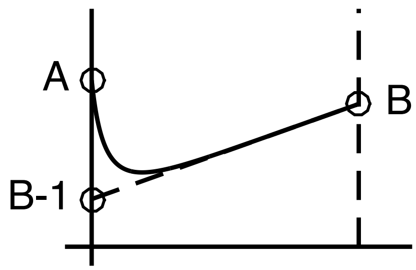
In case a the inner solution is $v(s) = k \tanh \frac{1}{2}k(s+s_0)\,,\; k \coth \frac{1}{2}k(s+s_0)$.



<u>preview</u>



<u>region 1</u> : $A > B - 1 > 0$, e.g. $A = 1.75$, $B = 1.5$



choose $u_0(x) = x + B - 1$: satisfies BC at $x = 1$

Then there is a boundary layer at $x = 0$.

$s = \dfrac{x}{\epsilon}$, $v(s) = k \coth \frac{1}{2}k(s + s_0)$

matching $\Rightarrow \lim_{s\to\infty} v(s) = \lim_{x\to 0} u_0(x) \Rightarrow k = B - 1$

$v(0) = A \Rightarrow (B-1)\coth\frac{1}{2}(B-1)s_0 = A \Rightarrow \exists\, s_0 > 0$, note : tanh fails

uniform asymptotic approximation

$$\bar{u}(x) = u_0(x) + v(s) - (B-1) = x + (B-1)\coth\left[\tfrac{1}{2}(B-1)\left(\frac{x}{\epsilon} + s_0\right)\right]$$

1. $x = 0 \Rightarrow \bar{u}(0) = A$

2. $0 < x \le 1 \Rightarrow \lim_{\epsilon\to 0} \bar{u}(x) = x + B - 1$

suppose we had chosen $u_0(x) = x + A$: satisfies BC at $x = 0$

Then there is a boundary layer at $x = 1$.

$s = \dfrac{x-1}{\epsilon}$, $v(s) = k \coth \frac{1}{2}k(s + s_0)$

matching $\Rightarrow \lim_{s\to -\infty} v(s) = \lim_{x\to 1} u_0(x) \Rightarrow -k = 1 + A > 0$: fails , tanh also fails

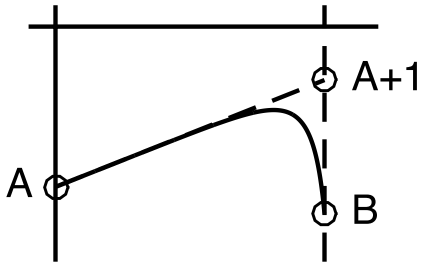
suppose $u_0(x) = 0$: satisfies neither BC

Then there are boundary layers at $x = 0, 1$.

for example at $x = 0$, $s = \dfrac{x}{\epsilon}$, $v(s) = k \coth \frac{1}{2}k(s + s_0)$

matching $\Rightarrow \lim_{s\to\infty} v(s) = \lim_{x\to 0} u_0(x) \Rightarrow k = 0$: fails , tanh also fails

<u>region 2</u> : $B < A+1 < 0$, e.g. $A = -1.5$, $B = -1.75$



choose $u_0(x) = A + x$: satisfies BC at $x = 0$

Then there is a boundary layer at $x = 1$.

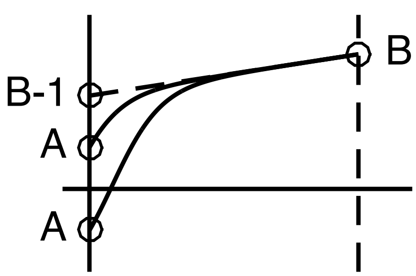
$s = \dfrac{x-1}{\epsilon}$, $v(s) = k \coth \tfrac{1}{2} k(s + s_0)$

$\lim\limits_{s \to -\infty} v(s) = \lim\limits_{x \to 1} u_0(x) \Rightarrow -k = A + 1 \Rightarrow k = -(A+1)$

$v(0) = B \Rightarrow (A+1) \coth \tfrac{1}{2}(A+1)s_0 = B \Rightarrow \exists\, s_0 < 0$, tanh fails

$$\bar{u}(x) = u_0(x) + v(s) - (A+1) = x - 1 + (A+1)\coth\left[\tfrac{1}{2}(A+1)\left(\frac{x-1}{\epsilon} + s_0\right)\right]$$

<u>region 3</u> : $0 < |A| < B - 1$, e.g. $A = \pm 1$, $B = 3.25$



choose $u_0(x) = x + B - 1$: satisfies BC at $x = 1$

Then there is a boundary layer at $x = 0$.

$s = \dfrac{x}{\epsilon}$, $v(s) = k \tanh \tfrac{1}{2} k(s + s_0)$

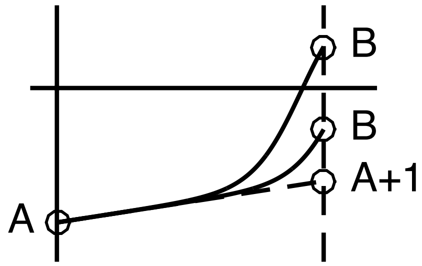
$\lim\limits_{s \to \infty} v(s) = \lim\limits_{x \to 0} u_0(x) \Rightarrow k = B - 1$

$v(0) = A \Rightarrow (B-1) \tanh \tfrac{1}{2}(B-1)s_0 = A \Rightarrow \exists \begin{cases} s_0 > 0 \text{ if } A > 0 \\ s_0 < 0 \text{ if } A < 0 \end{cases}$

$$\bar{u}(x) = u_0(x) + v(s) - (B-1) = x + (B-1)\tanh\left[\tfrac{1}{2}(B-1)\left(\frac{x}{\epsilon} + s_0\right)\right]$$

note : $\begin{cases} A > 0 \Rightarrow u(x) \text{ is concave down} \\ A < 0 \Rightarrow u(x) \text{ has an inflection point at } x = -\epsilon s_0 > 0 \end{cases}$

region 4 : $A+1<0\,,\ |B|<|A+1|$, e.g. $A=-3.25\,,\ B=\pm 1$



choose $u_0(x)=A+x$: satisfies BC at $x=0$

Then there is a boundary layer at $x=1$.

$s=\dfrac{x-1}{\epsilon}$, $v(s)=k\tanh\frac{1}{2}k(s+s_0)$

$\lim\limits_{s\to-\infty} v(s)=\lim\limits_{x\to 1} u_0(x) \Rightarrow -k=A+1 \Rightarrow k=-(A+1)$

$v(0)=B \Rightarrow (A+1)\tanh\frac{1}{2}(A+1)s_0=B \Rightarrow \exists \begin{cases} s_0>0 \text{ if } B>0 \\ s_0<0 \text{ if } B<0 \end{cases}$

$\bar{u}(x)=u_0(x)+v(s)-(A+1)=x-1+(A+1)\tanh\left[\frac{1}{2}(A+1)\left(\dfrac{x-1}{\epsilon}+s_0\right)\right]$

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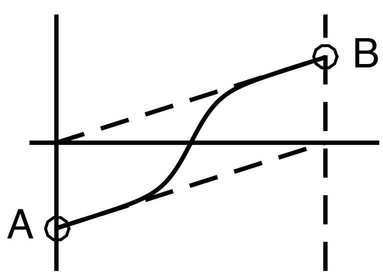
region 5 : $-1<A+B<1\,,\ B>A+1$, e.g. $A=-1\,,\ B=1$

note : $B>A+1\,,\ B>-(A+1) \Rightarrow B>|A+1|>0$



choose $u_0(x)=A+x$: satisfies BC at $x=0$

Then there is a boundary layer at $x=1$.

