1 Wed 1/4

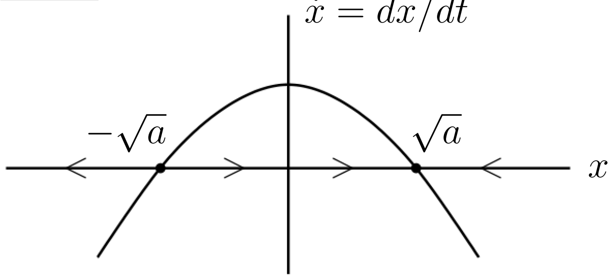
1. introduction

1.3 turning point

example : $\dfrac{dx}{dt} = a - x^2$, $x(0) = x_0$, goal : find $x(t)$

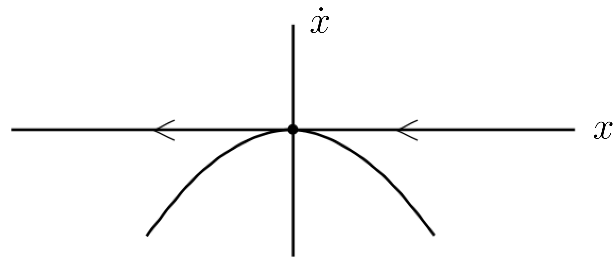
equilibrium : $\dfrac{dx}{dt} = 0 \Rightarrow a - x^2 = 0 \Rightarrow X = \begin{cases} \pm\sqrt{a} & , \ a > 0 \\ 0 & , \ a = 0 \\ \text{none} & , \ a < 0 \end{cases}$

case 1 : $a > 0$



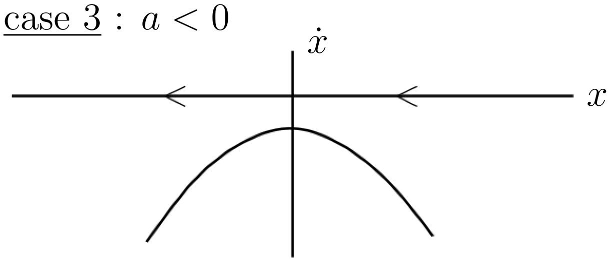
$X = \sqrt{a}$ is stable, $X = -\sqrt{a}$ is unstable

case 2 : $a = 0$

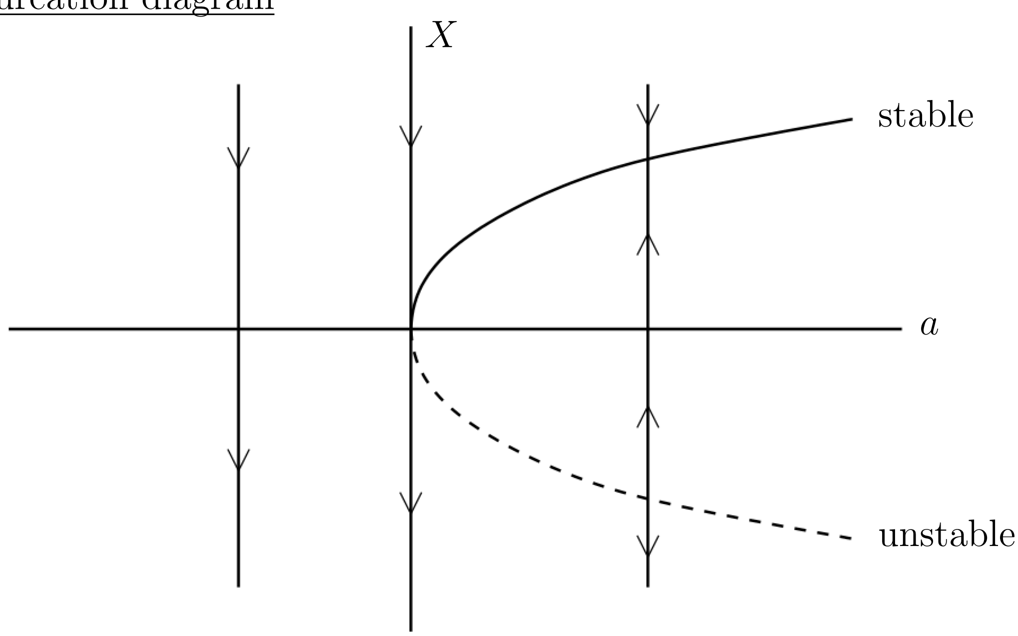


$X = 0$ is unstable

case 3 : $a < 0$



bifurcation diagram



There is a turning point at $(a, X) = (0, 0)$; this is also called saddle-node bifurcation. (more later)

<u>linear stability analysis</u> : define $x' = x - X$: <u>perturbation</u>

$$\frac{dx'}{dt} = \frac{dx}{dt} = a - x^2 = a - (x'+X)^2 = \cancel{a} - (\cancel{(x')^2} + 2x'X + \cancel{X^2})$$

<u>linearized equation</u> : $\frac{dx'}{dt} = -2Xx' \Rightarrow x'(t) = x'(0)e^{st}$, $s = -2X$: <u>growth rate</u>

$X = \sqrt{a} \Rightarrow s < 0 \Rightarrow \lim_{t\to\infty} x'(t) = 0$: stable

$X = -\sqrt{a} \Rightarrow s > 0 \Rightarrow \lim_{t\to\infty} x'(t) = \pm\infty$: unstable

$X = 0 \Rightarrow s = 0 \Rightarrow x'(t) = x'(0)$ for all t : <u>marginally stable</u>
nonlinear terms determine stability

<u>explicit solution</u>

<u>case 1</u> : $a = 0$

$$\frac{dx}{dt} = -x^2 \Rightarrow \frac{dx}{x^2} = -dt \Rightarrow -\frac{1}{x} = -t + c$$

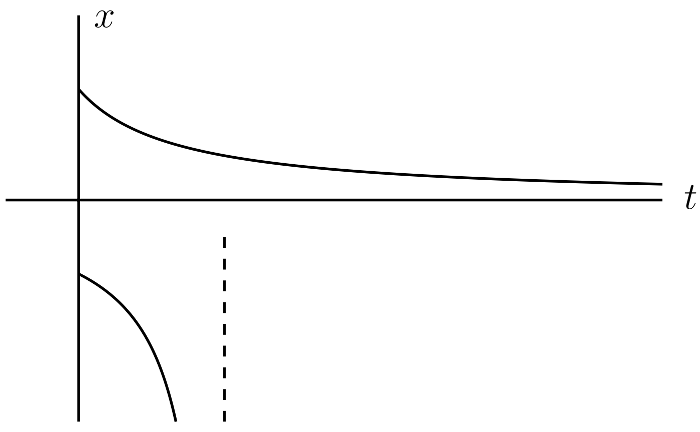
$$t = 0 \Rightarrow -\frac{1}{x_0} = c \Rightarrow -\frac{1}{x} = -t - \frac{1}{x_0} \Rightarrow x = \frac{1}{t + \frac{1}{x_0}}$$

$$x(t) = \frac{x_0}{x_0 t + 1}$$

$x_0 = 0 \Rightarrow x(t) = 0$ for all t

$x_0 > 0 \Rightarrow \lim_{t\to\infty} x(t) = 0$

$x_0 < 0 \Rightarrow x(t) \to -\infty$ as $t \to t_c = -1/x_0$: <u>blow-up</u>



<u>case 2</u> : $a > 0$

$$\frac{dx}{dt} = a - x^2\,,\; a = \alpha^2\,,\; \alpha > 0 \Rightarrow \frac{dx}{dt} = \alpha^2 - x^2 \Rightarrow \frac{dx}{\alpha^2 - x^2} = dt$$

$$\frac{1}{\alpha^2 - x^2} = \frac{1}{2\alpha}\left(\frac{1}{\alpha + x} + \frac{1}{\alpha - x}\right)$$

$$\frac{1}{2\alpha}\Big(\ln(\alpha + x) - \ln(\alpha - x)\Big) = t + c \Rightarrow \ln\left(\frac{\alpha + x}{\alpha - x}\right) = 2\alpha t + 2\alpha c$$

$$\frac{\alpha + x}{\alpha - x} = Ae^{2\alpha t}\,,\ A = \frac{\alpha + x_0}{\alpha - x_0}$$

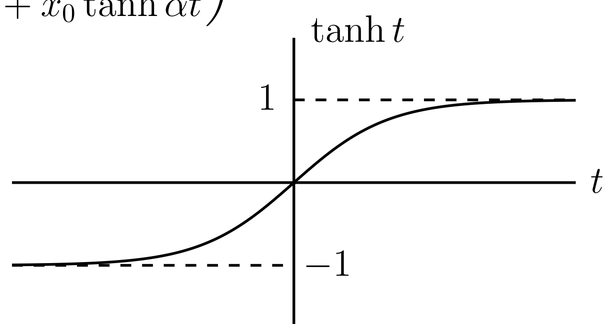
$$\alpha + x = (\alpha - x)Ae^{2\alpha t} \Rightarrow x(1 + Ae^{2\alpha t}) = \alpha(Ae^{2\alpha t} - 1)$$

$$x = \alpha\left(\frac{Ae^{2\alpha t} - 1}{1 + Ae^{2\alpha t}}\right) = \alpha\left(\frac{(\alpha + x_0)e^{2\alpha t} - (\alpha - x_0)}{(\alpha - x_0) + (\alpha + x_0)e^{2\alpha t}}\right)$$

$$= \alpha\left(\frac{(\alpha + x_0)e^{\alpha t} - (\alpha - x_0)e^{-\alpha t}}{(\alpha - x_0)e^{-\alpha t} + (\alpha + x_0)e^{\alpha t}}\right)$$

$$= \alpha\left(\frac{\alpha \sinh \alpha t + x_0 \cosh \alpha t}{\alpha \cosh \alpha t + x_0 \sinh \alpha t}\right) = \alpha\left(\frac{\alpha \tanh \alpha t + x_0}{\alpha + x_0 \tanh \alpha t}\right)$$

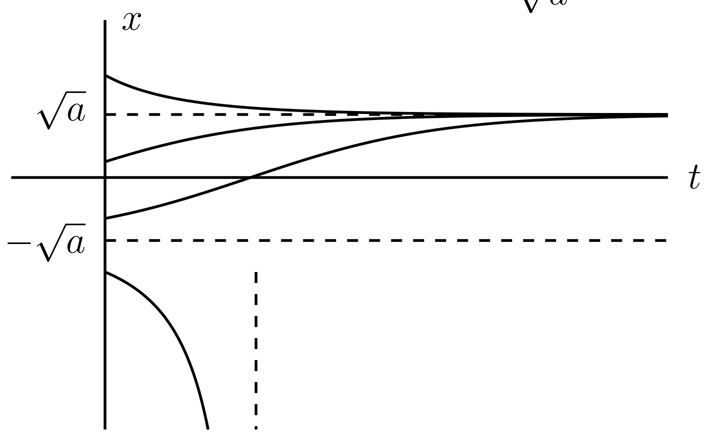
$$x(t) = \sqrt{a}\left(\frac{x_0 + \sqrt{a}\tanh\sqrt{a}\,t}{\sqrt{a} + x_0 \tanh\sqrt{a}\,t}\right)$$



$$x_0 = \pm\sqrt{a} \Rightarrow x(t) = \pm\sqrt{a} \text{ for all } t$$

$$x_0 > -\sqrt{a} \Rightarrow \lim_{t\to\infty} x(t) = \sqrt{a}$$

$$x_0 < -\sqrt{a} \Rightarrow x(t) \to -\infty \text{ as } t \to \frac{-1}{\sqrt{a}}\tanh^{-1}(\sqrt{a}/x_0)$$



<u>case 3</u> : $a < 0$

$$x(t) = \sqrt{-a}\left(\frac{x_0 - \sqrt{-a}\,\tan\sqrt{-a}\,t}{\sqrt{-a} + x_0 \tan\sqrt{-a}\,t}\right)\ ,\ \text{derivation} \ldots$$

$$x(t) \to -\infty \text{ as } t \to \frac{-1}{\sqrt{-a}}\tan^{-1}\left(\frac{\sqrt{-a}}{x_0}\right) : \text{ blow-up}$$

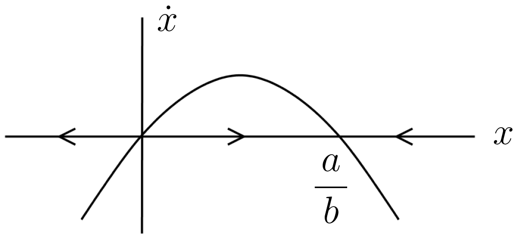
2 Mon 1/9

1.4 transcritical bifurcation

$$\frac{dx}{dt} = ax - bx^2 \ , \text{ assume } b > 0$$

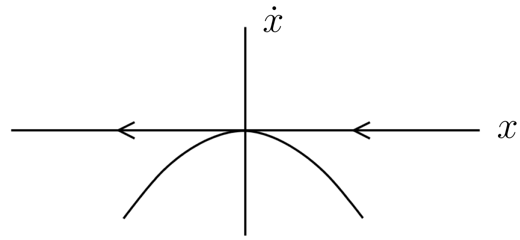
equilibrium : $ax - bx^2 = 0 \Rightarrow x(a - bx) = 0 \Rightarrow X = 0\,,\ a/b$

case 1 : $a > 0$



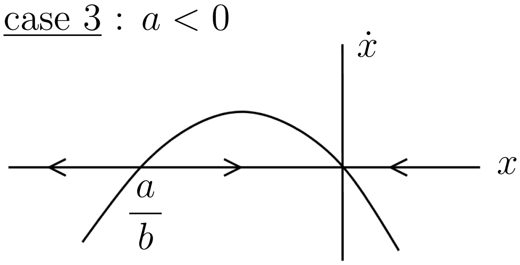
$X = 0$ is unstable, $X = \dfrac{a}{b}$ is stable

case 2 : $a = 0$



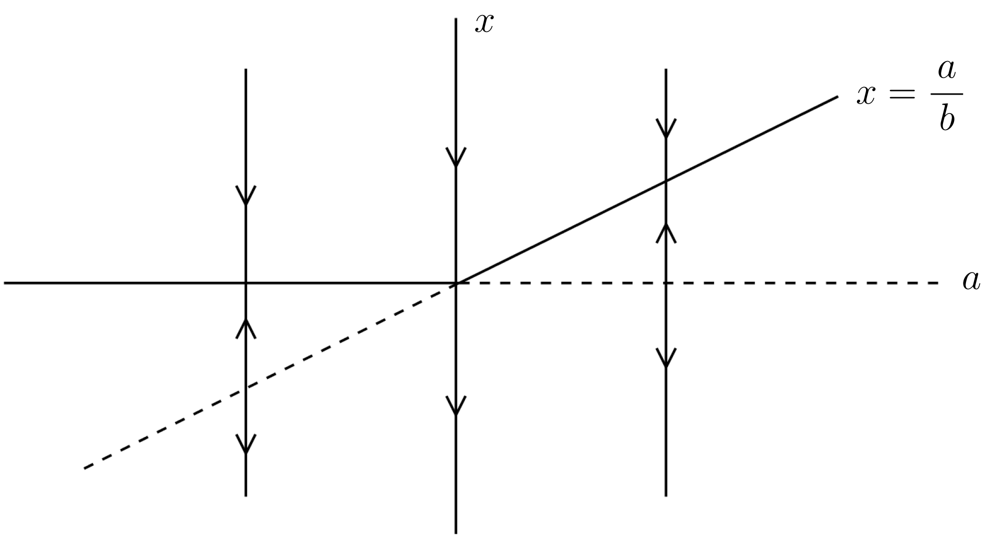
$X = 0$ is unstable

case 3 : $a < 0$



$X = 0$ is stable, $X = \dfrac{a}{b}$ is unstable

bifurcation diagram



This is a transcritical bifurcation; there is an exchange of stability at the bifurcation point $a = 0$.

