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show that $\int_x^\infty e^{it} t^{-m} dt \sim \frac{ie^{ix}}{x^m} \sum_{n=0}^{\infty} \frac{(-1)^n i^n (m+n-1)!}{(m-1)! x^n}$ as $x \to \infty$, $m = 1, 2, \ldots$

integration by parts : $u = t^{-m}$, $dv = e^{it}dt \Rightarrow du = -mt^{-(m+1)}$, $v = -ie^{it}$

$$\int_x^\infty e^{it} t^{-m} dt = t^{-m} \cdot -ie^{it}\Big|_x^\infty - im \int_x^\infty e^{it} t^{-(m+1)} dt = \frac{ie^{ix}}{x^m} - im \int_x^\infty e^{it} t^{-(m+1)} dt$$

we want to show that $\int_x^\infty e^{it} t^{-m} dt \sim \frac{ie^{ix}}{x^m}$ as $x \to \infty$

$$\int_x^\infty e^{it} t^{-m} dt = s_0(x) + r_0(x) \, , \; s_0(x) = \frac{ie^{ix}}{x^m} \, , \; r_0(x) = -im \int_x^\infty e^{it} t^{-(m+1)} dt$$

substitute $t = xs$, $dt = xds$

$$\int_x^\infty e^{it} t^{-(m+1)} dt = \int_1^\infty e^{ixs} (xs)^{-(m+1)} x ds = x^{-m} \int_1^\infty e^{ixs} s^{-(m+1)} ds$$

<u>Riemann-Lebesgue lemma</u>

1. The Fourier transform of an L^1 function vanishes at ∞.
2. If $\int_{-\infty}^\infty |f(x)| dx$ is finite, then $\hat{f}(k) = \int_{-\infty}^\infty f(x) e^{ikx} dx \to 0$ as $k \to \infty$.

The idea is that the integrand oscillates more and more rapidly as k increases, and the $\pm$ contributions to the integral cancel in the limit $k \to \infty$; there is no definite rate and the decay can be arbitrarily slow, but nonetheless it shows that $r_0(x) = o(x^{-m})$ as $x \to \infty$.

<u>pf</u>

case 1 : $f(x) = \chi_{(a,b)}(x)$: characteristic function of an interval

$$\int_{-\infty}^\infty f(x) e^{ikx} dx = \int_a^b e^{ikx} dx = \frac{e^{ikx}}{ik}\Big|_a^b = \frac{e^{ikb}}{ik} - \frac{e^{ika}}{ik} \to 0 \text{ as } k \to \infty$$

case 2 : $f(x) = \sum_{i=1}^{N} c_i \chi_{(a_i,b_i)}(x)$: <u>simple function</u> ...

case 3 : $f \in L^1$

given $\epsilon > 0$, there exists a simple function $g(x)$ st $\int_{-\infty}^\infty |f(x) - g(x)| dx < \epsilon$

there exists N st $k > N \Rightarrow \left| \int_{-\infty}^\infty g(x) e^{ikx} dx \right| < \epsilon$

$$\left| \int_{-\infty}^\infty f(x) e^{ikx} dx \right| = \left| \int_{-\infty}^\infty (f(x) - g(x) + g(x)) e^{ikx} dx \right| < \ldots < 2\epsilon \quad \underline{\text{ok}}$$