$s=\dfrac{x-1}{\epsilon}$, $v(s)=k\tanh\frac{1}{2}k(s+s_0)$

$\lim\limits_{s\to-\infty} v(s)=\lim\limits_{x\to 1} u_0(x) \Rightarrow -k=A+1 \Rightarrow k=-(A+1)$

$v(0)=B \Rightarrow (A+1)\tanh\frac{1}{2}(A+1)s_0=B$: fails try $v(s) = k \coth \frac{1}{2} k(s+s_0)$

$\lim_{s\to-\infty} v(s) = \lim_{x\to 1} u_0(x) \;\Rightarrow\; -k = A+1 \;\Rightarrow\; k = -(A+1)$

$v(0) = B \;\Rightarrow\; (A+1)\coth \frac{1}{2}(A+1)s_0 = B \;\Rightarrow\; \exists\, s_0 > 0$

$\bar{u}(x) = u_0(x) + v(s) - (A+1) = x - 1 + (A+1)\coth\Big[\frac{1}{2}(A+1)\Big(\frac{x-1}{\epsilon} + s_0\Big)\Big]$

This blows up because $-\frac{1}{\epsilon} + s_0 < 0 < s_0$ for $\epsilon \to 0$.

choose $u_0(x) = x + B - 1$: satisfies BC at $x = 1$, ... : fails

choose $u_L(x) = x + A$, $u_R(x) = x + B - 1$

Let x_0 be the position of an <u>internal layer</u>.

$s = \dfrac{x-x_0}{\epsilon}$, $v(s) = k \tanh \frac{1}{2} k(s+s_0)$

matching for $x \to x_0^-$: $\lim_{s\to-\infty} v(s) = \lim_{x\to x_0} u_L(x) \;\Rightarrow\; -k = x_0 + A$

matching for $x \to x_0^+$: $\lim_{s\to\infty} v(s) = \lim_{x\to x_0} u_R(x) \;\Rightarrow\; k = x_0 + B - 1$

$\Rightarrow\; x_0 = \dfrac{1-(A+B)}{2} \;\Rightarrow\; 0 < x_0 < 1$

$$\begin{aligned}\bar{u}(x) &= u_L(x) + v(s) - (x_0 + A) \\ &= x + A + (x_0+A)\Big(\tanh\Big[\tfrac{1}{2}(x_0+A)\Big(\frac{x-x_0}{\epsilon} + s_0\Big)\Big] - 1\Big)\end{aligned}$$

note : $x_0 + A = \dfrac{1+A-B}{2} < 0$

$0 < x < x_0 \Rightarrow \bar{u}(x) \sim x + A = u_L(x)$

$x_0 < x < 1 \Rightarrow \bar{u}(x) \sim x + A - 2(x_0+A) = x + A - (1+A-B) = x + B - 1 = u_R(x)$

An extra condition is needed to determine s_0.

for example : $\bar{u}(x_0) = 0 \;\Rightarrow\; u_L(x_0) + v(0) - (x_0 + A) = 0 \;\Rightarrow\; v(0) = 0 \;\Rightarrow\; s_0 = 0$

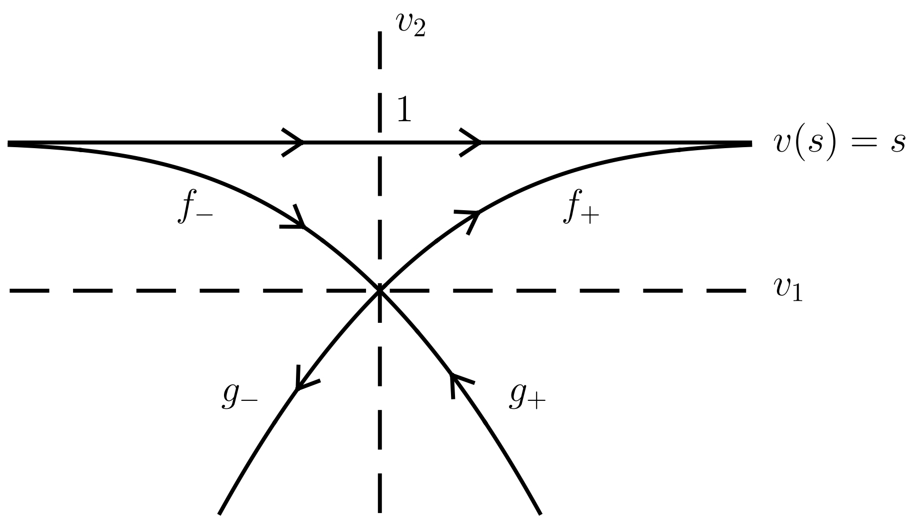
In the remaining cases we use a different inner solution.

case b : $s = \dfrac{x-x_0}{\sqrt{\epsilon}}$, $v(s) = \dfrac{1}{\sqrt{\epsilon}} u(x)$, $v'' + vv' - v = 0$

note : 1. $v(s) = s$ is a solution

2. If $v(s)$ is a solution, then so is $v(s+s_0)$ for any s_0.

define : $\left.\begin{array}{l} v_1 = v \\ v_2 = v' \end{array}\right\} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}' = \begin{pmatrix} v_2 \\ v_1(1-v_2) \end{pmatrix}$



fixed point : $(v_1, v_2) = (0,0)$, linearized system : $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}' = \begin{pmatrix} v_2 \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

e-values , e-vectors : $\lambda_1 = 1$, $q_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda_2 = -1$, $q_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$v'' + vv' - v = 0 \Rightarrow v'' + v(v'-1) = 0 \Rightarrow v''v' + v(v'-1)v' = 0$

$\Rightarrow \dfrac{v''v'}{v'-1} + vv' = 0 \Rightarrow v'' \left(\dfrac{v'-1+1}{v'-1} \right) + vv' = 0 \Rightarrow v'' - \dfrac{v''}{1-v'} + vv' = 0$