<u>linear stability</u>

$x' = x - X$

$$\frac{dx'}{dt} = \frac{dx}{dt} = ax - bx^2 = a(x'+X) - b(x'+X)^2 = ax' + \cancel{aX} - (\cancel{b(x')^2} + 2bx'X + \cancel{bX^2})$$

$$\frac{dx'}{dt} = (a - 2bX)x' \Rightarrow x'(t) = x'(0)e^{st}\,,\; s = a - 2bX$$

$$X = 0 \Rightarrow s = a : \begin{cases} \text{stable} & \text{if } a < 0 \\ \text{unstable} & \text{if } a > 0 \end{cases}$$

$$X = \frac{a}{b} \Rightarrow s = -a : \begin{cases} \text{unstable} & \text{if } a < 0 \\ \text{stable} & \text{if } a > 0 \end{cases}$$

<u>explicit solution</u>

$\frac{dx}{dt} = ax - bx^2$: separation of variables works, but here we use an alternative

$-\frac{1}{x^2}\frac{dx}{dt} = \frac{d}{dt}\left(\frac{1}{x}\right) = -a\left(\frac{1}{x}\right) + b$: linear equation for $\frac{1}{x}$

<u>case 1</u> : $a = 0$

$$\frac{1}{x} = bt + \frac{1}{x_0} = \frac{bx_0t + 1}{x_0} \text{ if } x_0 \neq 0$$

$x(t) = \frac{x_0}{bx_0t + 1}$ for all x_0 , check $X = 0$

$x_0 > 0 \Rightarrow x(t) \to 0$ as $t \to \infty$, $x_0 < 0 \Rightarrow x(t) \to -\infty$ as $t \to -1/bx_0$

<u>case 2</u> : $a \neq 0$

$$\frac{d}{dt}\left(\frac{1}{x} - \frac{b}{a}\right) = -a\left(\frac{1}{x} - \frac{b}{a}\right)$$

$$\frac{1}{x} - \frac{b}{a} = \left(\frac{1}{x_0} - \frac{b}{a}\right)e^{-at} \Rightarrow \frac{1}{x} = \frac{a - bx_0}{ax_0}e^{-at} + \frac{b}{a} = \frac{(a - bx_0)e^{-at} + bx_0}{ax_0}$$

$x(t) = \frac{ax_0}{(a - bx_0)e^{-at} + bx_0}$, check $X = 0\,,\; \frac{a}{b}$

$a > 0\,,\; x_0 > 0 \Rightarrow x(t) \to \frac{a}{b}$ as $t \to \infty$

$a > 0\,,\; x_0 < 0 \Rightarrow x(t) \to -\infty$ as $t \to \frac{1}{a}\ln\left(\frac{bx_0 - a}{bx_0}\right)$

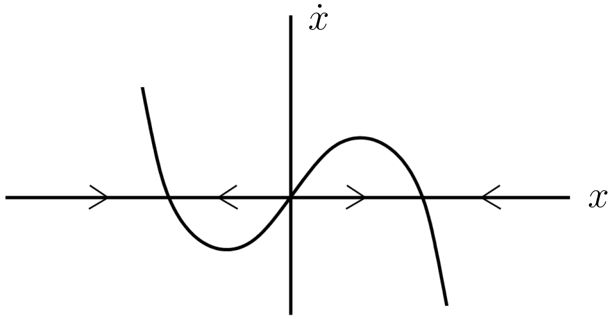
similarly for $a < 0$

1.5 pitchfork bifurcation

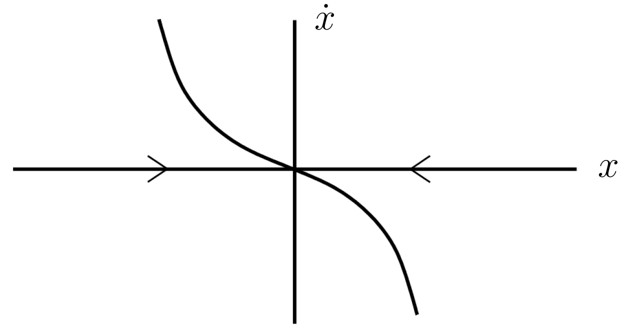
$\dfrac{dx}{dt} = ax - bx^3$: Landau equation

equilibrium : $x(a - bx^2) = 0 \Rightarrow X = 0$ or $X = \pm\sqrt{a/b}$ if a, b have the same sign

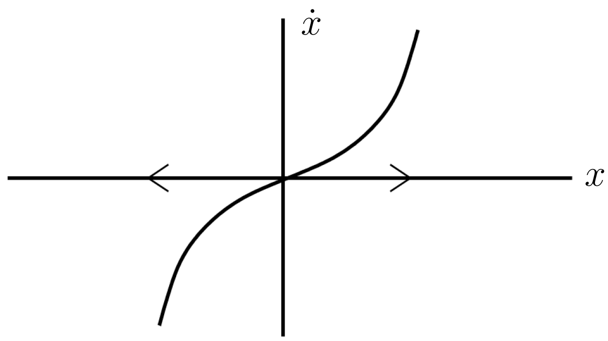
case 1 : $a > 0\,,\ b > 0$



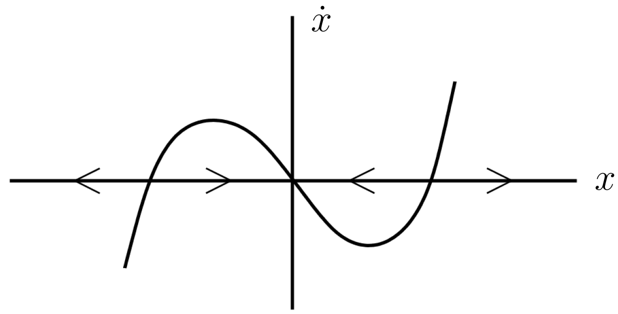
case 2 : $a < 0\,,\ b > 0$



case 3 : $a > 0\,,\ b < 0$

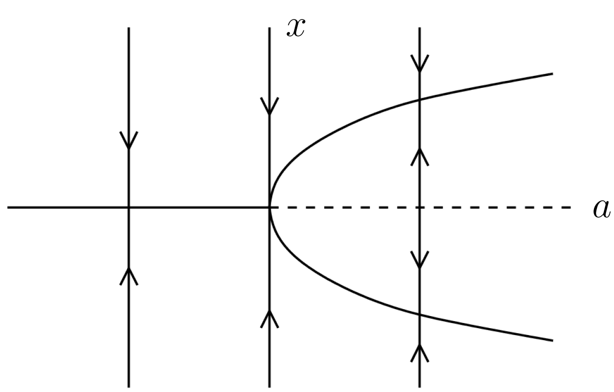


case 4 : $a < 0\,,\ b < 0$



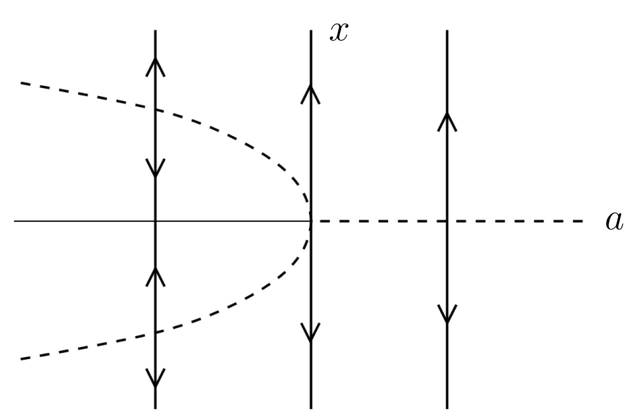
bifurcation diagram

case 1 and 2 : $b > 0$



supercritical pitchfork bifurcation

case 3 and 4 : $b < 0$



subcritical pitchfork bifurcation

The Landau equation is invariant under the transformation $Tx = -x$. The set of equilibrium points is either $\{0\}$ or $\{0, \pm\sqrt{a/b}\}$; both sets are invariant under T; all elements of the 1st set are invariant under T, but some elements of the 2nd set are not invariant under T; this is an example of symmetry breaking.

3
Wed
1/11

<u>linear stability</u>

$$x' = x - X$$

$$\frac{dx'}{dt} = \frac{dx}{dt} = ax - bx^3 = a(x'+X) - b(x'+X)^3$$

$$= ax' + \cancel{aX} - (b\cancel{(x')^3} + 3b\cancel{(x')^2}X + 3bx'X^2 + \cancel{bX^3})$$

$$\frac{dx'}{dt} = (a - 3bX^2)x' \ \Rightarrow x'(t) = x'(0)e^{st}\,,\ s = a - 3bX^2$$

$$X = 0 \ \Rightarrow s = a : \begin{cases} \text{stable} &, a < 0 \\ \text{unstable} &, a > 0 \end{cases}\,,\ X = \pm\sqrt{\frac{a}{b}} \Rightarrow s = -2a : \begin{cases} \text{stable} &, a > 0 \\ \text{unstable} &, a < 0 \end{cases}$$

<u>explicit solution</u>

$$\frac{dx}{dt} = ax - bx^3$$

$$-\frac{2}{x^3}\frac{dx}{dt} = \frac{d}{dt}\left(\frac{1}{x^2}\right) = -2a\left(\frac{1}{x^2}\right) + 2b \ \Rightarrow \frac{d}{dt}\left(\frac{1}{x^2} - \frac{b}{a}\right) = -2a\left(\frac{1}{x^2} - \frac{b}{a}\right)$$

$$\frac{1}{x^2} - \frac{b}{a} = \left(\frac{1}{x_0^2} - \frac{b}{a}\right)e^{-2at}$$

$$\frac{1}{x^2} = \begin{cases} \left(\dfrac{a - bx_0^2}{ax_0^2}\right)e^{-2at} + \dfrac{b}{a} & \text{if } a \neq 0 \\ 2bt + \dfrac{1}{x_0^2} & \text{if } a = 0 \end{cases}$$

$$x^2(t) = \begin{cases} \dfrac{ax_0^2}{(a - bx_0^2)e^{-2at} + bx_0^2} & \text{if } a \neq 0\ ,\ \text{check } X = 0\ ,\ \pm\sqrt{a/b} \\ \dfrac{x_0^2}{2bx_0^2 t + 1} & \text{if } a = 0\ ,\ \text{check } X = 0 \end{cases}$$

$$x(t) = \text{sign}(x_0)\sqrt{x^2(t)}$$

$$(a - bx_0^2)e^{-2at} + bx_0^2 = 0 \ \Rightarrow t_c = \frac{1}{2a}\ln\left(\frac{bx_0^2 - a}{bx_0^2}\right)\ ,\ \text{blow-up occurs if } t_c > 0$$

<u>case 1</u> : $a > 0\,,\ b > 0$

$t_c < 0$, no blow-up , $\lim_{t\to\infty} x(t) = \text{sign}(x_0)\sqrt{a/b}$

The system is <u>bistable</u>, i.e. there are two stable nonzero equilibrium points; a perturbation of $X = 0$ grows due to linear instability, but eventually <u>equilibrates</u> to $X = \pm\sqrt{a/b}$ due to nonlinearity.

<u>case 2</u> : $a < 0\,,\, b > 0$

$t_c < 0$, no blow-up , $\lim\limits_{t\to\infty} x(t) = 0$ for all x_0

Nonlinearity reinforces the linear stability of $X = 0$.

<u>case 3</u> : $a > 0\,,\, b < 0$

$$(a - bx_0^2)e^{-2at} + bx_0^2 = \begin{cases} a > 0 & \text{if } t = 0 \\ bx_0^2 < 0 & \text{if } t \to \infty \end{cases}$$

$t_c > 0$, blow-up occurs , $\lim\limits_{t\to t_c} x(t) = \text{sign}(x_0)\cdot\infty$

Nonlinearity reinforces the linear instability of $X = 0$.

<u>case 4</u> : $a < 0\,,\, b < 0$

$|x_0| > \sqrt{a/b} \Rightarrow x_0^2 > a/b \Rightarrow bx_0^2 - a < 0 \Rightarrow t_c > 0$, blow-up occurs

$|x_0| < \sqrt{a/b} \Rightarrow x_0^2 < a/b \Rightarrow a - bx_0^2 < 0 \Rightarrow (a - bx_0^2)e^{-2at} + bx_0^2 < 0$ for $t \geq 0$

$$\Rightarrow \text{no blow-up}\ ,\ \lim_{t\to\infty} x(t) = 0$$

$|x_0| = \sqrt{a/b}$ is a <u>threshold</u> for instability.

$X = 0$ is subject to a <u>finite amplitude instability</u>.

<u>1.6 Hopf bifurcation</u> 4 Wed 1/18

$$\frac{dx}{dt} = -y + \left(a - x^2 - y^2\right)x\ ,\ \frac{dy}{dt} = x + \left(a - x^2 - y^2\right)y$$

<u>equilibrium</u>

$$-y + \left(a - x^2 - y^2\right)x = 0 \Rightarrow -y^2 + \left(a - x^2 - y^2\right)xy = 0$$

$$x + \left(a - x^2 - y^2\right)y = 0 \Rightarrow \quad x^2 + \left(a - x^2 - y^2\right)xy = 0$$

$$\Rightarrow x^2 + y^2 = 0 \Rightarrow X = Y = 0$$

<u>linear stability</u>

$x' = x\ ,\ y' = y$

$$\frac{dx'}{dt} = -y' + ax'\ ,\ \frac{dy'}{dt} = x' + ay' \Rightarrow \frac{d}{dt}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

look for $\begin{pmatrix} x' \\ y' \end{pmatrix} = e^{st}\begin{pmatrix} x_0' \\ y_0' \end{pmatrix} \Rightarrow s\cancel{e^{st}}\begin{pmatrix} x_0' \\ y_0' \end{pmatrix} = \begin{pmatrix} a-s & -1 \\ 1 & a-s \end{pmatrix}\cancel{e^{st}}\begin{pmatrix} x_0' \\ y_0' \end{pmatrix}$

$\Rightarrow s$ is an eigenvalue , $\det\begin{pmatrix} a-s & -1 \\ 1 & a-s \end{pmatrix} = (a-s)^2 + 1 = 0 \Rightarrow s = a \pm i$

$e^{st} = e^{(a+i)t} = e^{at}e^{it} = e^{at}(\cos t + i \sin t)$

$x'(t), y'(t) \in \text{span}\{e^{at}\sin t, e^{at}\cos t\}$

$\Rightarrow \big(X, Y\big) = (0,0)$ is $\begin{cases} \text{stable} & \text{if } a < 0 \\ \text{unstable} & \text{if } a > 0 \end{cases}$

<u>explicit solution</u>

$x = r\cos\theta\,,\ y = r\sin\theta$

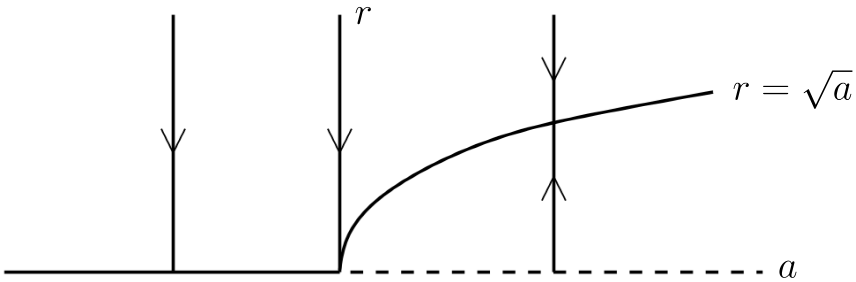
$x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$\frac{dx}{dt} + i\frac{dy}{dt} = -y + \big(a - x^2 - y^2\big)x + i(x + \big(a - x^2 - y^2\big)y)$$

$$= -y + ix + (a - x^2 - y^2)(x + iy) = (x + iy)(i + a - x^2 - y^2) = re^{i\theta}(i + a - r^2)$$

$$\frac{d}{dt}re^{i\theta} = re^{i\theta}i\frac{d\theta}{dt} + \frac{dr}{dt}e^{i\theta} = \left(ir\frac{d\theta}{dt} + \frac{dr}{dt}\right)e^{i\theta}$$

$$\frac{dr}{dt} = ar - r^3\ ,\ \text{equilibrium}: R = \begin{cases} 0\ ,\sqrt{a} & \text{if } a > 0 \\ 0 & \text{if } a < 0 \end{cases}\ ,\ \ r(t) = \ldots$$



There is a supercritical pitchfork bifurcation at $a = 0$.

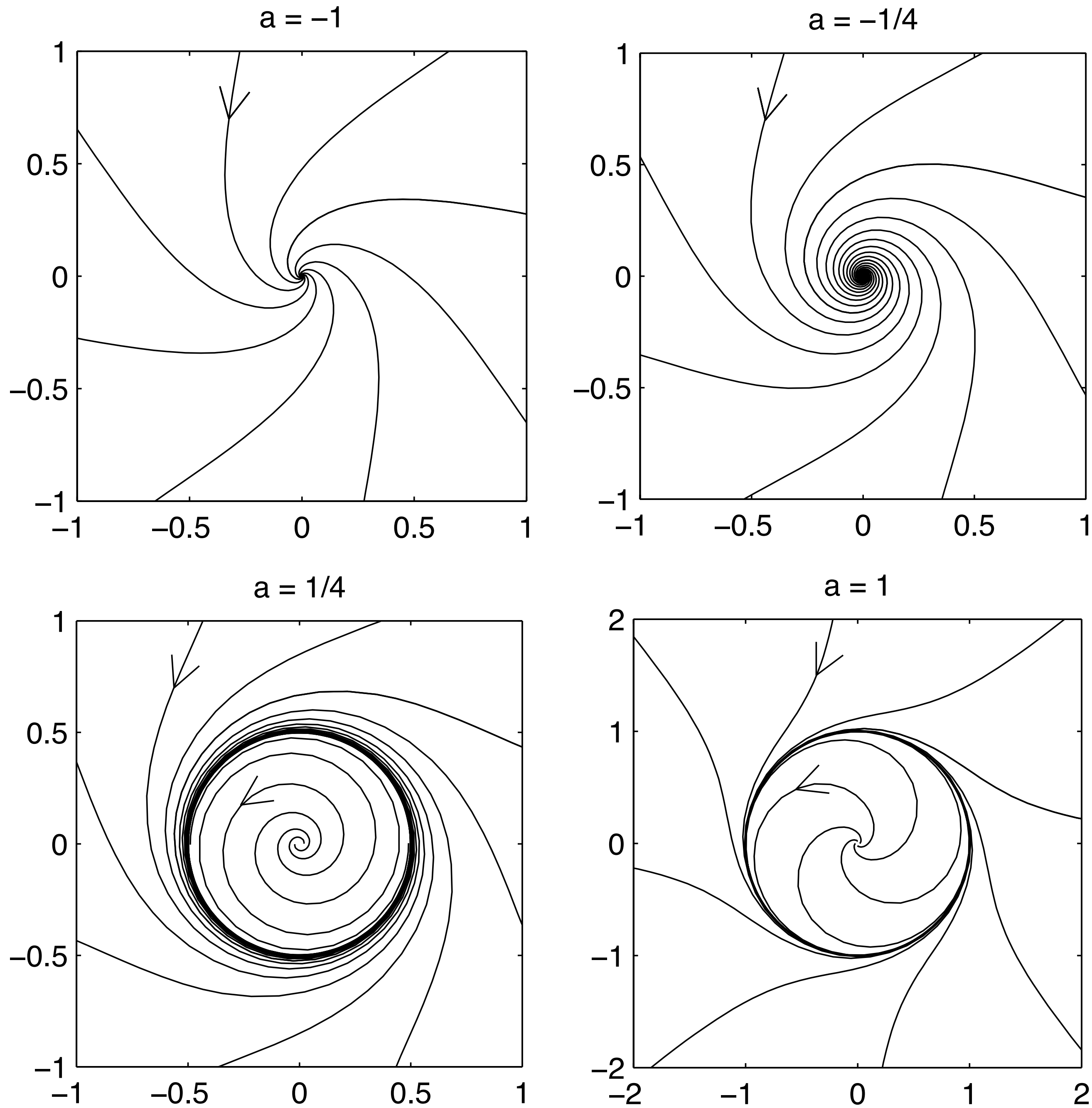
$$\frac{d\theta}{dt} = 1 \Rightarrow \theta(t) = t + \theta_0$$

For $a > 0$, the equilibrium $R = \sqrt{a}$ yields a time-dependent periodic solution of the original system given by $x(t) = \sqrt{a}\cos(t + \theta_0), y(t) = \sqrt{a}\sin(t + \theta_0)$.

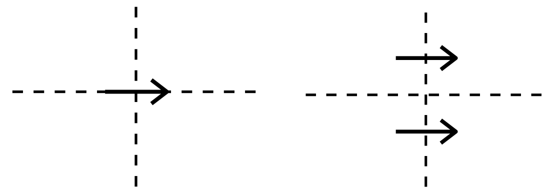
In general, $(x(t), y(t))$ defines an <u>orbit</u> or <u>trajectory</u> in the xy-plane.

