Math 557   Winter 2020   Homework 1   due: Tuesday, January 28

0. (optional) Give a brief description of your academic background and research interests. If you work in a lab or research group, please give your supervisor's name and describe your project. One paragraph is fine.

When a few terms in the asymptotic expansion are required on a homework problem, you should derive the result by hand. You may check your answer using symbolic software, but a step-by-step derivation should be given to receive full credit.

1. Let $\mathrm{Ei}(x) = s_n(x) + r_n(x)$ as in class. Derive the following results.

a) $\dfrac{r_n(x)}{s_n(x)} \to 0$ as $x \to \infty$ b) $|r_n(x)| \sim \dfrac{n! e^{-x}}{x^{n+1}}$ as $x \to \infty$

2. Find the asymptotic expansion of $f(x)$ as $x \to \infty$ with respect to the given sequence $\{\phi_n(x)\}$. Give the first three terms and the general term.

a) $f(x) = \log(1 + e^{-x})$, $\phi_n(x) = e^{-nx}$, $n \geq 1$

b) $f(x) = (x^2 + 1)^{-1/3}$, $\phi_n(x) = x^{-2(n+\frac{1}{3})}$, $n \geq 0$

3. page 11/9 (asymptotic expansion of Laplace transform of $(1 + t^2)^{-1}$)
Explain why it is necessary to assume $|\arg p| < \pi/2$. Make sure to verify that the expansion is asymptotic, i.e. if the approximation includes terms up to $O(p^{-(2n-1)})$, then the error is $O(p^{-(2n+1)})$ as $p \to \infty$.

4. Define the error function by $\mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2}\, dt$. You may assume that $\lim\limits_{x \to \infty} \mathrm{erf}(x) = 1$.

a) Find the Taylor series of $\mathrm{erf}(x)$ about $x = 0$. (there's an easy method that doesn't require computing the derivatives of $\mathrm{erf}(x)$)

b) Find the asymptotic expansion of $\mathrm{erf}(x)$ as $x \to \infty$ with respect to the following sequence.

$$\phi_n(x) = 1 , \quad \frac{e^{-x^2}}{x} , \quad \frac{e^{-x^2}}{x^3} , \quad \frac{e^{-x^2}}{x^5} , \quad \dots$$

In (a), (b), give the first three terms and the general term.

c) Find the interval of convergence for the series in (a) and (b).

d) Present a table with the following format. Print 15 digits (in Matlab use `format long`).
 column 1: n
 column 2: n-term Taylor approximation for $\mathrm{erf}(2)$;
 column 3: error in column 2
 column 4: n-term asymptotic approximation for $\mathrm{erf}(2)$
 column 5: error in column 4
Take $n = 1 : 18$. To compute the error you need the exact value; use `erf(2)` for that.

e) What is the smallest error in $\mathrm{erf}(2)$ that can be achieved using the asymptotic series? How many terms in the asymptotic series are used in that case? How many terms are needed in the Taylor series to obtain the same accuracy?

5. Find the first two terms in the asymptotic expansion as $x \to \infty$.

a) $\displaystyle\int_0^1 e^{-xt^3}\, dt$ b) $\displaystyle\int_0^\infty e^{-xt} \cos t\, dt$ c) $\displaystyle\int_0^\infty e^{-xt} \sin t\, dt$ (there's more than one way to do b, c)