$v' + \ln(1-v') + \frac{1}{2}v^2 = c = 0$ (at fixed point) , assumes $v' < 1$

$v_2 + \ln(1-v_2) + \frac{1}{2}v_1^2 = 0$

consider solutions such that $0 < v_2 < 1$: increasing functions of s

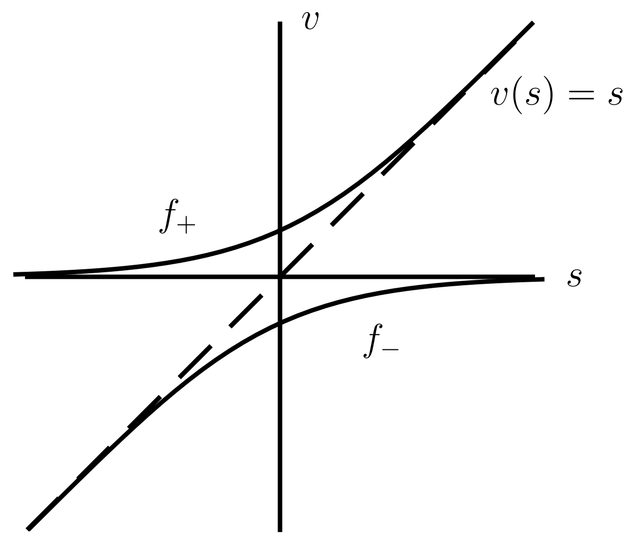
define $f_+(s) > 0$, $f_-(s) < 0$

$\lim_{s \to -\infty} f_+(s) = 0$, $f_+(s) \sim s$ as $s \to \infty$

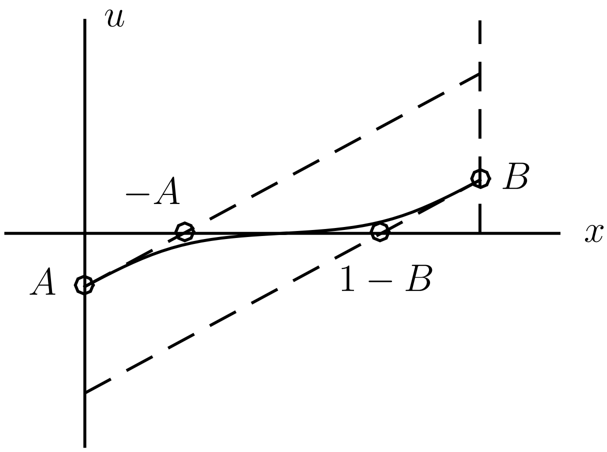
$\lim_{s \to \infty} f_-(s) = 0$, $f_-(s) \sim s$ as $s \to -\infty$

check that $f_+''(s) > 0$, $f_-''(s) < 0$

f_+ , f_- : <u>corner solutions</u>



<u>region 6</u> : $0 < -A < 1 - B < 1$, e.g. $A = -0.25$, $B = 0.25$



Use the 2 corner solutions to connect the 3 outer solutions.

$\bar{u}(x) = \sqrt{\epsilon}\, f_-\left(\dfrac{x+A}{\sqrt{\epsilon}}\right) + \sqrt{\epsilon}\, f_+\left(\dfrac{x+B-1}{\sqrt{\epsilon}}\right)$, note : $x_0 = -A$, $1-B$

$0 < x < -A \Rightarrow x + A < 0$, $x + B - 1 < 0 \Rightarrow \bar{u}(x) \sim x + A$

$-A < x < 1 - B \Rightarrow x + A > 0$, $x + B - 1 < 0 \Rightarrow \bar{u}(x) \sim 0$

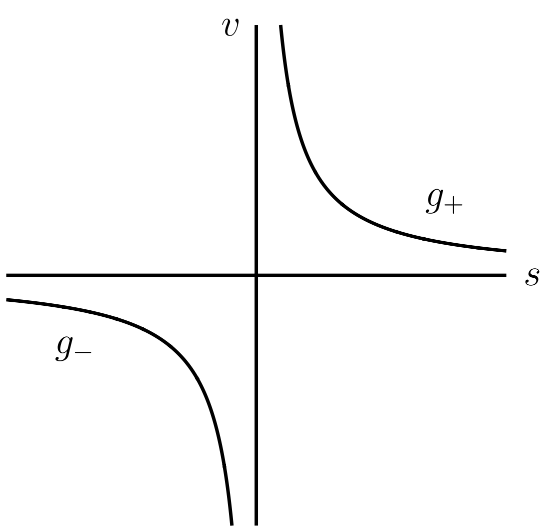
$1 - B < x < 1 \Rightarrow x + A > 0$, $x + B - 1 > 0 \Rightarrow \bar{u}(x) \sim x + B - 1$

---

recall : $v_2 + \ln(1 - v_2) + \frac{1}{2}v_1^2 = 0$, $v_1 = v$, $v_2 = v'$

consider solutions such that $v_2 < 0$: decreasing functions of s

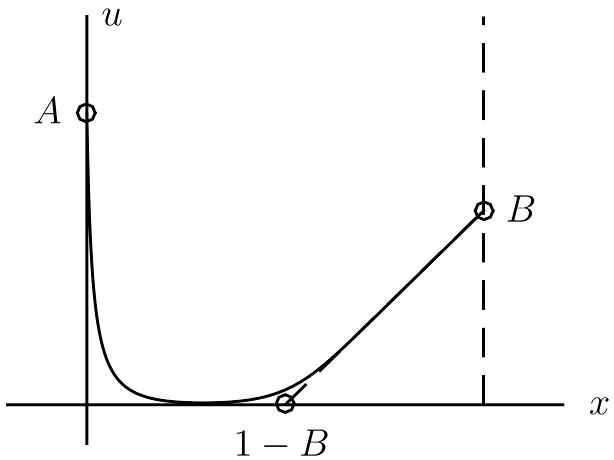
$v_2 \to -\infty \Rightarrow v_2 \sim -\frac{1}{2}v_1^2 \Rightarrow v' \sim -\frac{1}{2}v^2$, choose $v(s) = \dfrac{2}{s}$



$g_+(s) > 0 : \lim_{s\to\infty} g_+(s) = 0$, $\lim_{s\to 0^+} g_+(s) = \infty$

$g_-(s) < 0 : \lim_{s\to -\infty} g_-(s) = 0$, $\lim_{s\to 0^-} g_-(s) = -\infty$

<u>region 7</u> : $A > 0\,,\, 0 < B < 1$, e.g. $A = 0.75\,,\, B = 0.5$



need to combine $g_+(s)\,,\, f_+(s)$ and use freedom to choose s_0

$g_+(s)\,:\, x_0 = 0$, set s_0 such that $g_+(s_0) = \dfrac{2}{s_0} = \dfrac{A}{\sqrt{\epsilon}}$

$f_+(s)\,:\, x_0 = 1 - B$, set $s_0 = 0$

$$\bar{u}(x) = \sqrt{\epsilon}\, g_+\left(\frac{x}{\sqrt{\epsilon}} + s_0\right) + \sqrt{\epsilon}\, f_+\left(\frac{x + B - 1}{\sqrt{\epsilon}}\right)$$

$$\bar{u}(0) = \sqrt{\epsilon}\, g_+(s_0) + \sqrt{\epsilon}\, f_+\left(\frac{B - 1}{\sqrt{\epsilon}}\right) \sim A$$

$0 < x < 1 - B \;\Rightarrow\; \bar{u}(x) \sim 0$

$1 - B < x < 1 \;\Rightarrow\; \bar{u}(x) \sim x + B - 1$

<u>summary</u>

The various phenomena are as follows.