$a < 0$: all orbits approach $(0,0)$ as $t \to \infty$, $(0,0)$ is a stable <u>focus</u>

$a > 0$: the circle $x^2 + y^2 = a$ is a stable <u>limit cycle</u>, all orbits with $(x_0, y_0) \neq (0,0)$ approach the limit cycle as $t \to \infty$, $(0,0)$ is an unstable focus


a = −1
a = −1/4
a = 1/4
a = 1
1
0.5
0
−0.5
−1
−1 −0.5 0 0.5 1
2
1
0
−1
−2
−2 −1 0 1 2

summary



1. $dx/dt = a - x^2$: turning point , $s = \pm\sqrt{a}$
2. $dx/dt = ax - bx^2$: transcritical , $s = \pm a$
3. $dx/dt = ax - bx^3$: pitchfork , $s = \{a\,, -2a\}$
4. $\begin{aligned} dx/dt &= -y + (a - x^2 - y^2)x \\ dy/dt &= \ \ x + (a - x^2 - y^2)y \end{aligned}$: Hopf , $s = a \pm i$

In each case the bifurcation occurs at $a=0$, i.e. when the real part of s changes sign, where s is an eigenvalue of the linearized problem. Cases 1,2,3 are called zero-crossing bifurcations; in case 4 the bifurcation occurs when a complex conjugate pair of eigenvalues crosses the imaginary axis in the complex s-plane.

1.7 nonlinear oscillations in a conservative system

5 Mon 1/23

$\dfrac{d^2x}{dt^2} = f(x)$: Newton's 2nd law for particle motion $x(t)$ in a force field

initial data : $x(0)\,, dx/dt(0)$

equilibrium points : $f(X) = 0$

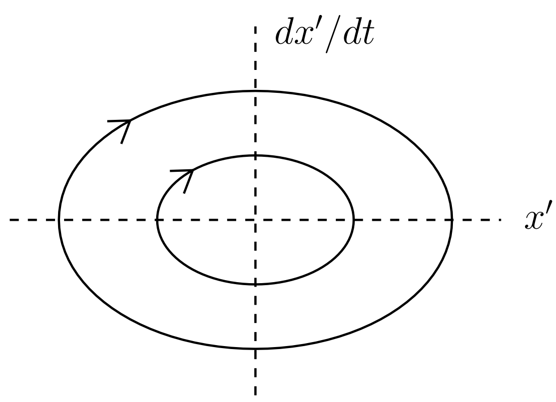
linear stability : $x' = x - X$

$$\frac{d^2x'}{dt^2} = \frac{d^2x}{dt^2} = f(x) = f(x' + X) = \cancel{f(X)} + f'(X)x' + \cdots \Rightarrow \frac{d^2x'}{dt^2} = f'(X)x'$$

case 1 : $f'(X) < 0 \Rightarrow f'(X) = -\alpha^2\,, \alpha > 0$

$x'(t) = a\cos\alpha t + b\sin\alpha t = A\sin(\alpha t + \theta_0)$: stable

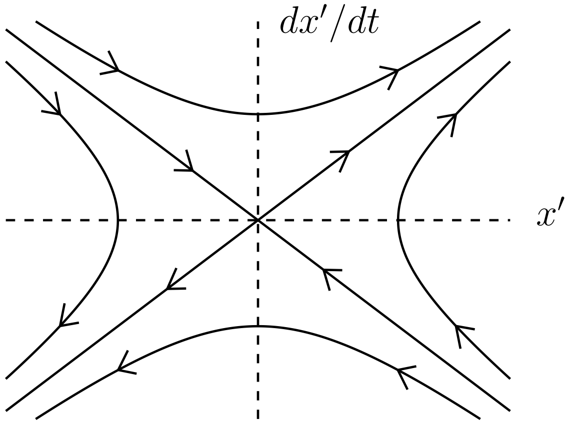
$\dfrac{dx'}{dt} = \alpha A\cos(\alpha t + \theta_0) \Rightarrow \dfrac{1}{A^2}(x')^2 + \dfrac{1}{\alpha^2 A^2}\left(\dfrac{dx'}{dt}\right)^2 = 1$: ellipse in phase plane



X is a center, the orbits are bounded linear oscillations with period $2\pi/\alpha$, this is called simple harmonic motion case 2 : $f'(X) > 0 \Rightarrow f'(X) = \alpha^2$, $\alpha > 0$

$$x'(t) = \begin{cases} c\cosh(\alpha t + \theta_0) & \text{if } |a| < |b| \\ c\sinh(\alpha t + \theta_0) & \text{if } |a| > |b| \\ ce^{\pm\alpha t} & \text{if } |a| = |b| \end{cases} , \quad \frac{dx'}{dt} = \begin{cases} \alpha c\sinh(\alpha t + \theta_0) & \text{if } |a| < |b| \\ \alpha c\cosh(\alpha t + \theta_0) & \text{if } |a| > |b| \\ \pm\alpha ce^{\pm\alpha t} & \text{if } |a| = |b| \end{cases}$$

$$\frac{1}{c^2}(x')^2 - \frac{1}{\alpha^2 c^2}\left(\frac{dx'}{dt}\right)^2 = \pm 1 : \text{ hyperbolas} , \quad \frac{dx'}{dt} = \pm\alpha x' : \text{ lines} , \text{ unstable}$$



X is a saddle point, the orbits are unbounded except for (0,0), some orbits have a turning point, note the 4 orbits that converge to (0,0) as $t \to \pm\infty$

Return to the nonlinear problem.

potential energy : $V(x) = -\int_{x_0}^{x} f(s)ds \Rightarrow f(x) = -V'(x)$

X is an equilibrium $\Leftrightarrow f(X) = 0 \Leftrightarrow V'(X) = 0$: X is a critical point of $V(x)$

$V''(X) > 0 \Rightarrow V(x)$ has a local min at $x = X$, $f'(X) < 0$: center

$V''(X) < 0 \Rightarrow V(x)$ has a local max at $x = X$, $f'(X) > 0$: saddle

kinetic energy : $\frac{1}{2}\left(\frac{dx}{dt}\right)^2$

theorem

The total energy $E = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x)$ of a solution $x(t)$ is constant in time; this is the sense in which the system is conservative.

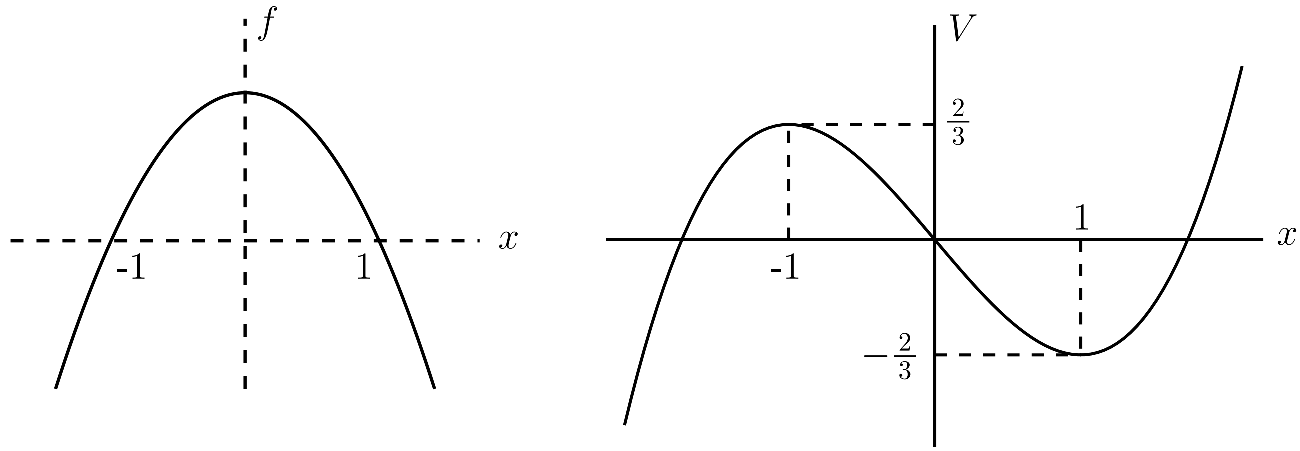
proof

$$\frac{dE}{dt} = \frac{d}{dt}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x)\right] = \frac{dx}{dt}\frac{d^2x}{dt^2} + V'(x)\frac{dx}{dt} = \frac{dx}{dt}\big(f(x) - f(x)\big) = 0 \quad \text{ok}$$

corollary

$V(x(t)) \leq E$ for all t , $V(x(t_1)) = E \Leftrightarrow \frac{dx}{dt}(t_1) = 0$

<u>example</u> : $\dfrac{d^2x}{dt^2} = 1 - x^2 \;\Rightarrow V(x) = -x + \dfrac{1}{3}x^3$



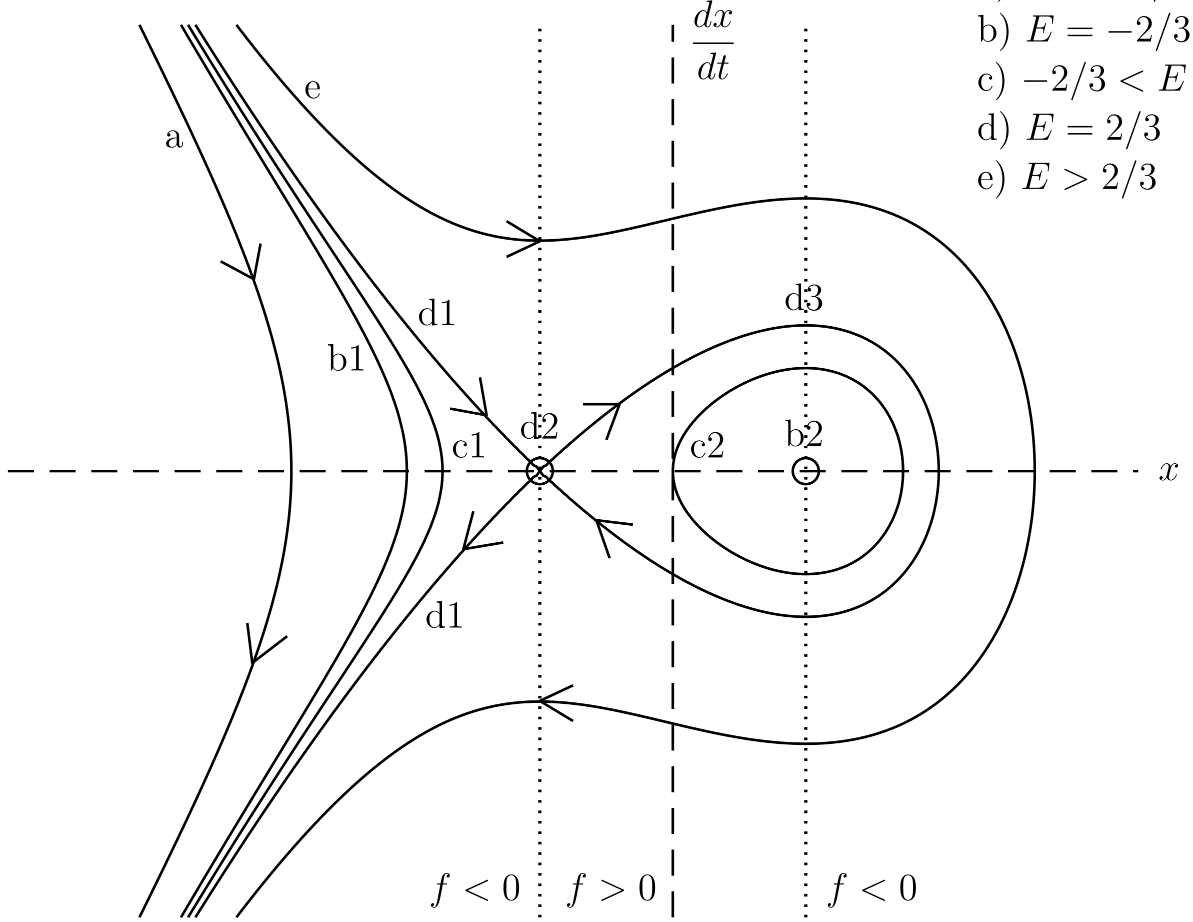
$x < -1\,,\; x > 1 \;\Rightarrow f(x) < 0\,,\; V'(x) > 0$: particle is decelerating , $x''(t) < 0$

$-1 < x < 1 \qquad \Rightarrow f(x) > 0\,,\; V'(x) < 0$: particle is accelerating , $x''(t) > 0$

equilibrium : $X = \pm 1$, linear stability : $f'(x) = -2x \begin{cases} f'(1) = -2 : \text{center} \\ f'(-1) = 2 : \text{saddle} \end{cases}$

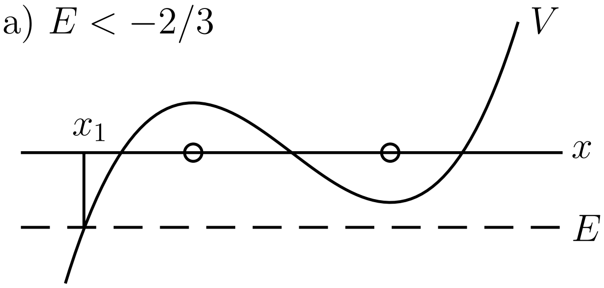
phase plane : $\dfrac{1}{2}\left(\dfrac{dx}{dt}\right)^2 - x + \dfrac{1}{3}x^3 = E$

a) $E < -2/3$
b) $E = -2/3$
c) $-2/3 < E < 2/3$
d) $E = 2/3$
e) $E > 2/3$



The set $\{x : V(x) = E\}$ has 1, 2, or 3 points, and at such points $x'(t) = 0$.

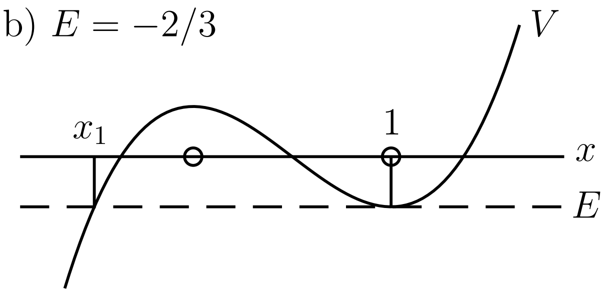
a) $E < -2/3$



let $x_1 = x(t_1)$ be such that $V(x_1) = E$, then $x''(t_1) = f(x_1) = -V'(x_1) < 0$, $x(t)$ has a local max at x_1 and x_1 is a turning point, $x(t)$ comes in from $-\infty$, turns around at $x_1 < -1$, then decreases back to $-\infty$

6
Fri
1/27

b) $E = -2/3$

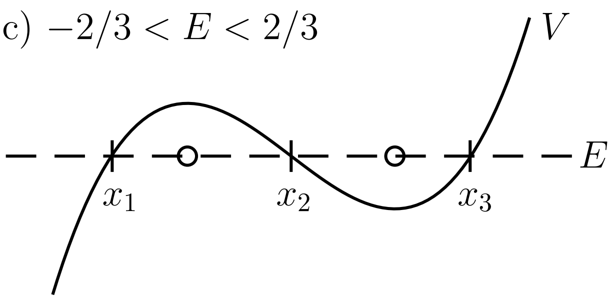


2 possibilities

b1) $x(t) \le x_1$, $x(t)$ is similar to (a)

b2) $x(t) = 1$ is the center

c) $-2/3 < E < 2/3$

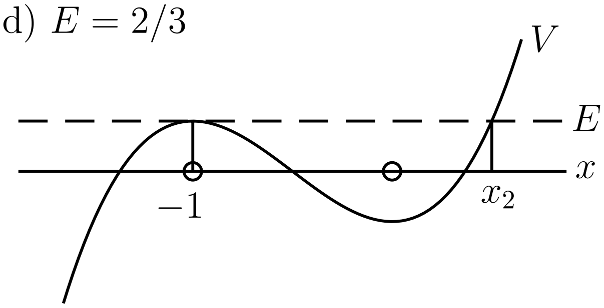


2 possibilities

c1) $x(t) \le x_1$, $x(t)$ is similar to (a)

c2) $x_2 \le x(t) \le x_3$, x_2 and x_3 are turning points, $x(t)$ is a <u>periodic orbit</u>

d) $E = 2/3$



3 possibilities

d1) $x(t) < -1$

$x'(t) > 0 \Rightarrow x(t) \to -1$ as $t \to \infty$

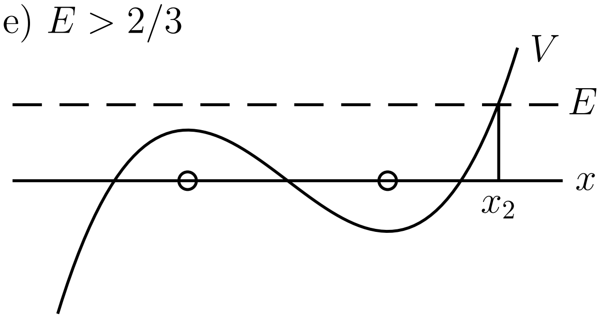
$x'(t) < 0 \Rightarrow x(t) \to -1$ as $t \to -\infty$

d2) $x(t) = -1$ is the saddle point

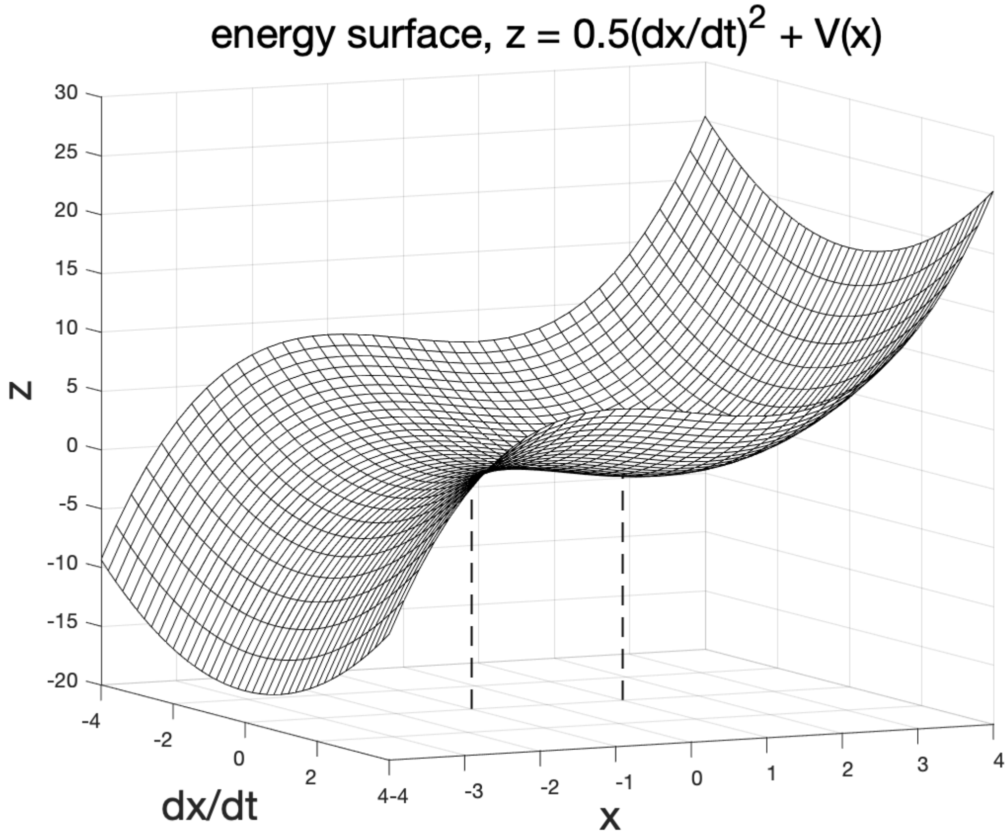
d3) $-1 < x(t) \le x_2$

In (d3), $x(t)$ has a turning point (local max) at x_2, $x(t) \to -1$ as $t \to \pm\infty$, $x(t)$ is a <u>homoclinic orbit</u>, also called a <u>separatrix</u> because it separates the bounded and unbounded orbits.

e) $E > 2/3$



$x(t)$ starts out similar to (a), but here it has sufficient energy to circle around the equilibrium at $X = 1$



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7 Mon 1/30

<u>period</u>

Consider a periodic orbit with $-2/3 < E < 2/3$.

$$T = \int_0^T dt = 2\int_{x_2}^{x_3} \frac{dx}{dx/dt} = 2\int_{x_2}^{x_3} \frac{dx}{\sqrt{2(E - V(x))}}$$

$V(x_2) = V(x_3) = E \Rightarrow$ improper integral, does it converge or diverge?

