Math 557 Winter 2020 Homework 3 due: Tuesday, February 25

1. page 38 / 1 (iv) (Laplace's method)

2. It follows from the work on page 1, chapter 3 of the lecture notes that

$$\int_{-\infty}^{\infty} e^{-\lambda y^2} \cos(2\lambda x_0 y)\, dy = e^{-\lambda x_0^2} \left(\frac{\pi}{\lambda}\right)^{1/2} \quad \text{for all } x_0 \in \mathbb{R} \text{ and all } \lambda > 0.$$

The method there used Cauchy's theorem to shift the path of integration. Now derive the formula using the following two methods. You may use the fact that $\int_{-\infty}^{\infty} e^{-\lambda y^2} dy = \left(\frac{\pi}{\lambda}\right)^{1/2}$.

method 1: Define $f(x_0, \lambda) = \int_{-\infty}^{\infty} e^{-\lambda y^2} \cos(2\lambda x_0 y)\, dy$, compute $\frac{\partial f}{\partial x_0}$, write the resulting formula as a differential equation for $f(x_0, \lambda)$, solve the equation to obtain $f(x_0, \lambda)$.

method 2: Expand $\cos(2\lambda x_0 y)$ in a Taylor series around $y = 0$ and integrate term-by-term.

Is there another method? If you know one, please tell me.

3. Use the method of steepest descent to find the first three terms in the asymptotic expansion of $\int_C \frac{e^{\lambda(z^2-1)}}{z - 1/2}\, dz$ as $\lambda \to \infty$, where C is the vertical line from $z = 1 - i\infty$ to $z = 1 + i\infty$.
Draw the z-plane with the curve C and paths of steepest ascent and descent, as in class. Note that the process of deforming C into the steepest descent path yields a residue term due to the pole at $z = \frac{1}{2}$.

4. page 69 / 3 $\big(\mathrm{Bi}(\lambda)$ as $\lambda \to \infty\big)$

5. page 69 / 4 $\big(\mathrm{Ai}(-\lambda),\ \mathrm{Bi}(-\lambda)$ as $\lambda \to \infty\big)$

In the last problem also draw a plot of the z-plane, indicate the saddle points and paths of steepest descent of ϕ, and shade the regions in which $\phi > \phi_0$. Explain how the curves C_1, C_2, C_3 are deformed so that Laplace's method can be applied.

Note that the asymptotic properties of $\mathrm{Ai}(\lambda)$ and $\mathrm{Bi}(\lambda)$ for $\lambda \to \pm\infty$ will be used when we study turning points later in the course.