1. boundary layer of width $O(\epsilon)$: tanh, coth , w/wo inflection points
2. internal layer of width $O(\epsilon)$: tanh
3. corner layer of width $O(\sqrt{\epsilon})$: $f_\pm$
4. boundary layer of width $O(\sqrt{\epsilon})$: $g_\pm$

7.2 linear and nonlinear oscillations

ex : $u'' + 2\epsilon u' + u = 0\ ,\ 0 < \epsilon << 1$

general solution : $u(t) = e^{\alpha t} \Rightarrow \alpha^2 + 2\epsilon\alpha + 1 = 0 \Rightarrow \alpha = -\epsilon \pm i\sqrt{1-\epsilon^2}$

$u(t,\epsilon) = e^{-\epsilon t}\left(A\cos\sqrt{1-\epsilon^2}t + B\sin\sqrt{1-\epsilon^2}t\right)$: damped oscillation

$u(t,0) = A\cos t + B\sin t$: no damping

initial conditions : $u(0,\epsilon) = 1\ ,\ u'(0,\epsilon) = 0$

$u(t,\epsilon) = e^{-\epsilon t}\left(\cos\sqrt{1-\epsilon^2}t + \frac{\epsilon}{\sqrt{1-\epsilon^2}}\sin\sqrt{1-\epsilon^2}t\right)$, check ...

regular perturbation series

$u(t,\epsilon) \sim u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \cdots$ as $\epsilon \to 0$

$u_0'' + \epsilon u_1'' + \cdots + 2\epsilon(u_0' + \cdots) + u_0 + \epsilon u_1 + \cdots = 0$

order ϵ^0 : $u_0'' + u_0 = 0\ ,\ u_0(0) = 1\ ,\ u_0'(0) = 0 \Rightarrow u_0(t) = \cos t$

order ϵ^1 : $u_1'' + 2u_0' + u_1 = 0 \Rightarrow u_1'' + u_1 = 2\sin t\ ,\ u_1(0) = u_1'(0) = 0$

particular solution : $u_p(t) = -t\cos t$

check : $u_p = -t\cos t \Rightarrow u_p' = t\sin t - \cos t \Rightarrow u_p'' = t\cos t + 2\sin t$

$\Rightarrow u_p'' + u_p = 2\sin t$ ok

$u_1(t) = A\sin t + B\cos t - t\cos t$

$u_1(0) = 0 \Rightarrow B = 0$

$u_1'(0) = 0 \Rightarrow A - 1 = 0 \Rightarrow u_1(t) = \sin t - t\cos t$

$u(t,\epsilon) = \cos t + \epsilon(\sin t - t\cos t) + O(\epsilon^2)$ as $\epsilon \to 0$

This is valid for $t = O(1)$, but it fails for $t = O(\epsilon^{-1})$ due to the secular term $\epsilon t\cos t$. To derive a uniform asymptotic approximation we could use the exponential method of chapter 6, but here we will use an alternative that also works for nonlinear problems.

<u>method of multiple scales</u>

note : the exact solution has two time scales , ϵt : slow , $\sqrt{1-\epsilon^2}t$: fast

define : $s = \epsilon t$, $r = (1 + w_2\epsilon^2 + w_3\epsilon^3 + \cdots)t$

$$u(t,\epsilon) = u_0(r,s) + \epsilon u_1(r,s) + \epsilon^2 u_2(r,s) + \cdots$$

$$\frac{d}{dt} = (1 + w_2\epsilon^2 + \cdots)\partial_r + \epsilon\partial_s$$

$$\frac{d^2}{dt^2} = (1 + 2w_2\epsilon^2 + \cdots)\partial_r^2 + 2\epsilon(1 + w_2\epsilon^2 + \cdots)\partial_r\partial_s + \epsilon^2\partial_s^2$$

$$u' = (1 + w_2\epsilon^2 + \cdots)\partial_r(u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots) + \epsilon\partial_s(u_0 + \epsilon u_1 + \cdots)$$

$$= \partial_r u_0 + \epsilon\,(\partial_r u_1 + \partial_s u_0) + \epsilon^2\,(w_2\partial_r u_0 + \partial_r u_2 + \partial_s u_1) + \cdots$$

$$u'' = (1 + 2w_2\epsilon^2 + \cdots)\partial_r^2(u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots) + 2\epsilon\partial_r\partial_s(u_0 + \epsilon u_1 + \cdots)$$

$$+ \epsilon^2\partial_s^2 u_0 + \cdots$$

$$= \partial_r^2 u_0 + \epsilon\left(\partial_r^2 u_1 + 2\partial_r\partial_s u_0\right) + \epsilon^2\left(2w_2\partial_r^2 u_0 + \partial_r^2 u_2 + 2\partial_r\partial_s u_1 + \partial_s^2 u_0\right) + \cdots$$

---

recall : $u'' + 2\epsilon u' + u = 0$, $u(0) = 1$, $u'(0) = 0$

order ϵ^0 : $\partial_r^2 u_0 + u_0 = 0$, $u_0(0,0) = 1$, $\partial_r u_0(0,0) = 0$

$$u_0(r,s) = A_0(s)\cos r + B_0(s)\sin r \ , \ A_0(0) = 1 \ , \ B_0(0) = 0$$

To determine $A_0(s), B_0(s)$, we need to consider the next order.

order ϵ^1 : $\partial_r^2 u_1 + 2\partial_r\partial_s u_0 + 2\partial_r u_0 + u_1 = 0$

$$u_1(0,0) = 0 \ , \ \partial_r u_1(0,0) + \partial_s u_0(0,0) = 0$$

$$\partial_r u_0 + \partial_r\partial_s u_0 = -A_0(s)\sin r + B_0(s)\cos r - A_0'(s)\sin r + B_0'(s)\cos r$$

$$\partial_r^2 u_1 + u_1 = 2\left((A_0(s) + A_0'(s))\sin r - (B_0(s) + B_0'(s))\cos r\right)$$

note : the same operator appears at all orders

To suppress secular terms in r, set $A_0(s) + A_0'(s) = 0, B_0(s) + B_0'(s) = 0$.

$\Rightarrow$ $A_0(s) = e^{-s}$, $B_0(s) = 0$

<u>summary 1</u>

$$u(t,\epsilon) \sim u_0(r,s) = e^{-s}\cos r \sim e^{-\epsilon t}\cos t \text{ as } \epsilon \to 0$$

$$u(t,\epsilon) = e^{-\epsilon t}\left(\cos\sqrt{1-\epsilon^2}t + \frac{\epsilon}{\sqrt{1-\epsilon^2}}\sin\sqrt{1-\epsilon^2}t\right)$$

The approximation has error $O(\epsilon)$ and is valid for $t = O(\epsilon^{-1})$.

$u_1(r,s) = A_1(s)\cos r + B_1(s)\sin r$

$u_1(0,0) = 0 \Rightarrow A_1(0) = 0$

$\partial_r u_1(0,0) + \partial_s u_0(0,0) = 0 \Rightarrow B_1(0) - 1 = 0 \Rightarrow B_1(0) = 1$

To determine $A_1(s), B_1(s)$, we need to consider the next order.

order ϵ^2 : $2w_2\partial_r^2 u_0 + \partial_r^2 u_2 + 2\partial_r\partial_s u_1 + \partial_s^2 u_0 + 2(\partial_r u_1 + \partial_s u_0) + u_2 = 0$

$u_2(0,0) = 0$, $w_2\partial_r u_0(0,0) + \partial_r u_2(0,0) + \partial_s u_1(0,0) = 0$

$$2w_2\partial_r^2 u_0 + 2\partial_r\partial_s u_1 + \partial_s^2 u_0 + 2(\partial_r u_1 + \partial_s u_0)$$

$$= 2w_2 \cdot -e^{-s}\cos r + 2(A_1'(s)\cdot -\sin r + B_1'(s)\cos r) + e^{-s}\cos r$$

$$+ 2(A_1(s)\cdot -\sin r + B_1(s)\cos r - e^{-s}\cos r)$$

$$\partial_r^2 u_2 + u_2 = (2w_2 e^{-s} - 2B_1'(s) - e^{-s} - 2B_1(s) + 2e^{-s})\cos r + 2(A_1'(s) + A_1(s))\sin r$$

To suppress secular terms in r, set $A_1'(s) + A_1(s) = 0, B_1'(s) + B_1(s) = (w_2 + \frac{1}{2})e^{-s}$.