<u>recall</u> : $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_0^1 = 2$: converges , $\displaystyle\int_0^1 \frac{dx}{x} = 2\ln x\Big|_0^1 = \infty$: diverges

$$E - V(x) = \cancel{E} - (\cancel{V(x_2)} + V'(x_2)(x - x_2) + \tfrac{1}{2}V''(x_2)(x - x_2)^2 + \cdots)$$

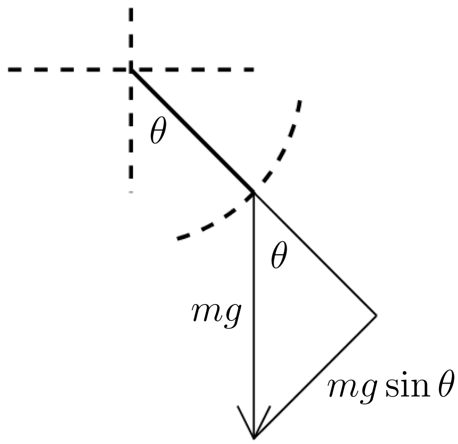
$$\frac{1}{\sqrt{E - V(x)}} \sim \frac{1}{\sqrt{-V'(x_2)(x - x_2)}}$$: converges , similar for x_3

however if $E \to 2/3$, then $x_2 \to -1$, $V'(x_2) \to 0$

$$\frac{1}{\sqrt{E - V(x)}} \sim \frac{1}{\sqrt{-\frac{1}{2}V''(x_2)(x - x_2)^2}}$$: diverges , hence $T \to \infty$ as $E \to 2/3$

On the homoclinic orbit (where $E = 2/3$), $x(t) \to -1$ as $t \to \pm\infty$; this is consistent with the exponential behavior of the orbits of the linearized problem at a saddle point.

<u>example</u> : nonlinear pendulum



angular displacement → velocity → acceleration

$$\ell d\theta \to \ell\frac{d\theta}{dt} \to \ell\frac{d^2\theta}{dt^2}$$

Newton's 2nd law (tangential component)

$$m \cdot \ell\frac{d^2\theta}{dt^2} = -mg\sin\theta \Rightarrow \frac{d^2\theta}{dt^2} = -(g/\ell)\sin\theta$$

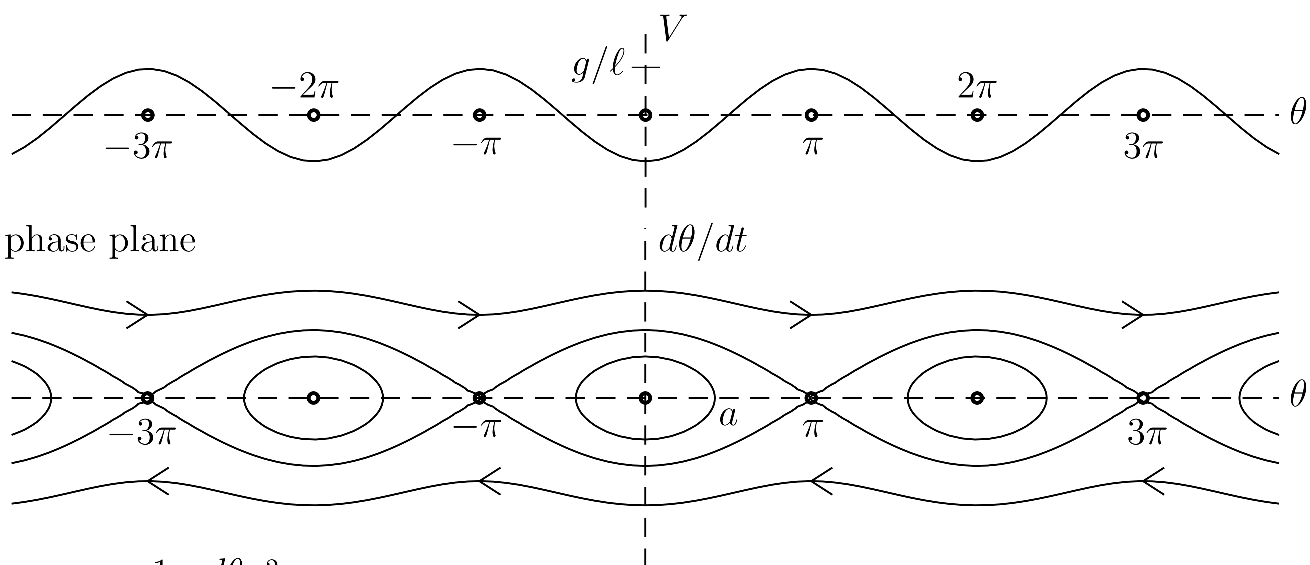
equilibrium points : $\sin\theta = 0 \Rightarrow \theta = n\pi\,,\ n = 0\,,\ \pm1\,,\ \pm2\,,\ \ldots$

linear stability : $\theta' = \theta - n\pi$

$f(\theta) = -(g/\ell)\sin\theta \Rightarrow f'(\theta) = -(g/\ell)\cos\theta \Rightarrow f'(n\pi) = (g/\ell)(-1)^{n+1}$

$$\frac{d^2\theta'}{dt^2} = (g/\ell)(-1)^{n+1}\theta' \ : \ \begin{cases}\text{stable if } n \text{ is even : center}\\ \text{unstable if } n \text{ is odd : saddle point}\end{cases}$$

potential energy : $V(\theta) = -(g/\ell)\cos\theta$, satisfies $f(\theta) = -V'(\theta)$



energy : $\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - (g/\ell)\cos\theta = E$

$E < -g/\ell$: no orbits

$E = -g/\ell$: center at $\theta = 2n\pi$

$-g/\ell < E < g/\ell$: bounded periodic orbits

$E = g/\ell$: saddle point or <u>heteroclinic orbit</u>

$E > g/\ell$: unbounded periodic orbits

<u>period</u> : bounded periodic orbits

$-g/\ell < E < g/\ell \Rightarrow E = -g/\ell \cos a$, where $0 \le a \le \pi$, a : amplitude of orbit

$$T = 2\int_{-a}^{a} \frac{d\theta}{\sqrt{2(E - V(\theta))}} = 4\int_0^a \frac{d\theta}{\sqrt{2(g/\ell)(\cos\theta - \cos a)}}$$

$$E - V(\theta) = -\frac{g}{\ell}\cos a + \frac{g}{\ell}\cos\theta = (g/\ell)(\cos\theta - \cos a)$$

$$\cos\theta - \cos a = \cos(2\cdot\theta/2) - \cos(2\cdot a/2) = 1 - 2\sin^2(\theta/2) - (1 - 2\sin^2(a/2))$$

$$= 2(\sin^2(a/2) - \sin^2(\theta/2))$$

substitute $\sin(\theta/2) = \sin(a/2)\sin\phi$

$$\cos\theta - \cos a = 2\sin^2(a/2)(1 - \sin^2\phi) = 2\sin^2(a/2)\cos^2\phi$$

$$-\sin\theta d\theta = 2\sin^2(a/2)\cdot 2\cos\phi\cdot -\sin\phi d\phi$$

$$\sin\theta = 2\sin(\theta/2)\cos(\theta/2) = 2\sin(a/2)\sin\phi\sqrt{1 - \sin^2(a/2)\sin^2\phi}$$

$$d\theta = \frac{4\sin^2(a/2)\cos\phi\sin\phi\, d\phi}{2\sin(a/2)\sin\phi\sqrt{1 - \sin^2(a/2)\sin^2\phi}} = \frac{2\sin(a/2)\cos\phi\, d\phi}{\sqrt{1 - \sin^2(a/2)\sin^2\phi}}$$

$$= \frac{\sqrt{2(\cos\theta - \cos a)}\, d\phi}{\sqrt{1 - \sin^2(a/2)\sin^2\phi}}$$

$$T = 4\sqrt{\ell/g}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \sin^2(a/2)\sin^2\phi}} = 4\sqrt{\ell/g}\, K(\sin^2(a/2))$$

$K(m) = \displaystyle\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - m\sin^2\phi}}$: <u>complete elliptic integral</u>

$\dfrac{1}{\sqrt{1-x}} = 1 + \dfrac{x}{2} + O(x^2)$ as $x \to 0$

$$T = 4\sqrt{\ell/g}\int_0^{\pi/2}\Big(1 + \frac{1}{2}\sin^2(a/2)\sin^2\phi + O(a^4)\Big)d\phi \text{ as } a \to 0$$

$$= 4\sqrt{\ell/g}\Big(\frac{\pi}{2} + \frac{1}{2}\cdot\frac{a^2}{4}\cdot\frac{\pi}{4} + O(a^4)\Big) = 2\pi\sqrt{\ell/g}\Big(1 + \frac{a^2}{16} + O(a^4)\Big)$$

note : $a \to \pi \Rightarrow T \to 4\sqrt{\ell/g}\displaystyle\int_0^{\pi/2}\frac{d\phi}{\cos\phi} = \infty$

8 Wed 2/1

1.8 maps

$x_{n+1} = F(x_n)\ ,\ n = 0, 1, 2, \dots$

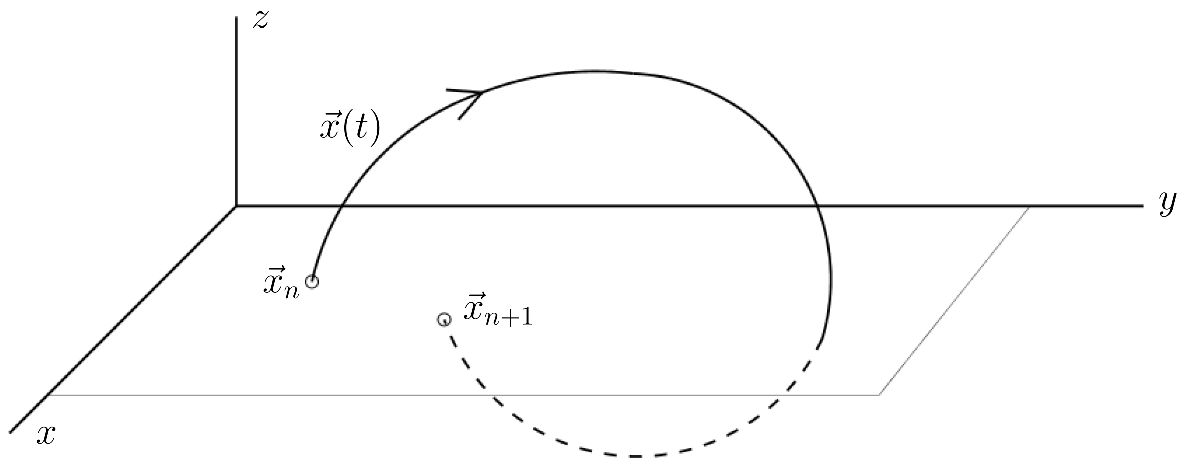
1. Euler's method for $\dfrac{dx}{dt} = f(x)$

$$\frac{x_{n+1} - x_n}{\Delta t} = f(x_n) \Rightarrow x_{n+1} = x_n + \Delta t f(x_n)$$

2. Newton's method for $f(x) = 0$

$$f(x_{n+1}) = f(x_n) + f'(x_n)(x_{n+1} - x_n) + \cdots \Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

3. Poincaré map for $\dfrac{d\vec{x}}{dt} = \vec{f}(\vec{x})\ ,\ \vec{x} = (x, y, z)$



4. logistic map

$x_{n+1} = ax_n - bx_n^2\ ,\ x_n$: population in nth generation

$b = 0 \Rightarrow x_n = a^n x_0$: exponential growth , $b > 0$: decay , finite resources

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definition

If $X = F(X)$, then $x_0 = X \Rightarrow x_n = X$ for $n \geq 1$ and X is called a fixed point.

linear stability : $x'_n = x_n - X$

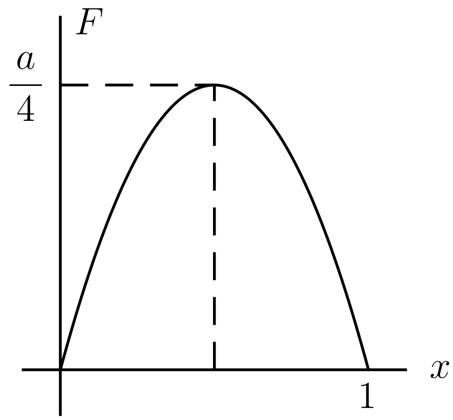
$$x'_{n+1} = x_{n+1} - X = F(x_n) - F(X) = F(x'_n + X) - F(X) = \cdots$$

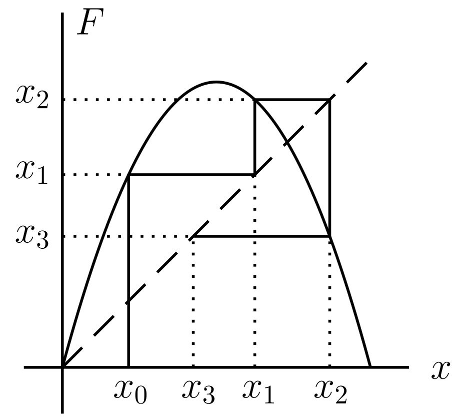
$$x'_{n+1} = F'(X)x'_n \Rightarrow x'_n = (F'(X))^n x'_0$$

$$\lim_{n\to\infty} x'_n = \begin{cases} 0 & \text{if } |F'(X)| < 1 \text{ : stable} \\ \text{unbounded} & \text{if } |F'(X)| > 1 \text{ : unstable} \\ \text{bounded} & \text{if } |F'(X)| = 1 \end{cases}$$

example : logistic map

consider $x_{n+1} = ax_n(1-x_n) = F(x_n)$, $F(x) = ax(1-x)$, assume $a > 0$





note : if $0 \le x \le 1$ and $0 \le a \le 4$, then $0 \le F(x) \le 1$, i.e. $F : [0,1] \to [0,1]$

fixed points : $X = aX(1-X) \Rightarrow aX^2 + (1-a)X = X(aX+1-a) = 0$

$$X = 0\,,\,\frac{a-1}{a}$$

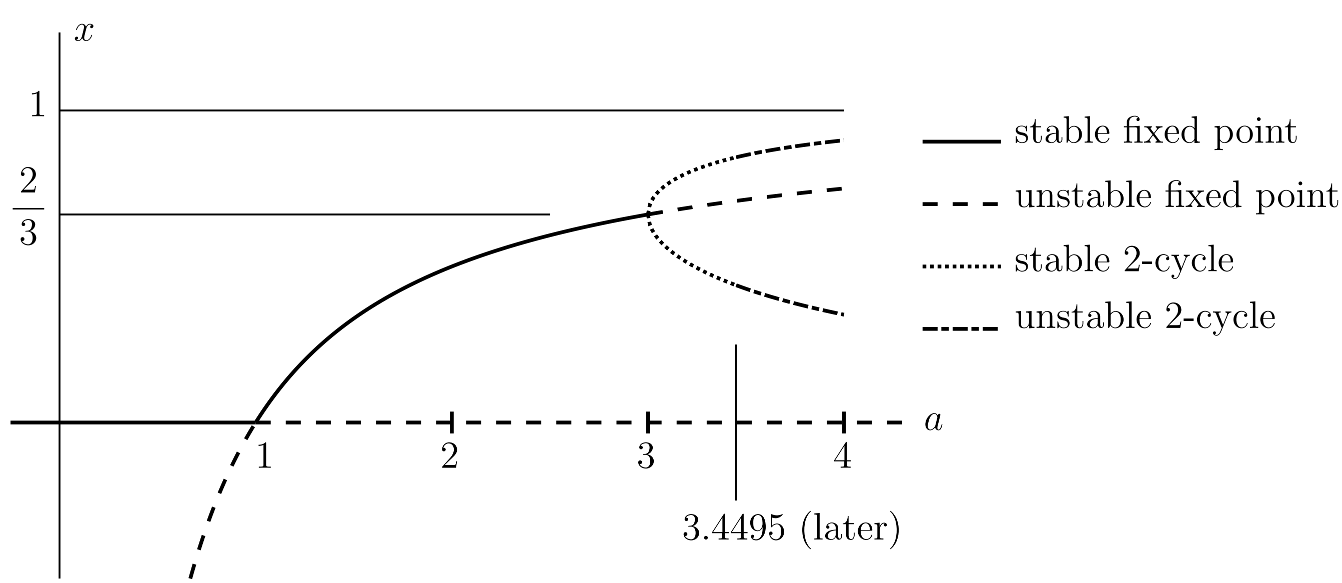
linear stability : $F'(x) = a - 2ax = a(1-2x)$

$$F'(0) = a \Rightarrow X = 0 : \begin{cases} \text{stable} & \text{for } 0 \le a < 1 \\ \text{unstable} & \text{for } 1 < a \le 4 \end{cases}$$

$$F'\Big(\frac{a-1}{a}\Big) = a\Big(1 - 2\Big(\frac{a-1}{a}\Big)\Big) = a - 2(a-1) = 2-a$$

$$\Rightarrow X = \frac{a-1}{a} : \begin{cases} \text{stable} & \text{for } 1 < a < 3 \\ \text{unstable} & \text{for } 0 \le a < 1 \text{ and } 3 < a \le 4 \end{cases}$$

bifurcation diagram (partial)



$(0,1)$: transcritical bifurcation , $(3, 2/3)$: flip bifurcation

<u>definition</u> : If there exist $X_1 \neq X_2$ such that $X_2 = F(X_1)$ and $X_1 = F(X_2)$, then $\{x_n\} = \{X_1, X_2, X_1, X_2, \ldots\}$ is an orbit of period 2 or a <u>2-cycle</u>.