$A_1(0) = 0 \Rightarrow A_1(s) = 0$

To suppress secular terms in s, set $w_2 = -\frac{1}{2}$.

$B_1(0) = 1 \Rightarrow B_1(s) = e^{-s} \Rightarrow u_1(r,s) = e^{-s}\sin r$

<u>summary 2</u>

$$u(t,\epsilon) \sim u_0(r,s) + \epsilon u_1(r,s) = e^{-s}\cos r + \epsilon e^{-s}\sin r$$

$$\sim e^{-\epsilon t}\left(\cos(1 - \tfrac{1}{2}\epsilon^2)t + \epsilon\sin(1 - \tfrac{1}{2}\epsilon^2)t\right) + O(\epsilon^2) \text{ as } \epsilon \to 0$$

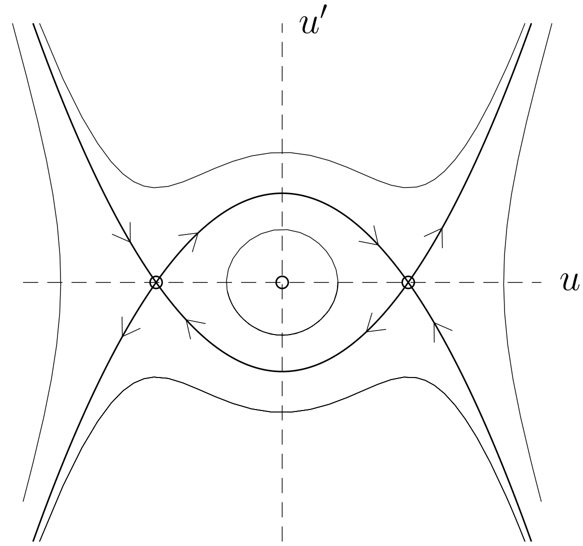
$$u(t,\epsilon) = e^{-\epsilon t}\left(\cos\sqrt{1-\epsilon^2}\,t + \frac{\epsilon}{\sqrt{1-\epsilon^2}}\sin\sqrt{1-\epsilon^2}\,t\right)$$

The approximation has error $O(\epsilon^2)$ and is valid for $t = O(\epsilon^{-2})$.

ex : $u'' + u = \epsilon u^3$: unforced undamped Duffing equation (159/3)

$\epsilon = 0$: simple harmonic motion , $\epsilon > 0$: weakly nonlinear oscillation

1st order form



$u' = v$

$v' = -u(1 - \epsilon u^2)$

fixed points : $u = 0\,, \pm\,\epsilon^{-1/2}$

linear stability analysis

case 1 : $(u, v) = (0, 0)$

$u(t) \sim \delta u_1(t) + \delta^2 u_2(t) + \cdots$ as $\delta \to 0$

$v(t) \sim \delta v_1(t) + \delta^2 v_2(t) + \cdots$

$\delta u_1' + \cdots = \delta v_1 + \cdots$

$\delta v_1' + \cdots = -(\delta u_1 + \cdots)(1 - \epsilon(\delta u_1 + \cdots)^2)$

order δ : $\begin{array}{l} u_1' = v_1 \\ v_1' = -u_1 \end{array} \Rightarrow \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \Rightarrow \lambda = \pm i$

$\Rightarrow$ $(0, 0)$: elliptic fixed point (stable)

case 2 : $(u, v) = (\epsilon^{-1/2}, 0)$

$u(t) \sim \epsilon^{-1/2} + \delta u_1(t) + \delta^2 u_2(t) + \cdots$ as $\delta \to 0$

$v(t) \sim \delta v_1(t) + \delta^2 v_2(t) + \cdots$

$\delta u_1' + \cdots = \delta v_1 + \cdots$

$$\delta v_1' + \cdots = -(\epsilon^{-1/2} + \delta u_1 + \cdots)(1 - \epsilon(\epsilon^{-1/2} + \delta u_1 + \cdots)^2)$$
$$= -(\epsilon^{-1/2} + \delta u_1 + \cdots)(1 - \epsilon(\epsilon^{-1} + 2\epsilon^{-1/2}\delta u_1 + \cdots))$$
$$= -(\epsilon^{-1/2} + \delta u_1 + \cdots)(\not{1} - \not{1} - 2\epsilon^{1/2}\delta u_1 + \cdots)) = 2\delta u_1 + \cdots$$

order δ : $\begin{array}{l} u_1' = v_1 \\ v_1' = 2u_1 \end{array} \Rightarrow \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \Rightarrow \lambda = \pm\sqrt{2}$, $q = \begin{pmatrix} \sqrt{1/3} \\ \pm\sqrt{2/3} \end{pmatrix}$

$\Rightarrow$ $(\epsilon^{-1/2}, 0)$: hyperbolic fixed point (unstable) , also for $(-\epsilon^{-1/2}, 0)$

global analysis

$u'' + u = \epsilon u^3 \Rightarrow u''u' + uu' = \epsilon u^3 u' \Rightarrow \frac{1}{2}(u')^2 + \frac{1}{2}u^2 = \epsilon \cdot \frac{1}{4}u^4 + c$

$\Rightarrow v^2 + u^2 = \frac{1}{2}\epsilon u^4 + 2c$: consider solutions for different values of c

$(u, v) = (0, 0) \Rightarrow c = 0$: elliptic fixed point

$(u, v) = (\pm\epsilon^{-1/2}, 0) \Rightarrow \epsilon^{-1} = \frac{1}{2}\epsilon \cdot \epsilon^{-2} + 2c \Rightarrow c = \frac{1}{4}\epsilon^{-1}$: hyperbolic fixed points

$v^2 = \frac{1}{2}\epsilon(u^4 - 2\epsilon^{-1}u^2 + \epsilon^{-2}) = \frac{1}{2}\epsilon(u^2 - \epsilon^{-1})^2 \Rightarrow v = \pm(\epsilon/2)^{1/2}(u^2 - \epsilon^{-1})$: parabola