Look for a 2-cycle of the logistic map.

$X_2 = aX_1(1 - X_1)$

$X_1 = aX_2(1 - X_2) = a \cdot aX_1(1 - X_1)(1 - aX_1(1 - X_1))$

$X(a^2(X - 1)(1 - aX + aX^2) + 1) = 0$

4th degree equation for X

we already know two roots, $X = 0\,,\ \dfrac{a-1}{a} = 1 - \dfrac{1}{a}$

$$aX((X - 1 + \frac{1}{a} - \frac{1}{a})(a^2X^2 - a^2X + a) + \frac{1}{a}) = 0$$

$$aX((X - 1 + \frac{1}{a})(a^2X^2 - a^2X + a) - aX^2 + aX - 1 + \frac{1}{a}) = 0$$

$$aX((X - 1 + \frac{1}{a})(a^2X^2 - a^2X + a) + (X - 1 + \frac{1}{a})(-aX + 1)) = 0$$

$$aX(X - 1 + \frac{1}{a})(a^2X^2 - a^2X + a - aX + 1) = 0$$

$$aX(X - 1 + \frac{1}{a})(a^2X^2 - a(a+1)X + a + 1) = 0$$

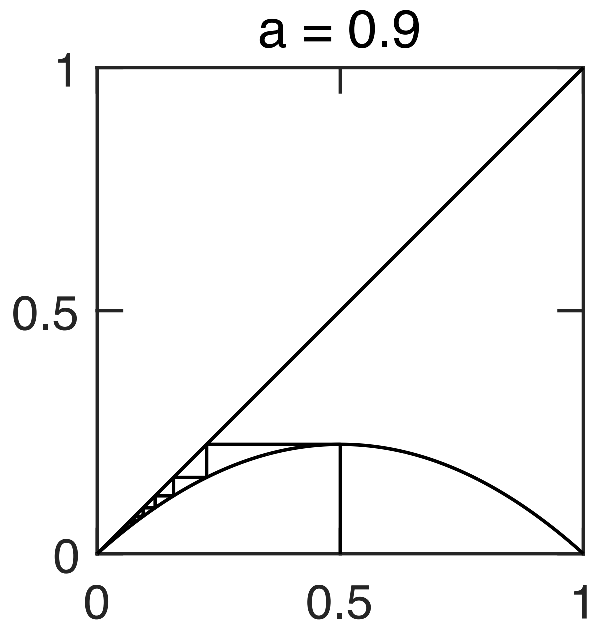
$$X = \frac{a(a+1) \pm \sqrt{a^2(a+1)^2 - 4a^2(a+1)}}{2a^2}$$

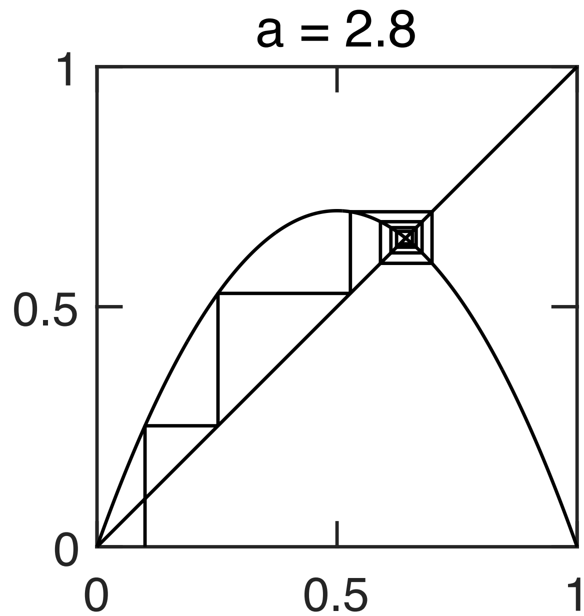
$$= \frac{a + 1 \pm \sqrt{a^2 + 2a + 1 - 4a - 4)}}{2a}$$

$$= \frac{a-1}{a} + \frac{3 - a \pm \sqrt{(a+1)(a-3)}}{2a}$$

Hence the 2-cycle exists for $a < -1$ and $a > 3$.

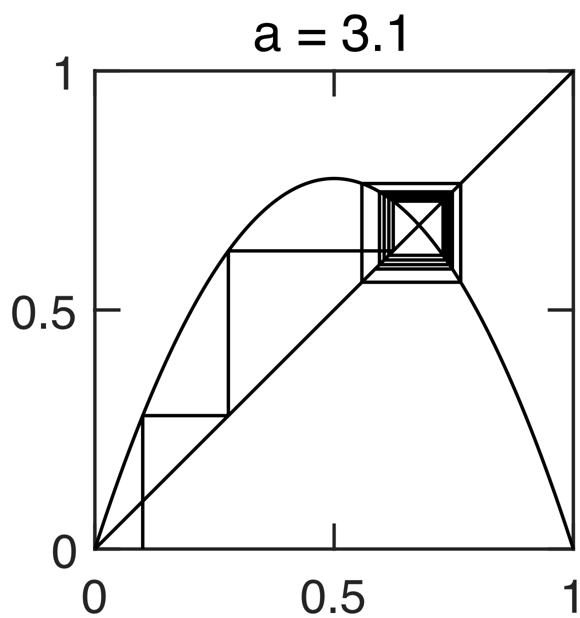
<u>logistic map</u> : $x_{n+1} = a\,x_n\,(1 - x_n)$

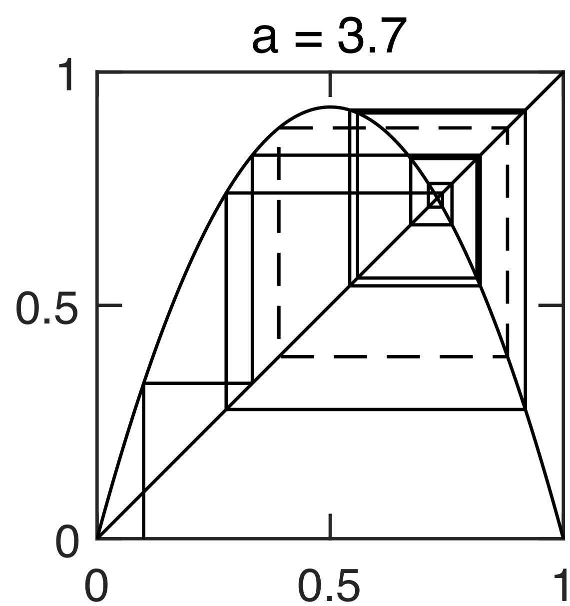




$a = 0.9$: a single stable fixed point at $X = 0$

$a = 2.8$: two fixed points, $X = 0$ is unstable and $X = \dfrac{a-1}{a}$ is stable





$a = 3.1$: two unstable fixed points, a stable 2-cycle

$a = 3.7$: two unstable fixed points, an unstable 2-cycle (more later)

2.1, 2.2 classification of bifurcations (local behavior) 9 Mon 2/6

consider $dx/dt = F(a,x)$ or $x_{n+1} = F(a,x_n)$

$F : \mathbb{R}^\ell \times \mathbb{R}^m \to \mathbb{R}^m$, $a \in \mathbb{R}^\ell$: control parameters , $x \in \mathbb{R}^m$: state space

define : $\mathcal{E} = \{(a,x) : F(a,x) = 0\}$: equilibrium set

A point $(a_0, X_0) \in \mathcal{E}$ is called a bifurcation point if the number of solutions of $F(a,x) = 0$ changes as a varies in a neighborhood of a_0.

$m = 1, \ell = 1$

$\mathcal{E}$ is a curve in the (a,x)-plane; we will classify the points $(a_0, X_0) \in \mathcal{E}$.

question : Given $(a_0, X_0) \in \mathcal{E}$, when does there exist a function $x = X(a)$ such that $X_0 = X(a_0)$ and $F(a, X(a)) = 0$?

notation : $F_0 = F(a_0, X_0), F_{x0} = F_x(a_0, X_0), F_{a0} = F_a(a_0, X_0), \ldots$

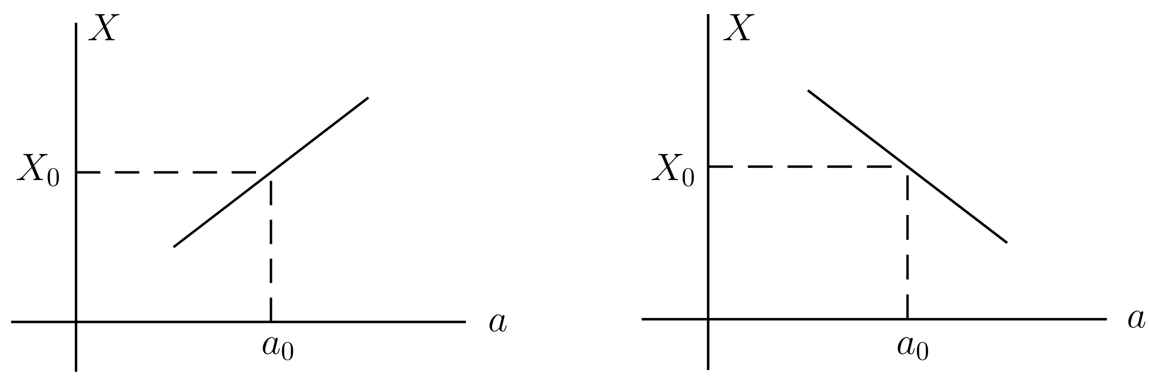
answer : implicit function theorem : If $F_{x0} \neq 0$, then ...

definition : If $F_0 = 0$ and $F_{x0} \neq 0$ or $F_{a0} \neq 0$, then $(a_0, X_0) \in \mathcal{E}$ is a regular point.

case 1 : $F_{x0} \neq 0, F_{a0} \neq 0$: regular point, but no bifurcation

$F(a,x) = \cancel{F_0} + F_{x0}(x - X_0) + F_{a0}(a - a_0) + \cdots = 0$

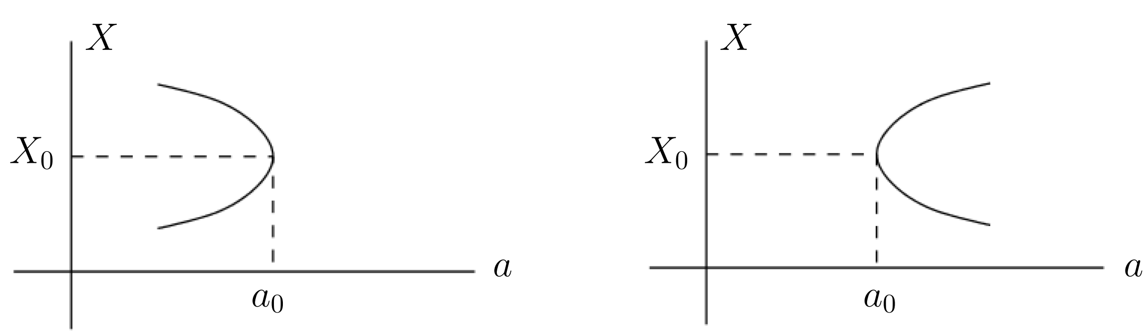
IFT $\Rightarrow X(a) = X_0 - (F_{a0}/F_{x0})(a - a_0) + \cdots$



case 2 : $F_{x0} = 0, F_{a0} \neq 0, F_{xx0} \neq 0$: regular turning point

$F(a,x) = \cancel{F_0} + \cancel{F_{x0}}(x - X_0) + F_{a0}(a - a_0) + \frac{1}{2}F_{xx0}(x - X_0)^2 + \cdots = 0$

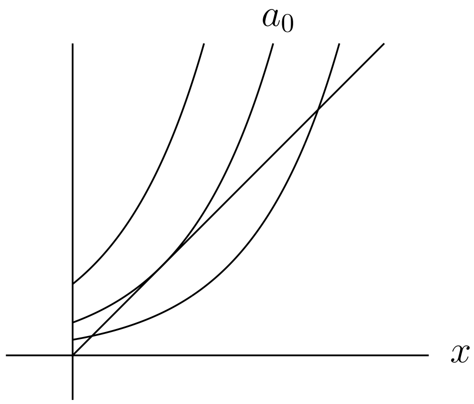
IFT as above does not apply, but $X(a) = X_0 \pm \sqrt{-2(F_{a0}/F_{xx0})(a - a_0)} + \cdots$

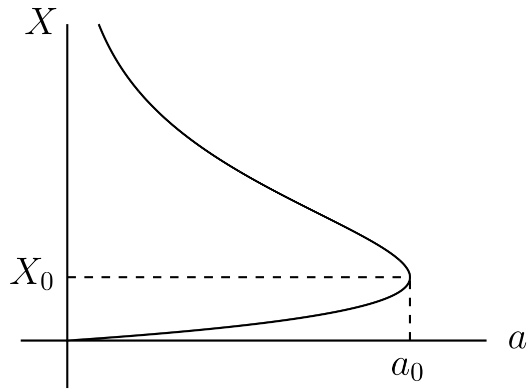


note : IFT can be applied, $F_{a0} \neq 0 \Rightarrow A(x) = a_0 - \frac{1}{2}(F_{xx0}/F_{a0})(x - X_0)^2 + \cdots$

<u>example</u> : $x_{n+1} = ae^{x_n}$: <u>exponential map</u> , $a > 0$

equilibrium set : $F(a,x) = ae^x - x = 0 \Leftrightarrow x = ae^x \Leftrightarrow a = xe^{-x}$





$F(a,x) = ae^x - x = 0\,,\ F_x(a,x) = ae^x - 1 = 0 \Rightarrow X_0 = 1\,,\ a_0 = e^{-1}$

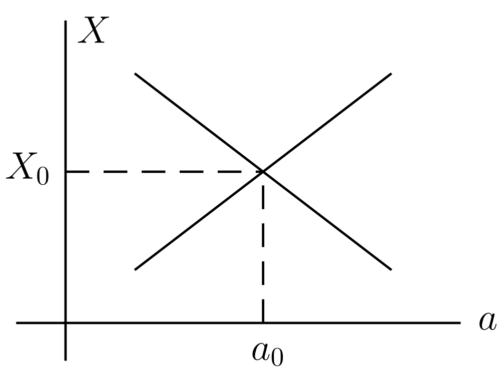
$F_{x0} = 0, F_{a0} = e \neq 0, F_{xx0} = 1 \neq 0$: regular turning point

$X(a) = X_0 \pm \sqrt{-2(F_{a0}/F_{xx0})(a - a_0)} + \cdots = 1 \pm \sqrt{2(1 - ea)} + \cdots$

<u>definition</u> : If $F_0 = F_{x0} = F_{a0} = 0$, then $(a_0, X_0) \in \mathcal{E}$ is a <u>singular point</u>.

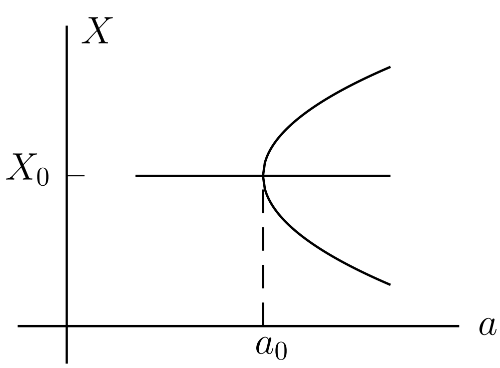
<u>basic examples</u>

1. $F(a,x) = (x - X_0)^2 - (a - a_0)^2 = 0$: transcritical bifurcation



$X(a) = X_0 \pm (a - a_0)$

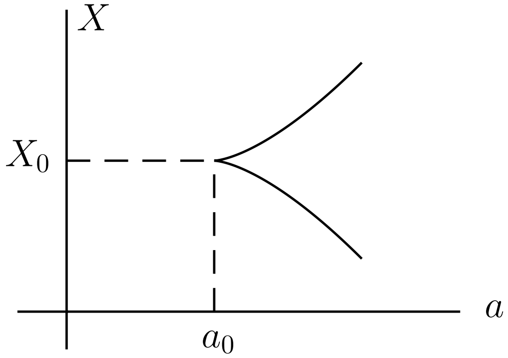
2. $F(a,x) = (x - X_0)((x - X_0)^2 - (a - a_0)) = 0$: singular turning point
pitchfork bifurcation



$X(a) = X_0, X(a) = X_0 \pm \sqrt{a - a_0}$

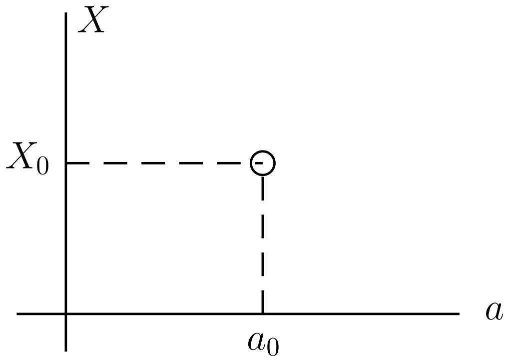
note : In a pitchfork bifurcation, dx/da changes sign along a branch of $X(a)$; this is not true for a transcritical bifurcation.