1. The solution connecting the 2 hyperbolic points is called a heteroclinic orbit.

2. $0 < c < \frac{1}{4}\epsilon^{-1}$: bounded periodic solutions

consider $u(0) = 1, u'(0) = 0$

regular perturbation series

$u(t) \sim u_0(t) + \epsilon u_1(t) + \cdots$ as $\epsilon \to 0$

$u'' + u = \epsilon u^3$

$u_0'' + \epsilon u_1'' + \cdots + u_0 + \epsilon u_1 + \cdots = \epsilon(u_0 + \epsilon u_1 + \cdots)^3$

order ϵ^0 : $u_0'' + u_0 = 0\,,\ u_0(0) = 1\,,\ u_0'(0) = 0 \Rightarrow u_0(t) = \cos t$

order ϵ^1 : $u_1'' + u_1 = u_0^3 = \cos^3 t = \frac{3}{4}\cos t + \frac{1}{4}\cos 3t$

step 1 : $u = a\cos 3t \Rightarrow u' = -3a\sin 3t \Rightarrow u'' = -9a\cos 3t$

$u'' + u = \frac{1}{4}\cos 3t \Rightarrow -8a = \frac{1}{4} \Rightarrow a = -\frac{1}{32} \Rightarrow u_p(t) = -\frac{1}{32}\cos 3t$

step 2 : $u = at\sin t \Rightarrow u' = a(t\cos t + \sin t) \Rightarrow u'' = a(-t\sin t + 2\cos t)$

$u'' + u = \frac{3}{4}\cos t \Rightarrow 2a = \frac{3}{4} \Rightarrow a = \frac{3}{8} \Rightarrow u_p(t) = \frac{3}{8}t\sin t$

$u_1(t) = -\frac{1}{32}\cos 3t + \frac{3}{8}t\sin t + a\cos t + b\sin t$

$u_1(0) = 0 \Rightarrow -\frac{1}{32} + a = 0 \Rightarrow a = \frac{1}{32}$

$u_1'(0) = 0 \Rightarrow b = 0$

$u_1(t) = \frac{1}{32}(\cos t - \cos 3t) + \frac{3}{8}t\sin t$

$u(t) \sim \cos t + \epsilon(\frac{1}{32}(\cos t - \cos 3t) + \frac{3}{8}t\sin t) + \cdots$: unbounded as $t \to \infty$

<u>Poincaré-Lindstedt method</u> (method of strained coordinates)

$t = s + \epsilon f_1(s) + \epsilon^2 f_2(s) + \cdots$ as $\epsilon \to 0$

$u(t) \sim u_0(s) + \epsilon u_1(s) + \epsilon^2 u_2(s) + \cdots$

$$u' = u_0' \frac{ds}{dt} + \epsilon u_1' \frac{ds}{dt} + \cdots$$

$$1 = \frac{ds}{dt} + \epsilon f_1' \frac{ds}{dt} + \cdots = \frac{ds}{dt}(1 + \epsilon f_1' + \cdots)$$

$$\frac{ds}{dt} = (1 + \epsilon f_1' + \cdots)^{-1} = 1 - \epsilon f_1' + \cdots$$

$$u' = u_0'(1 - \epsilon f_1' + \cdots) + \epsilon u_1' + \cdots = u_0' + \epsilon(u_1' - u_0' f_1') + \cdots$$

$$u'' = u_0'' \frac{ds}{dt} + \epsilon\Big(u_1'' \frac{ds}{dt} - u_0' f_1'' \frac{ds}{dt} - u_0'' \frac{ds}{dt} f_1'\Big) + \cdots$$

$$= (u_0'' + \epsilon(u_1'' - u_0' f_1'' - u_0'' f_1') + \cdots)(1 - \epsilon f_1' + \cdots)$$

$$u'' = u_0'' + \epsilon(u_1'' - u_0' f_1'' - 2u_0'' f_1') + \cdots$$

$$u'' + u = \epsilon u^3$$

order ϵ^0 : $u_0'' + u_0 = 0\,,\ u_0(0) = 1\,,\ u_0'(0) = 0 \ \Rightarrow\ u_0(s) = \cos s$

order ϵ^1 : $u_1'' - u_0' f_1'' - 2u_0'' f_1' + u_1 = u_0^3$, $u_1(0) = u_1'(0) = 0$

$$u_1'' + u_1 = \cos^3 s - f_1'' \sin s - 2f_1' \cos s$$

$$u_1'' + u_1 = (\tfrac{3}{4} - 2f_1') \cos s - f_1'' \sin s + \tfrac{1}{4} \cos 3s$$

to suppress secular terms set $f_1' = \frac{3}{8} \ \Rightarrow\ f_1(s) = \frac{3}{8}s$

$$u_1'' + u_1 = \tfrac{1}{4} \cos 3s \ \Rightarrow\ u_1(s) = -\tfrac{1}{32} \cos 3s + \tfrac{1}{32} \cos s$$

$$t \sim s + \tfrac{3}{8}\epsilon s + \cdots = (1 + \tfrac{3}{8}\epsilon + \cdots)s$$

$$s \sim t(1 + \tfrac{3}{8}\epsilon + \cdots)^{-1} \sim t(1 - \tfrac{3}{8}\epsilon + \cdots)$$

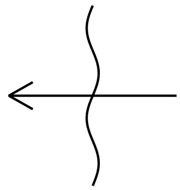
$$u(t) \sim u_0(s) + \epsilon u_1(s) + \cdots$$

$$u(t) \sim \cos[(1 - \tfrac{3}{8}\epsilon)t] + \tfrac{1}{32}\epsilon\Big(\cos[(1 - \tfrac{3}{8}\epsilon)t] - \cos[3(1 - \tfrac{3}{8}\epsilon)t]\Big) + \cdots$$

The period of the solution is $T = 2\pi(1 + \frac{3}{8}\epsilon) + O(\epsilon^2)$.

hw7 : find the $O(\epsilon^2)$ term

ex : premixed flame front (Keener, p. 530)

unburnt gas  burnt gas : determine flame structure/speed

gas : mixture of fuel and oxidizer

$u(x,t)$: gas temperature , $-\infty < x < \infty$

$c(x,t)$: fuel concentration

$u_t = \alpha u_{xx} + Bck(u)$: reaction-diffusion equation

$c_t = \beta c_{xx} \ - \ ck(u)$

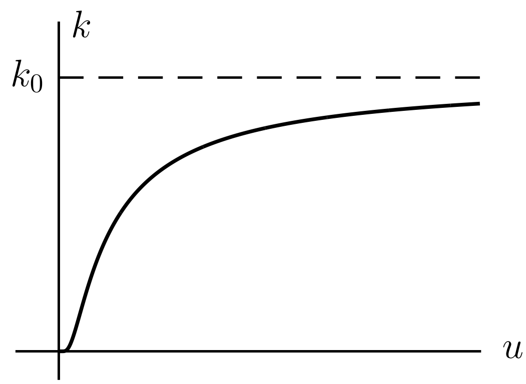
α , β : diffusion coefficients

$B > 0$: exothermic reaction (releases heat)