3. $F(a,x) = (x - X_0)^2 - (a - a_0)^3 = 0$: <u>cusp</u>



$$X(a) = X_0 \pm (a - a_0)^{3/2}$$

4. $F(a,x) = (x - X_0)^2 + (a - a_0)^2 = 0$: <u>conjugate point</u>



<u>example</u> : $F(a,x) = x^3 + a^3 - 3ax = 0$: <u>folium of Descartes</u> 10 Wed 2/8

$F_x(a,x) = 3x^2 - 3a = 0 \Rightarrow a = x^2$

$x^3 + a^3 - 3ax = x^3 + x^6 - 3x^3 = x^3(x^3 - 2) = 0 \Rightarrow X_0 = 0\,,\ 2^{1/3}$

$F_{xx}(a,x) = 6x\,,\ F_a(a,x) = 3a^2 - 3x\,,\ F_{ax}(a,x) = -3\,,\ F_{aa}(a,x) = 6a$

<u>$(a_0, X_0) = (0,0)$</u>

$F_0 = F_{x0} = F_{a0} = 0$: singular point

$F_{xx0} = 0\,,\ F_{ax0} = -3\,,\ F_{aa0} = 0$

$F(a,x) \sim F_{ax0}(a - a_0)(x - X_0) = -3ax = 0$

$\Rightarrow a = 0$ or $x = 0$: transcritical or pitchfork , locally $a = x^2$ or $x = a^2$

<u>$(a_0, X_0) = (4^{1/3}, 2^{1/3})$</u>

$F_0 = F_{x0} = 0\,,\ F_{a0} = 3(16^{1/3} - 2^{1/3}) \neq 0$: regular point

$F_{xx0} = 6 \cdot 2^{1/3} \neq 0 \Rightarrow F(a,x) \sim F_{a0}(a - a_0) + \frac{1}{2} F_{xx0}(x - X_0)^2 = 0$

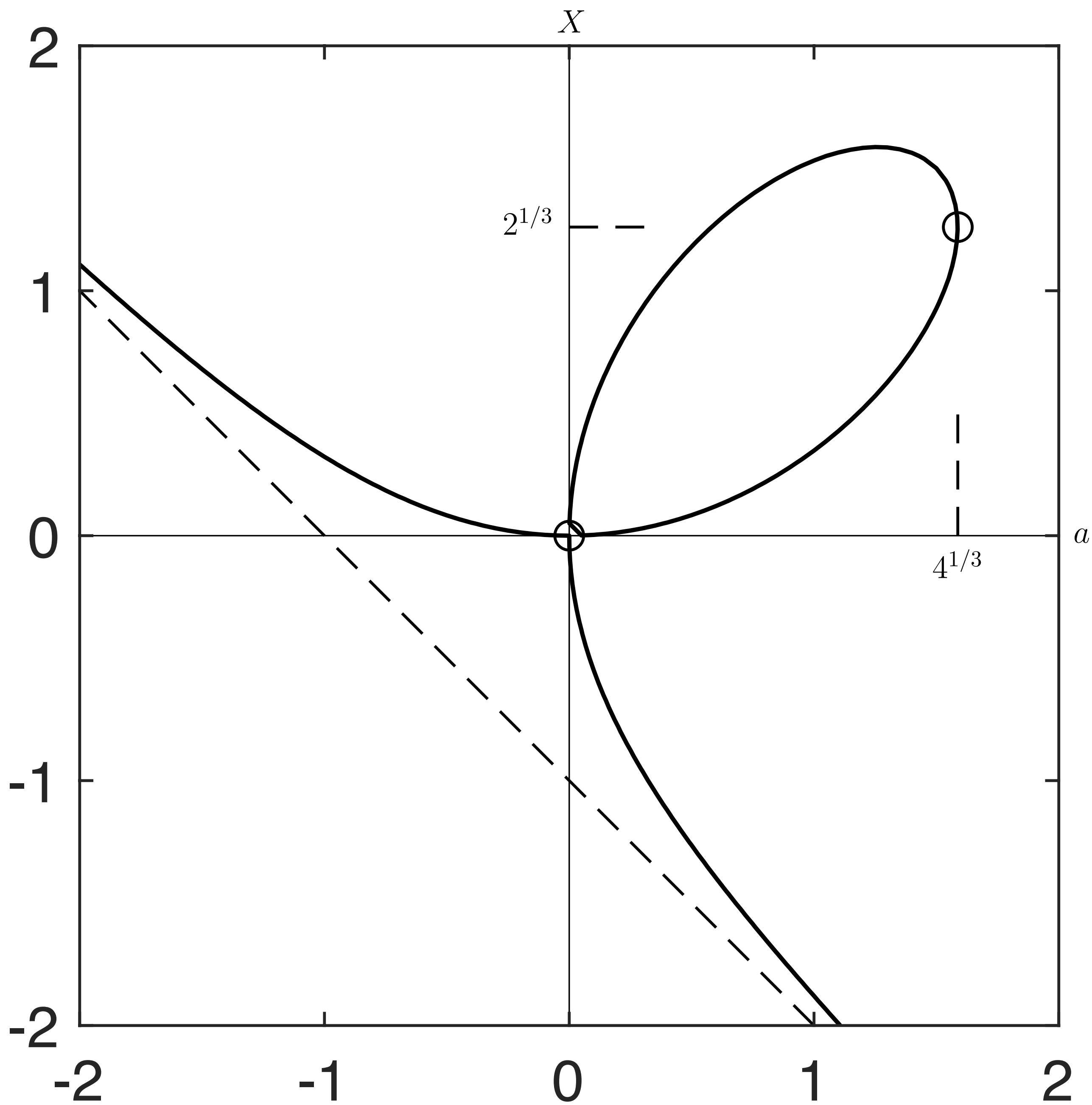
$X(a) = X_0 \pm \sqrt{-2(F_{a0}/F_{xx0})(a - a_0)} \Rightarrow a < a_0$: turning point

<u>folium of Descartes</u>

$F(a, x) = x^3 + a^3 - 3ax = 0$

$(a_0, X_0) = (0, 0)$: singular point , pitchfork

$(a_0, X_0) = (4^{1/3}, 2^{1/3})$: regular turning point



<u>$m = 1, \ell = 2$</u>

The <u>equilibrium set</u> $\mathcal{E} = \{(a, b, x) : F(a, b, x) = 0$ defines a surface $x = X(a, b)$ in (a, b, x)-space (which may be multi-valued).

Assume $(a_0, b_0, X_0) \in \mathcal{E}$.

$F(a, b, x) = \not{F_0} + F_{x0}(x - X_0) + F_{a0}(a - a_0) + F_{b0}(b - b_0) + \cdots = 0$

If $F_{x0} \neq 0$, then $X(a, b) = X_0 - \dfrac{1}{F_{x0}}(F_{a0}(a - a_0) + F_{b0}(b - b_0)) + \cdots$ and there is no bifurcation at (a_0, b_0, X_0); hence bifurcation can only occur when $F_{x0} = 0$.

The two equations $F(a, b, x) = 0, F_x(a, b, x) = 0$ define a curve in the $(a.b)$-plane called the <u>bifurcation set</u>. (For $\ell = 1$ this is just a set of points on the a-axis.)

<u>example</u> : $F(a, b, x) = 4x^3 - 2ax + b = 0$

For each (a, b) the surface $x = X(a, b)$ has 1, 2, or 3 values.

$F_x(a, b, x) = 12x^2 - 2a = 0 \Rightarrow x = \pm(a/6)^{1/2}$: bifurcation only occurs for $a > 0$

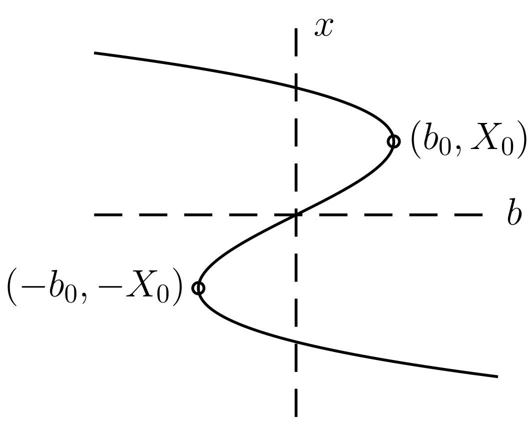
$F(a, b, x) = 0 \Rightarrow 4 \cdot \pm(a/6)^{3/2} - 2a \cdot \pm(a/6)^{1/2} + b = 0$

$\Rightarrow \pm 4(a/6)^{3/2} - 12 \cdot \pm(a/6)^{3/2} + b = 0$

$\Rightarrow \pm 8(a/6)^{3/2} = b \Rightarrow 64a^3 = 6^3 b^2 \Rightarrow 8a^3 = 27b^2$: bifurcation set

The bifurcation set itself has a bifurcation; this can be analyzed as $f(a, b) = 0$, or observe that $b = \pm\sqrt{8/27}a^{3/2}$, so the bifurcation set has a cusp at $(a, b) = (0, 0)$.

Now consider a plane (a_0, b, x) with $a_0 > 0$; the intersection of the surface with this plane is a curve $x = X(a_0, b)$ with turning points at $b = \pm b_0 = \pm\sqrt{8/27}a_0^{3/2}$; this follows from the relation $b = -4x^3 + 2a_0 x$ sketched below.



$$b(x) = b_0 + b_0'(x - X_0) + \tfrac{1}{2}b_0''(x - X_0)^2 + \cdots$$
$$= b_0 - 12X_0(x - X_0)^2 + \cdots$$
$$x(b) = X_0 \pm \sqrt{-(1/12X_0)(b - b_0)} + \cdots$$

As a_0 varies there are two curves of turning points on the surface, each of which corresponds to a <u>fold</u> and which project onto the cusp in the (a, b)-plane.

The cusp separates regions of the (a, b)-plane where $X(a, b)$ has 1, 2, or 3 values; $X(a, b)$ has 2 values on the cusp.

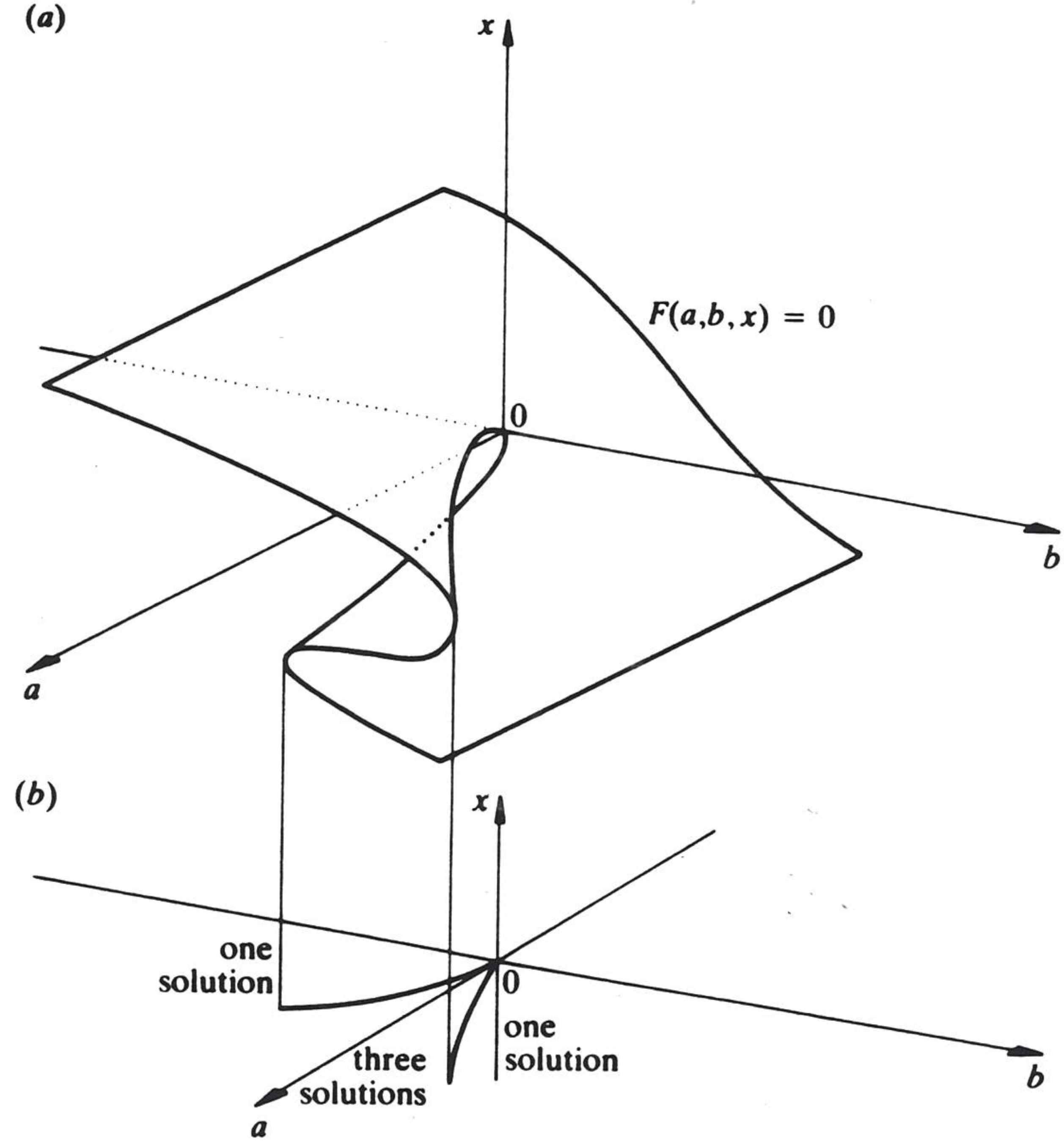


Fig. 2.3 The cusp catastrophe. (*a*) The folded surface $F(a, b, x) = 0$ in the (a, b, x)-space and (*b*) the projection in the (a, b)-plane of its points with 'vertical' tangents, and with a cusp at $(0, 0)$.

11
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2/13

2.3 structural stability

How is a bifurcation affected by an imperfection in the system?

$F(a,x,\delta)=0$, $\delta=0$: perfect system , $\delta\neq 0$: imperfect system

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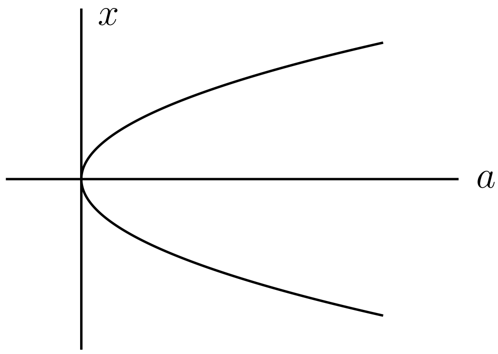
example 1 : $F(a,x,0)=x^2-a=0$

$(a_0,X_0)=(0,0)\Rightarrow F_0=0, F_{x0}=0, F_{a0}=-1, F_{xx0}=2$: regular turning point

$x=\pm\sqrt{a}$, $a=x^2$



$$F(a,x,\delta)=F(a,x,0)+F_\delta(a,x,0)\delta+O(\delta^2)$$
$$=x^2-a+F_{\delta 0}\delta+F_{x\delta 0}x\delta+F_{a\delta 0}a\delta+\tfrac{1}{2}F_{xx\delta 0}x^2\delta+O(\delta^2,a^2\delta,ax\delta,x^3\delta)$$
$$=x^2(1+\tfrac{1}{2}F_{xx\delta 0}\delta)+F_{x\delta 0}x\delta-a(1-F_{a\delta 0}\delta)+F_{\delta 0}\delta+O(\delta^2,\ldots)$$
$$=(1+\tfrac{1}{2}F_{xx\delta 0}\delta)(x+\tfrac{1}{2}F_{x\delta 0}\delta)^2-(1-F_{a\delta 0}\delta)(a-F_{\delta 0}\delta)+O(\delta^2,\ldots)$$

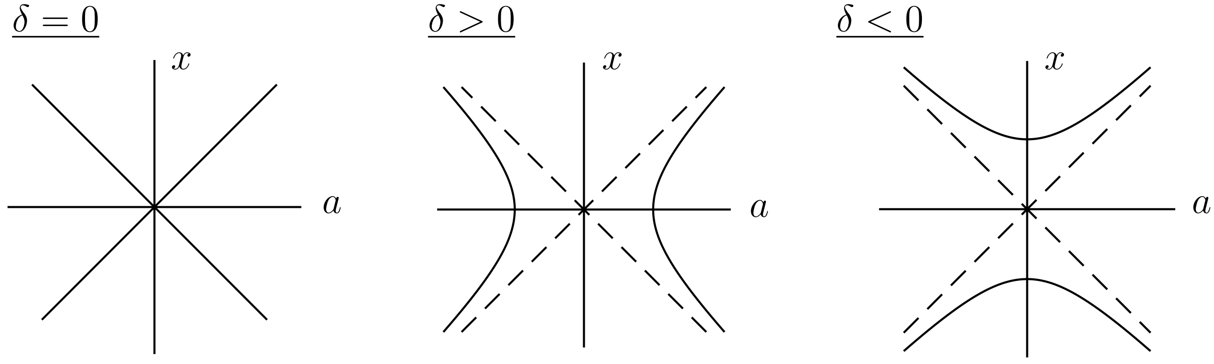
$\Rightarrow$ A turning point is structurally stable.

---

example 2 : $F(a,x,\delta)=x^2-a^2+\delta=0$

$\delta=0\ \Rightarrow x^2-a^2=0\ \Rightarrow x=\pm a$: transcritical bifurcation at $(0,0)$

$\delta\neq 0\ \Rightarrow x=\pm\sqrt{a^2-\delta}$, $a=\pm\sqrt{x^2+\delta}$

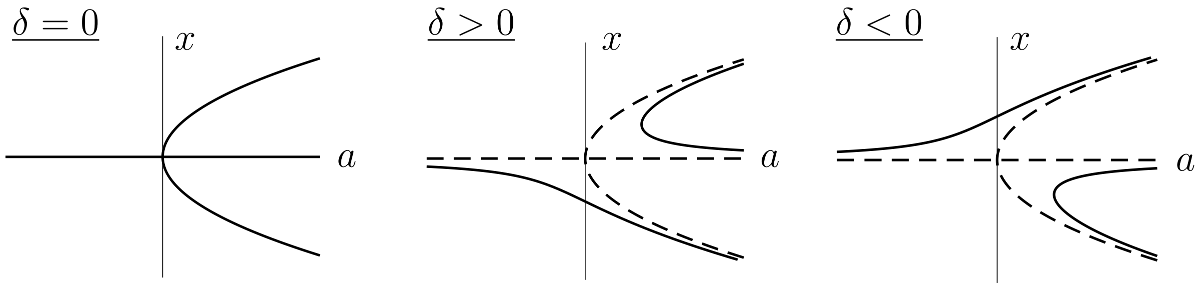


A transcritical bifurcation is not structurally stable.

<u>example 3</u> : $F(a,x,\delta) = x(x^2-a)+\delta$

$\delta = 0 \ \Rightarrow x(x^2-a)=0 \ \Rightarrow x=0\,,\ x=\pm\sqrt{a}$

$\delta \neq 0 \ \Rightarrow a = x^2 + \delta/x$



A pitchfork bifurcation is <u>not</u> structurally stable.

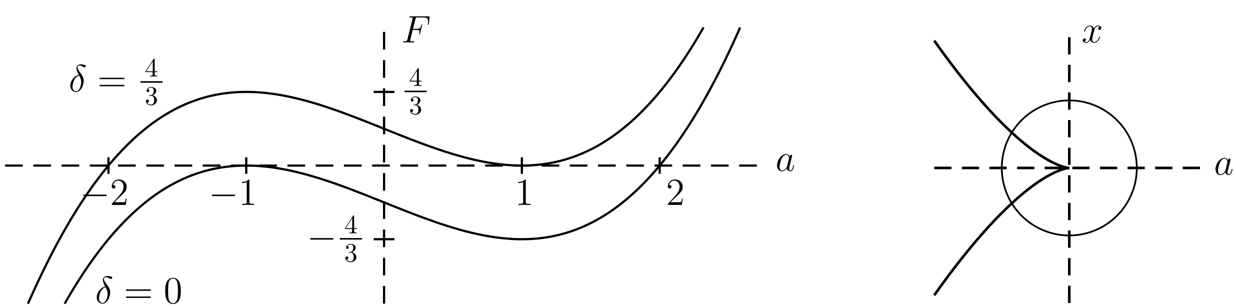
<u>example 4</u> : $F(a,x,\delta) = x^2 + a^2 - \delta = 0$

$\delta = 0$: conjugate point , $\delta > 0$: circle of radius $\sqrt{\delta}$, $\delta < 0$: no solution

A conjugate point is <u>not</u> structurally stable.

<u>example</u> : $F(a,x,\delta) = \frac{1}{2}x^2 + \frac{1}{3}a^3 - a - \frac{2}{3} + \delta = 0$

$f(a) = \frac{1}{3}a^3 - a - \frac{2}{3}, f'(a) = a^2 - 1 = 0 \Rightarrow a = \pm 1, f(-1) = 0\,,\ f(1) = -\frac{2}{3}, \ldots$



<u>$\delta = 0$</u> : $F(a,x,0) = \frac{1}{2}x^2 + \frac{1}{3}(a+1)^2(a-2) \ \Rightarrow X_0 = 0\,,\ a_0 = -1, 2$

$(-1,0) \Rightarrow F \sim \frac{1}{2}x^2 - (a+1)^2$: transcritical

$(2,0) \Rightarrow F \sim \frac{1}{2}x^2 + 3(a-2)$: turning point , $a < 2$

<u>$\delta = \frac{4}{3}$</u> : $F(a,x,\frac{4}{3}) = \frac{1}{2}x^2 + \frac{1}{3}(a-1)^2(a+2) \ \Rightarrow X_0 = 0\,,\ a_0 = 1, -2$

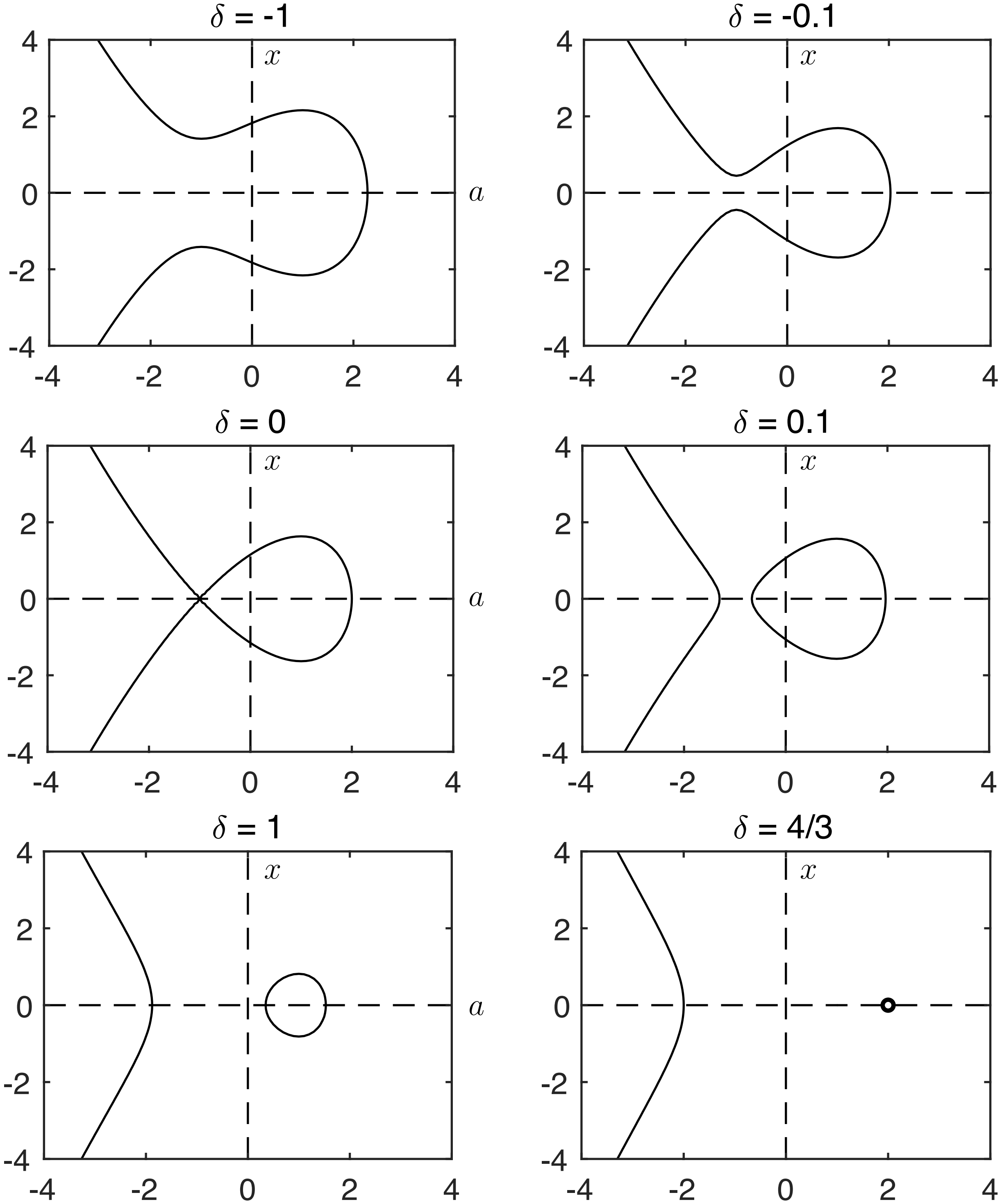
$(1,0) \Rightarrow F \sim \frac{1}{2}x^2 + (a-1)^2$: conjugate point

$(-2,0) \Rightarrow F \sim \frac{1}{2}x^2 + 3(a+2)$: turning point , $a < -2$

$x \to \pm\infty \ \Rightarrow a \to -\infty\,,\ F \sim \frac{1}{2}x^2 + \frac{1}{3}a^3$: cusp , $a < 0$

<u>example</u> : $F(a,x,\delta) = \frac{1}{2}x^2 + \frac{1}{3}a^3 - a - \frac{2}{3} + \delta = 0$

Each frame shows the equilibrium set in the (a,x)-plane.



$\delta = 0$: The transcritical bifurcation at $(-1,0)$ is structurally unstable and the turning point at $(2,0)$ is structurally stable.

$\delta = \frac{4}{3}$: The turning point at $(-2,0)$ is structurally stable and the conjugate point at $(1,0)$ is structurally unstable.

12 Wed 2/15

2.4 bifurcations in higher dimensions

$\vec{F}(\vec{a},\vec{x}) = 0\,,\ F : \mathbb{R}^\ell \times \mathbb{R}^m \to \mathbb{R}^m\,,\ m \geq 2 :$ vector field

consider a point in the equilibrium set : $\vec{F}(\vec{a}_0, \vec{X}_0) = \vec{0}$

$$F_i = \cancel{F_{i0}} + \sum_{j=1}^{\ell} \frac{\partial F_i}{\partial a_j}\bigg|_0 \cdot (a_j - a_{j0}) + \sum_{k=1}^{m} \frac{\partial F_i}{\partial x_k}\bigg|_0 \cdot (x_k - X_{k0}) + \cdots = 0\,,\ i = 1 : m$$

$A_0(\vec{a} - \vec{a}_0) + J_0(\vec{x} - \vec{X}_0) = \vec{0} \Rightarrow \vec{X}(\vec{a}) = \vec{X}_0 - J_0^{-1}A_0(\vec{a} - \vec{a}_0) + \cdots$

Hence bifurcation can occur only if J_0 is singular; recall for $\ell = m = 1$, $J_0 = F_{x0}$.

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example : $\vec{F}(a,\vec{x}) = \begin{pmatrix} \sin y - \tan x \\ ax - y - x^2 y \end{pmatrix}\,,\ \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$: $\ell = 1, m = 2$ a one-parameter family of vector fields on $\mathbb{R}^2$

$\vec{F}(a,\vec{0}) = \vec{0} \Rightarrow \vec{X}_0 = \vec{0}$ is a solution for all a

$$J = \begin{pmatrix} -\sec^2 x & \cos y \\ a - 2xy & -1 - x^2 \end{pmatrix}\bigg|_0 = \begin{pmatrix} -1 & 1 \\ a & -1 \end{pmatrix}\,,\ \det J = 1 - a\,,\ \text{bifurcation at } a_0 = 1$$

$$J_0 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, A_0 = \begin{pmatrix} 0 \\ x \end{pmatrix}\bigg|_0 = \vec{0}\,,\ \cancel{A_0}(a - a_0) + J_0(\vec{x} - \vec{X}_0) = \vec{0} \Rightarrow J_0\vec{x} = \vec{0}$$

Hence we expect that the solution branch bifurcating from $(1,\vec{0})$ behaves like

$\vec{X}(a) \sim \epsilon(a)\vec{v}$, where $\epsilon(a) \to 0$ as $a \to 1$ and $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \text{null}(J_0)$.

Set $\epsilon(a) = \vec{v} \cdot \vec{X}(a) = X(a) + Y(a)$ and look for a perturbation series solution.

$X(a) = \epsilon X_1 + \epsilon^2 X_2 + \cdots\ ,\ Y(a) = \epsilon Y_1 + \epsilon Y_2 + \cdots\ ,\ a = 1 + \epsilon a_1 + \epsilon^2 a_2 + \cdots$

$$X(a) + Y(a) = \epsilon\ ,\ \begin{pmatrix} \sin Y(a) - \tan X(a) \\ aX(a) - Y(a) - X^2(a)Y(a) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

ϵ^0 : ok , $\epsilon^1 : X_1 + Y_1 = 1\ ,\ Y_1 - X_1 = 0\ ,\ X_1 - Y_1 = 0 \Rightarrow X_1 = Y_1 = \frac{1}{2}$

$\epsilon^2 : X_2 + Y_2 = 0\ ,\ Y_2 - X_2 = 0\ ,\ X_2 + a_1 X_1 - Y_2 = 0 \Rightarrow X_2 = Y_2 = 0\,,\ a_1 = 0$

$\epsilon^3 : X_3 + Y_3 = 0\ ,\ Y_3 - \frac{1}{6}Y_1^3 - (X_3 + \frac{1}{6}X_1^3) = 0\ ,\ X_3 + \cancel{a_1 X_2} + a_2 X_1 - Y_3 - X_1^2 Y_1 = 0$

$\Rightarrow Y_3 - X_3 = \frac{1}{6}(Y_1^3 + X_1^3) = \frac{1}{6}(\frac{1}{8} + \frac{1}{8}) = \frac{1}{24} \Rightarrow X_3 = -\frac{1}{48}\,,\ Y_3 = \frac{1}{48}\,,$

$-\frac{1}{48} + \frac{1}{2}a_2 - \frac{1}{48} - \frac{1}{8} = 0 \Rightarrow a_2 = \frac{1}{3}$

$X(a) = \frac{1}{2}\epsilon - \frac{1}{48}\epsilon^3 + \cdots\ ,\ Y(a) = \frac{1}{2}\epsilon + \frac{1}{48}\epsilon^3 + \cdots\ ,\ a = 1 + \frac{1}{3}\epsilon^2 + \cdots$

$$\epsilon = \pm\sqrt{3(a-1)} + \cdots \Rightarrow \vec{X}(a) = \pm\frac{1}{2}\sqrt{3(a-1)}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \cdots \text{ as } a \to 1$$

Hence $(a, \vec{X}(a))$ has a supercritical pitchfork bifurcation at $(1,\vec{0})$, and this shows why a perturbation series in powers of $a - 1$ is bound to fail.

singularity theory, catastrophe theory

theorem (René Thom)

Let $\vec{F}(\vec{a},\vec{x}) = \vec{0}$, where $\vec{F} : \mathbb{R}^\ell \times \mathbb{R}^m \to \mathbb{R}^m, \vec{F} = -\nabla V$. There are 7 distinct structurally stable bifurcations of co-dimension $\ell \leq 4$.

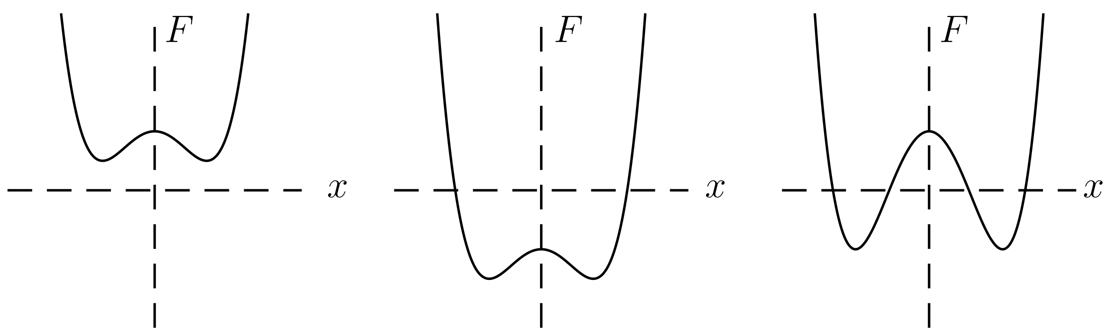
ℓ	m	V	name
1	1	$\frac{1}{3}x^3 + ax$	turning point
2	1	$\frac{1}{4}x^4 + \frac{1}{2}ax^2 + bx$	cusp
3	1	$\frac{1}{5}x^5 + \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + cx$	swallow tail
3	2	$x^3 + y^3 + cxy - ax - by$	hyperbolic umbilic
3	2	$x^3 - 3xy^2 + c(x^2+y^2) - ax - by$	elliptic umbilic
4	1	$\frac{1}{6}x^6 + \frac{1}{4}ax^4 + \frac{1}{3}bx^3 + \frac{1}{2}cx^2 + dx$	butterfly
4	2	$x^2y + \frac{1}{4}y^4 + cx^2 + dy^2 - ax - by$	parabolic umbilic

swallow tail : $F(a,b,c,x) = x^4 + ax^2 + bx + c = 0$ (Q2.13)

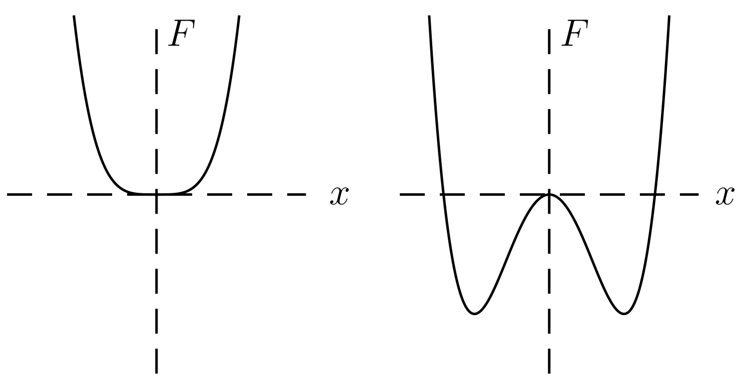
The equilibrium set is a 3D volume in (a,b,c,x)-space and the bifurcation set is a 2D surface in (a,b,c)-space.