$k(u) = k_0 e^{-E/Ru}$: Arrhenius rate function

E : activation energy

R : universal gas constant



u_0 , u_1 : temperature of unburnt , burnt gas , $u_0 \le u \le u_1$

c_0 : concentration of fuel in unburnt gas , $0 \le c \le c_0$

assume constant states : (u_0, c_0) as $t \to -\infty$: unburnt

$(u_1, 0)$ as $t \to \infty$: burnt

then $u_{xx} = c_{xx} = 0 \ \Rightarrow \ u_t + Bc_t = 0 \ \Rightarrow \ u_0 + Bc_0 = u_1$

non-dimensionalization

$$u^* = \frac{u - u_0}{u_1 - u_0} \ , \ c^* = \frac{c}{c_0} \ \Rightarrow \ 0 \le u^*, c^* \le 1$$

$$u \ = \ u_0 + (u_1 - u_0)u^* \ , \ c = c_0 c^*$$

$$\frac{1}{u} = \frac{1}{u_0 + (u_1 - u_0)u^*} = \frac{1}{u_1} \cdot \frac{1}{\frac{u_0}{u_1} + (1 - \frac{u_0}{u_1})u^*} \quad , \ \text{set } \gamma = 1 - \frac{u_0}{u_1}$$

$$= \frac{1}{u_1} \cdot \frac{1}{\frac{u_0}{u_1} - 1 + 1 + \gamma u^*} = \frac{1}{u_1} \cdot \frac{1}{1 + \gamma(u^* - 1)} = \frac{1}{u_1} - \frac{1}{1 + \gamma(u^* - 1)} \cdot \frac{\gamma(u^* - 1)}{u_1}$$

$$\frac{-E}{Ru} = \frac{-E}{Ru_1} + \frac{E\gamma(u^* - 1)}{Ru_1} \cdot \frac{1}{1 + \gamma(u^* - 1)}$$

define : $\frac{1}{\epsilon} = \frac{E\gamma}{Ru_1} = \frac{E(u_1 - u_0)}{Ru_1^2}$: <u>Zeldovitch number</u>

$\epsilon \to 0 \Leftrightarrow E \to \infty$: large activation energy asymptotics

$$k(u) = k_0 e^{-E/Ru_1} \exp\left(\frac{(u^* - 1)}{\epsilon} \cdot \frac{1}{1 + \gamma(u^* - 1)}\right)$$

$$t^* = k_0 \epsilon^2 e^{-E/Ru_1} t \ , \ x^* = \epsilon \left(\frac{k_0 e^{-E/Ru_1}}{\alpha}\right)^{1/2} x$$

$$u_t = \alpha u_{xx} + Bck(u)$$

$$(u_1 - u_0) u^*_{t^*} k_0 \epsilon^2 e^{-E/Ru_1} = \alpha(u_1 - u_0) u^*_{x^* x^*} \epsilon^2 \frac{k_0 e^{-E/Ru_1}}{\alpha} + Bc_0 c^* k_0 e^{-E/Ru_1} \exp\left(\frac{(u^* - 1)}{\epsilon} \cdot \frac{1}{1 + \gamma(u^* - 1)}\right)$$

$$c_t = \beta c_{xx} - ck(u)$$

$$c_0 c^*_{t^*} k_0 \epsilon^2 e^{-E/Ru_1} = \beta c_0 c^*_{x^* x^*} \epsilon^2 \frac{k_0 e^{-E/Ru_1}}{\alpha} - c_0 c^* k_0 e^{-E/Ru_1} \exp\left(\frac{(u^* - 1)}{\epsilon} \cdot \frac{1}{1 + \gamma(u^* - 1)}\right)$$

drop $*$

$$u_t = u_{xx} + c\,\frac{f(u)}{\epsilon^2} \ , \ f(u) = \exp\left(\frac{(u - 1)}{\epsilon} \cdot \frac{1}{1 + \gamma(u - 1)}\right)$$

$c_t = L^{-1} c_{xx} - c\,\frac{f(u)}{\epsilon^2}$, $L = \frac{\alpha}{\beta}$: <u>Lewis number</u>

<u>traveling wave</u>

$s = x + vt$, $u(x,t) = \bar{u}(s)$, $c(x,t) = \bar{c}(s)$, assume $v > 0$

$s \to -\infty$: unburnt $\Rightarrow \lim_{s \to -\infty} \bar{u}(s) = 0$, $\lim_{s \to -\infty} \bar{c}(s) = 1$

$s \to \infty$: burnt $\Rightarrow \lim_{s \to \infty} \bar{u}(s) = 1$, $\lim_{s \to \infty} \bar{c}(s) = 0$

drop bars : $u'' - vu' + c\dfrac{f(u)}{\epsilon^2} = 0$, $L^{-1}c'' - vc' - c\dfrac{f(u)}{\epsilon^2} = 0$

consider $\epsilon << 1$, $L = O(1)$

$0 \le u < 1 \Rightarrow \dfrac{f(u)}{\epsilon^2} \to 0$ as $\epsilon \to 0$, $u = 1 \Rightarrow \dfrac{f(u)}{\epsilon^2} = \dfrac{1}{\epsilon^2}$

$\Rightarrow \dfrac{f(u)}{\epsilon^2}$ is important only when $u = 1$

<u>outer solution</u>

$u(s,\epsilon) \sim u_0(s) + \epsilon u_1(s) + \cdots$ note : $u_0(s)$ vs u_0

$c(s,\epsilon) \sim c_0(s) + \epsilon c_1(s) + \cdots$

$v \sim v_0 + \epsilon v_1 + \cdots$

order ϵ^0 : $u_0'' - v_0 u_0' + c_0\dfrac{f(u_0)}{\epsilon^2} = 0$, $L^{-1}c_0'' - v_0 c_0' - c_0\dfrac{f(u_0)}{\epsilon^2} = 0$

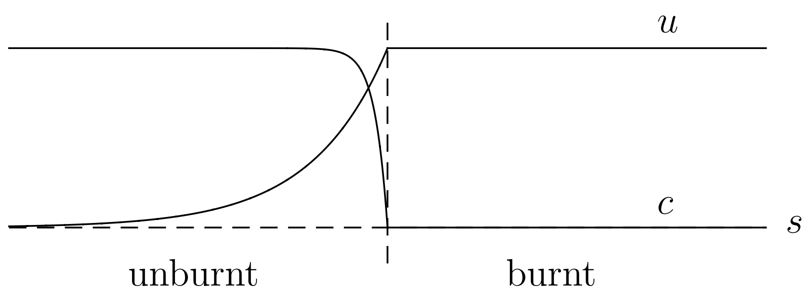
one possibility is $c_0(s) = 0$, $u_0(s) = 1$: ok for $s \to \infty$

otherwise $0 \le u_0(s) < 1$, which gives the following for $\epsilon \to 0$

$$\left.\begin{array}{l} u_0'' - v_0 u_0' = 0 \Rightarrow u_0(s) = a e^{v_0 s} \\ L^{-1}c_0'' - v_0 c_0' = 0 \Rightarrow c_0(s) = 1 - b e^{L v_0 s} \end{array}\right\} : \text{ ok for } s \to -\infty$$

choose a, b by matching at $s = 0$

$$u_0(s) = \begin{cases} e^{v_0 s} , & s < 0 \\ 1 , & s > 0 \end{cases} , \quad c_0(s) = \begin{cases} 1 - e^{L v_0 s} , & s < 0 \\ 0 , & s > 0 \end{cases}$$



1. $L > 1 \Rightarrow$ <u>reaction zone</u> is thinner than <u>preheat zone</u>

2. The kink in the profiles at $s = 0$ is unphysical and the flame speed v_0 is still undetermined.

<u>inner solution</u>

$r = \frac{s}{\epsilon}$, $u(s) = 1 + \epsilon\theta(r)$, $c(s) = \epsilon y(r)$: rescaled variables

$$\epsilon\frac{\theta''}{\epsilon^2} - v\epsilon\frac{\theta'}{\epsilon} + \epsilon y \frac{f(1+\epsilon\theta)}{\epsilon^2} = 0 \quad , \quad L^{-1}\epsilon\frac{y''}{\epsilon^2} - v\epsilon\frac{y'}{\epsilon} - \epsilon y\frac{f(1+\epsilon\theta)}{\epsilon^2} = 0$$

$$\theta'' - \epsilon v\theta' + yf(1+\epsilon\theta) = 0 \quad , \quad L^{-1}y'' - \epsilon v y' - yf(1+\epsilon\theta) = 0$$

$$f(u) = \exp\left(\left(\frac{u-1}{\epsilon}\right)\frac{1}{1+\gamma(u-1)}\right) \Rightarrow f(1+\epsilon\theta) = \exp\left(\theta\,\frac{1}{1+\epsilon\gamma\theta}\right) = e^{\theta} + O(\epsilon)$$

order ϵ^0 : $\theta(r) = \theta_0(r) + \epsilon\theta_1(r)\cdots$, $y(r) = y_0(r) + \epsilon y_1(r) + \cdots$

$$\theta_0'' + y_0e^{\theta_0} = 0 \quad , \quad L^{-1}y_0'' - y_0e^{\theta_0} = 0$$

$$\Rightarrow\ \theta_0'' + L^{-1}y_0'' = 0 \ \Rightarrow\ \theta_0(r) + L^{-1}y_0(r) = ar + b$$

matching as $r \to \infty$, $\lim_{r\to\infty}\theta_0(r)\,,\, y_0(r) = 0 \ \Rightarrow\ \theta_0(r) + L^{-1}y_0(r) = 0$

$$\theta_0'' - L\theta_0e^{\theta_0} = 0 \ \Rightarrow\ \theta_0'\theta_0'' - L\theta_0'\theta_0e^{\theta_0} = 0$$

$\Rightarrow\ \frac{1}{2}\left(\theta_0'\right)^2 - L(\theta_0 - 1)e^{\theta_0} = L$, using $\lim_{r\to\infty}\theta_0(r)\,,\,\theta_0'(r) = 0$

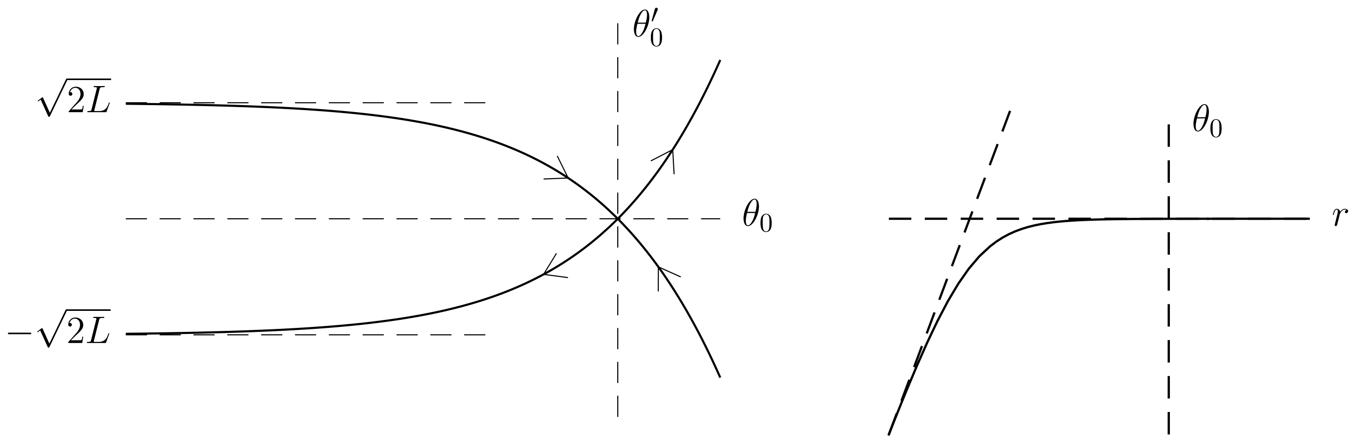
$\Rightarrow\ \left(\theta_0'\right)^2 = 2L\left(1 + (\theta_0 - 1)e^{\theta_0}\right)$, we're interested in $\theta_0 \le 0$

<u>phase plane</u>

$\begin{pmatrix}\theta_0\\ \theta_0'\end{pmatrix}' = \begin{pmatrix}\theta_0'\\ L\theta_0e^{\theta_0}\end{pmatrix}$, fixed point : $\begin{pmatrix}\theta_0\\ \theta_0'\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$

<u>linear stability</u>

$\begin{pmatrix}\theta_0\\ \theta_0'\end{pmatrix}' \sim \begin{pmatrix}\theta_0'\\ L\theta_0\end{pmatrix} = \begin{pmatrix}0 & 1\\ L & 0\end{pmatrix}\begin{pmatrix}\theta_0\\ \theta_0'\end{pmatrix}$, $\lambda = \pm\sqrt{L}$: hyperbolic point



note : 1. $\theta_0 \leq 0 \Rightarrow (\theta_0')^2 \leq 2L \Rightarrow -\sqrt{2L} \leq \theta_0' \leq \sqrt{2L}$

2. $\theta_0 \to -\infty \Rightarrow \theta_0' \to \pm\sqrt{2L}$

The orbit connecting $(\theta_0, \theta_0') = (-\infty, \sqrt{2L})$ with $(\theta_0, \theta_0') = (0, 0)$ defines a corner layer to be inserted near $s = 0$.

recall : $r = \frac{s}{r}$, $u(s) = 1 + \epsilon\theta(r) \Rightarrow u_0(s) \sim 1 + \epsilon\theta_0\left(\frac{s}{\epsilon}\right)$

$\Rightarrow u_0'(s) \sim \epsilon\theta_0'\left(\frac{s}{\epsilon}\right) \cdot \frac{1}{\epsilon} = \theta_0'(r)$

<u>matching of inner and outer solutions</u>

$\lim\limits_{s\to 0^-} u_0'(s) = \lim\limits_{r\to -\infty} \theta_0'(r) \Rightarrow v_0 = \sqrt{2L}$

express wave speed in dimensional variables

$$s = x^* + v_0 t^* = \epsilon\left(\frac{k_0 e^{-E/Ru_1}}{\alpha}\right)^{1/2} x + v_0 k_0 \epsilon^2 e^{-E/Ru_1} t$$

$$s = \epsilon\left(\frac{k_0 e^{-E/Ru_1}}{\alpha}\right)^{1/2} \left(x + v_0 \epsilon \left(\alpha k_0 e^{-E/Ru_1}\right)^{1/2} t\right)$$

$$V_0 = \epsilon\sqrt{2L}\left(\alpha k_0 e^{-E/Ru_1}\right)^{1/2}, \quad \frac{1}{\epsilon} = \frac{E(u_1 - u_0)}{Ru_1^2}, \quad L = \frac{\alpha}{\beta}$$

Hence as the fuel diffusivity $\beta \to 0$, the reaction zone becomes thinner and the flame propagates faster.

question : is the flame front stable?