$S(n) = \{(a,b,c) : F(a,b,c,x) = 0 \text{ has } n \text{ solutions}\} : n = 0,1,2,3,4$

generic cases : $S(0), S(2), S(4)$ are open sets in (a,b,c)-space



special cases : $S(1), S(3)$ are surfaces in (a,b,c)-space

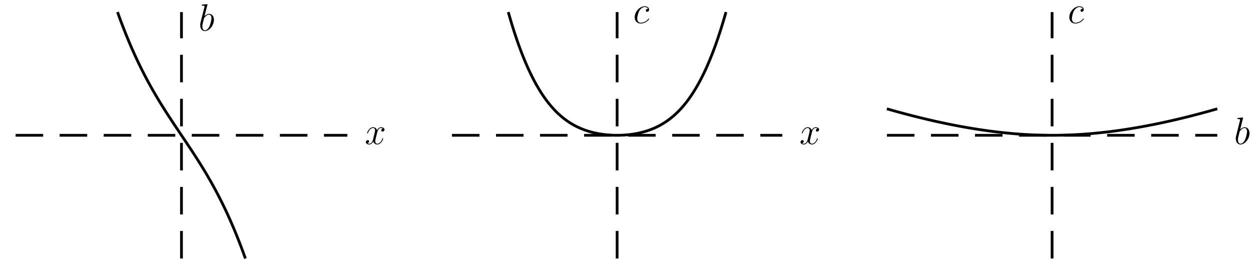


bifurcations occur when $F_x(a,b,c,x) = 4x^3 + 2ax + b = 0 \Rightarrow b = -4x^3 - 2ax$

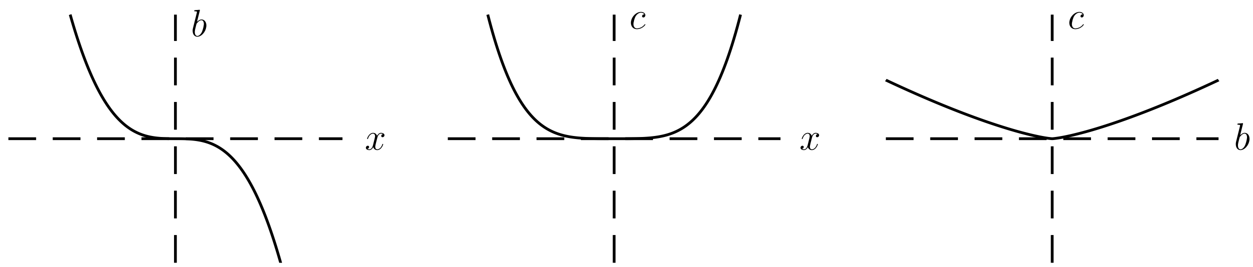
$F(a,b,c,x) = 0 \Rightarrow c = -x^4 - ax^2 - bx = -x^4 - ax^2 - x(-4x^3 - 2ax) = 3x^4 + ax^2$

$\Rightarrow$ the bifurcation set is a parametrized surface in (a,b,c)-space $\begin{cases} b = -4x^3 - 2ax \\ c = 3x^4 + ax^2 \end{cases}$

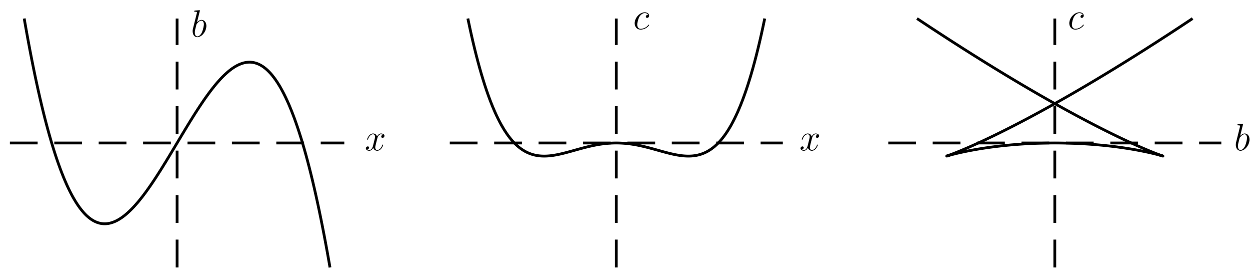
$\underline{a > 0}$: $b \sim -x$, $c \sim x^2$ as $x \to 0$ $\Rightarrow c \sim b^2$ as $b \to 0$



$\underline{a = 0}$: $b = -4x^3$, $c = 3x^4$ $\Rightarrow c \sim |b|^{4/3}$ as $b \to 0$



$\underline{a < 0}$

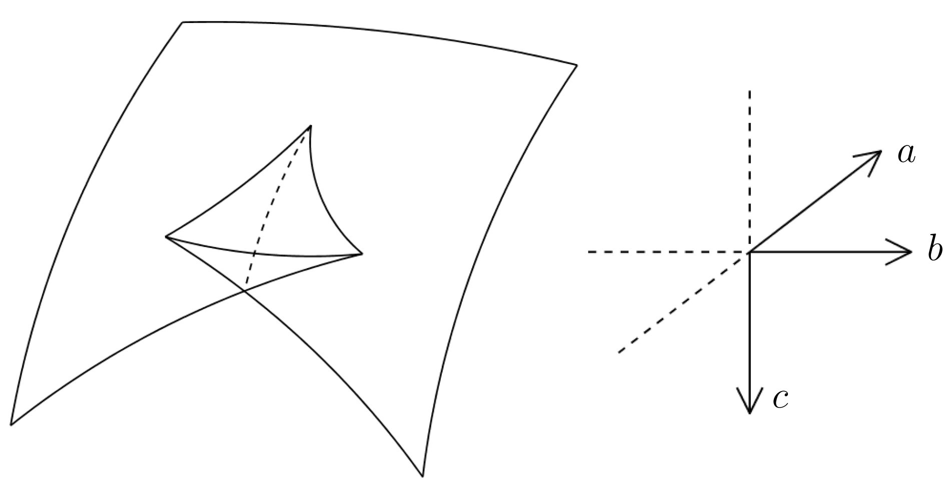


The curve $(c(x), b(x))$ in the (b, c)-plane has 2 cusps and a self-intersection point.

$b_x = -12x^2 - 2a = -2(6x^2 + a)$, $c_x = 12x^3 + 2ax = 2x(6x^2 + a)$

$\Rightarrow b_x = c_x = 0$ for $6x_0^2 + a = 0$ or $x_0 = \pm\sqrt{-a/6}$

as $x \to x_0^\pm$, $c_b = c_x/b_x \to 0/0 = -x_0$, $c_{bb} = (c_x x - c_b \cdot b_{xx})/b_x^2 \to 0/0 = \pm\infty$



The swallow tail surface has a <u>cusp ridge</u> and a <u>self-intersection curve</u>.

13 Mon 2/20

3. iterated maps : $x_{n+1} = F(x_n)$, $F : \mathbb{R}^m \to \mathbb{R}^m$

3.1 stability of fixed points

definition : A fixed point $X = F(X)$ is (Lyapunov) stable if for any $\epsilon > 0$, there exists $\delta > 0$, such that $|x_0 - X| < \delta \Rightarrow |x_n - X| < \epsilon$ for all $n \geq 1$, i.e. any orbit x_n starting sufficiently close to X remains arbitrarily close to X for all $n \geq 1$.

A fixed point X is asymptotically stable if X is stable and $x_n \to X$ as $n \to \infty$ for all x_0 in a neighborhood of X.

note : stability $\not\Leftarrow$ asymptotic stability

example : $x_{n+1} = -x_n$, $X = 0$ is stable, but not asymptotically stable

linear stability : $x'_{n+1} = J(X)x'_n$, $J(X) = \left(\left.\frac{\partial F_i}{\partial x_j}\right|_X \right) \Rightarrow x'_n = J(X)^n x'_0$

Assume $J(X)$ has m linearly independent eigenvectors u_j with eigenvalues λ_j.

$$x'_0 = \sum_{j=1}^m c_j u_j \Rightarrow x'_n = \sum_{j=1}^m c_j \lambda_j^n u_j : X \text{ is } \begin{cases} \text{linearly stable if } |\lambda_j| < 1 \text{ for all } j \\ \text{linearly unstable if } |\lambda_j| > 1 \text{ for some } j \end{cases}$$

Hartman-Grobman theorem

linear stability + hyperbolicity (the eigenvalues of $J(X)$ have nonzero real part) $\Rightarrow$ stability, asymptotic stability

In general $X = X(a)$; then if $X(a)$ is linearly stable for $a < a_0$ and $|\lambda_j(a)| \to 1$ as $a \to a_0$ for some j, we expect that a_0 is a bifurcation point and $X(a)$ is unstable for $a > a_0$.

---

example : $x_{n+1} = ax_n(1 - x_n)$, $0 \leq a \leq 4$

$X = 0$: fixed point for all $a > 0$, $F(x) = ax(1-x) \Rightarrow J(x) = a(1-2x) = \lambda$

$X = 0$: $\lambda = a \Rightarrow X$ is stable for $0 < a < 1$

$a = 1 \Rightarrow \lambda = 1 \Rightarrow x'_n = x'_0$

$a = 1$ is a transcritical bifurcation point $\begin{cases} X \text{ is unstable for } a > 1 \\ \text{new fixed point appears for } a > 1 \end{cases}$

$X = (a-1)/a$: $\lambda = 2 - a \Rightarrow X$ is stable for $1 < a < 3$

$a = 3 \Rightarrow \lambda = -1 \Rightarrow x'_n = (-1)^n x'_0$

$a = 3$ is a flip bifurcation point $\begin{cases} X \text{ is unstable for } a > 3 \\ \text{2-cycle appears for } a > 3 \end{cases}$

summary

$\lambda = 1$: a new fixed point appears (transcritical, pitchfork, ...)

$\lambda = -1$: a new 2-cycle appears (flip)

3.2 stability of periodic orbits

definition : If $X_1, \dots, X_p$ are distinct points such that $F(X_1) = X_2, F(X_2) = X_3, \dots, F(X_p) = X_1$, then $S = \{X_1, \dots, X_p\}$ is a periodic orbit of F of period p or p-cycle and $F(S) = S$, so S is an invariant set.

Define $S_\epsilon = \{x : |x - X_j| < \epsilon \text{ for some } j = 1 : p\}$. Then S is stable if for any $\epsilon > 0$, there exists $\delta > 0$ such that $x_0 \in S_\delta \Rightarrow x_n \in S_\epsilon$ for all $n \geq 1$.

$$G(x) = \underbrace{F \circ F \circ \cdots \circ F}_{p \text{ times}}(x) : p\text{-th generation map}$$

$X \in p$-cycle of $F \Rightarrow X$ is a fixed point of G (but $\Leftarrow$ does not hold, for example a fixed point of G may be a fixed point of F).

stability of a p-cycle of $F \Leftrightarrow$ stability as a fixed point of G

example : $p = 2$, 2-cycle

$S = \{X, Y\}$, $X \neq Y$, $F(X) = Y$, $F(Y) = X$

$G(x) = F(F(x))$: X, Y are fixed points of G

$$x'_{n+1} = K(X)x'_n \ , \ K(X) = \left(\frac{\partial G_i}{\partial x_j}\bigg|_X \right) \ , \ \frac{\partial G_i}{\partial x_j}\bigg|_X = \sum_{k=1}^{m} \frac{\partial F_i}{\partial x_k}\bigg|_{F(X)} \frac{\partial F_k}{\partial x_j}\bigg|_X$$

$\Rightarrow K(X) = J(Y)J(X)$, $K(Y) = J(X)J(Y)$

In general $K(X) \neq K(Y)$, but $K(X) = J(Y)K(Y)J(Y)^{-1}$, so $K(X)$ and $K(Y)$ have the same eigenvalues and the stability of the 2-cycle is well-defined.

---

example : $F(x) = ax(1 - x)$

2-cycle : $\{X, Y\} = \left\{ \dfrac{a + 1 \pm \sqrt{(a+1)(a-3)}}{2a} \right\}$ for $a > 3$, is it stable?

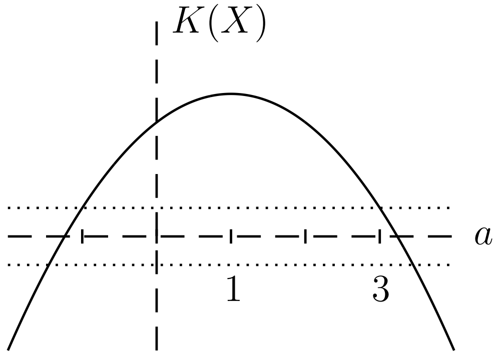
$J(x) = a(1 - 2x)$

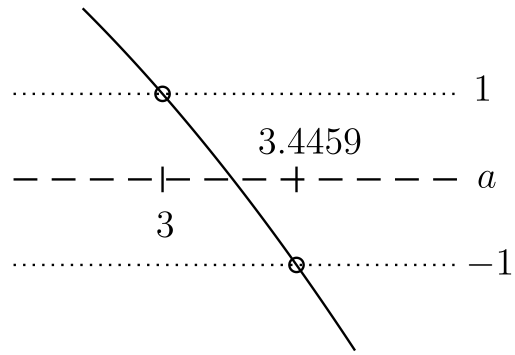
$$K(X) = J(Y)J(X) = a(1 - 2Y)a(1 - 2X) = a^2(1 - 2(X + Y) + 4XY)$$

$$= a^2\left(1 - 2\frac{(a+1)}{a} + 4\left(\frac{(a+1)^2 - (a+1)(a-3)}{4a^2}\right)\right)$$

$$= a^2 - 2a(a+1) + (a+1)(\cancel{a} + 1 - \cancel{a} + 3) = a^2 - 2a^2 - 2a + 4a + 4$$

$$K(X) = -a^2 + 2a + 4 = 5 - (a-1)^2$$





$$a = 3 \Rightarrow K(X) = 1 \begin{cases} G \text{ has a pitchfork bifurcation, 2 new fixed points appear} \\ F \text{ has a flip bifurcation, a 2-cycle appears} \end{cases}$$

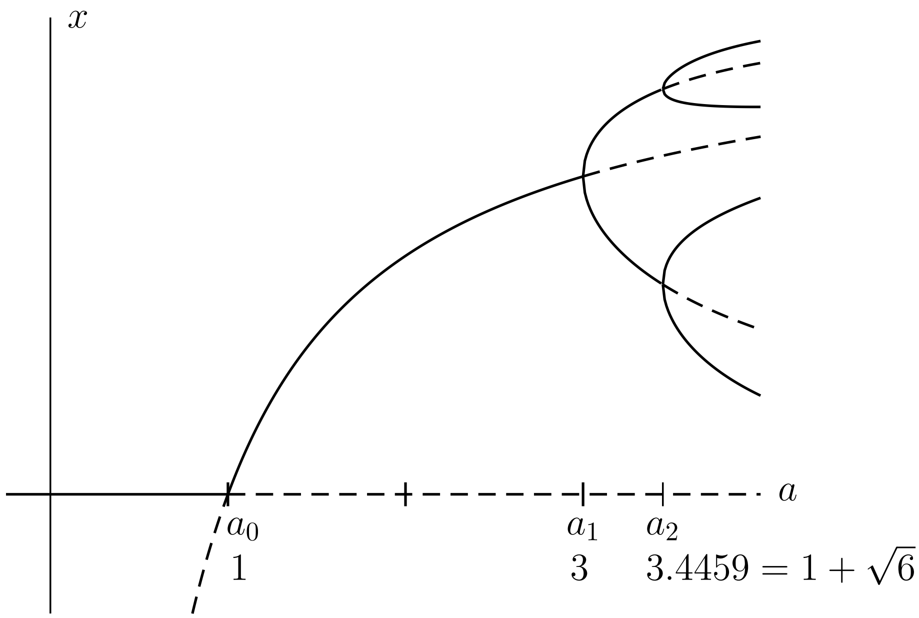
$K(X) = -1 \Leftrightarrow 5 - (a-1)^2 = -1 \Leftrightarrow a = 1 + \sqrt{6} \approx 3.4459$

G has a flip bifurcation at $a = 1 + \sqrt{6}$

the 2-cycle of F is $\begin{cases} \text{stable for } 3 < a < 1 + \sqrt{6} \\ \text{unstable for } a > 1 + \sqrt{6} \end{cases}$

F has a 4-cycle for $a > 1 + \sqrt{6}$ $\begin{cases} \text{stable for } 1 + \sqrt{6} < a < a_3 \\ \text{unstable for } a > a_3 \end{cases}$

<u>bifurcation diagram</u> (partial, schematic)



The bifurcation diagrams of F and G look the same, but they have different meanings; for example a 2-cycle of F corresponds to 2 fixed points of G.

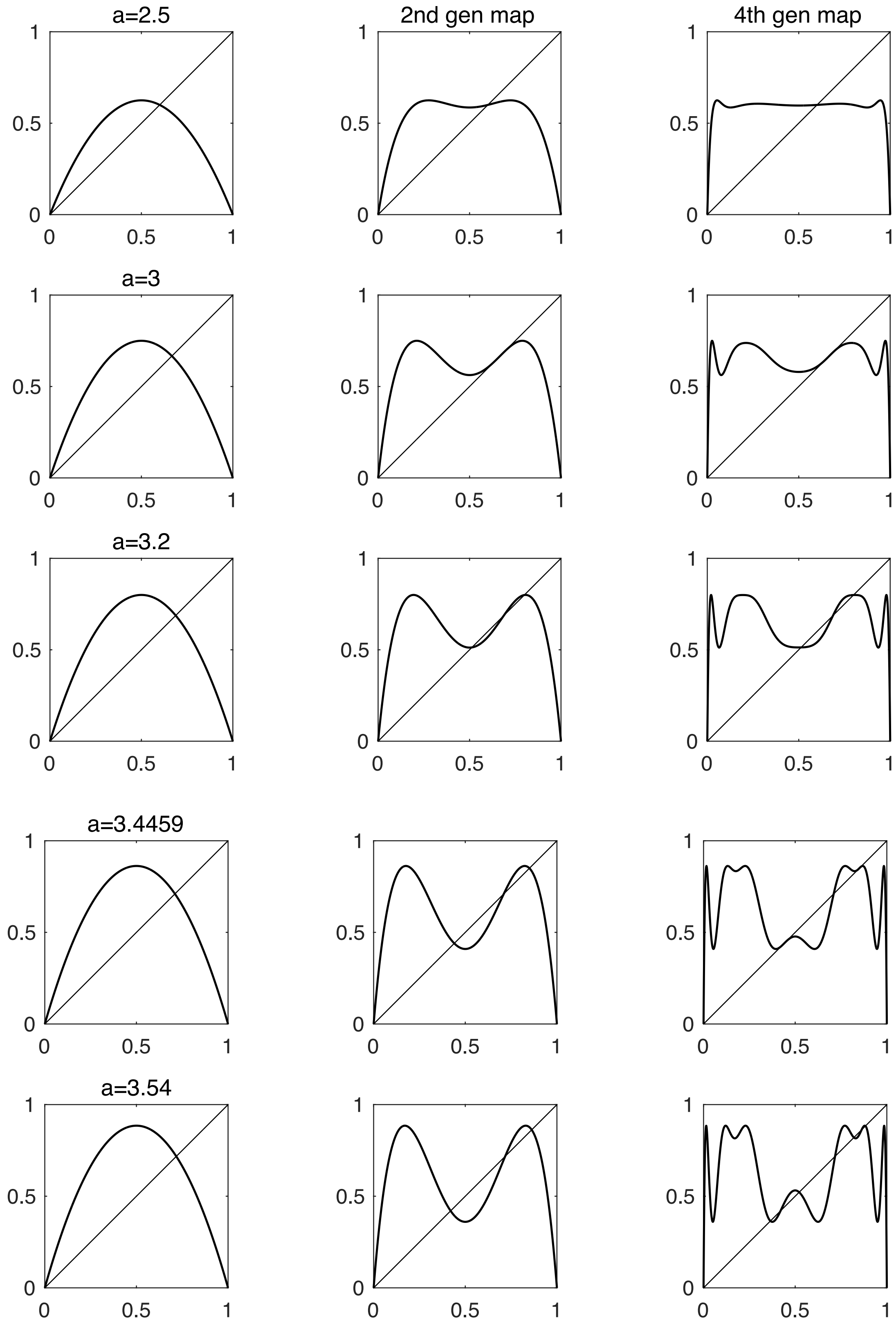
<u>preview</u> (Feigenbaum, 1970s) 14 Wed 2/22

$0 < a_1 < a_2 < \cdots$: infinite sequence of flip bifurcations , period-doubling

$a_r \sim a_\infty - A\delta^{-r}$ as $r \to \infty$

$a_\infty = 3.57$ for the logistic map , $\delta = 4.6692$: universal constant , $a > a_\infty$: later

<u>period-doubling</u> : creation of a 2^{r+1}-cycle from a 2^r-cycle by a flip bifurcation



2 fixed points

flip bifurcation

2 fixed points
2-cycle

flip bifurcation

2 fixed points
2-cycle
4-cycle

F has degree 2, 2nd gen map has degree 4, 4th gen map has degree 16
A flip bifurcation occurs when $\lambda = -1$ at a fixed point in the 2^r generation map.
This occurs in a self-similar manner.

definition

Given a map $F(x)$ and an initial point x_0, exactly one of the following holds.

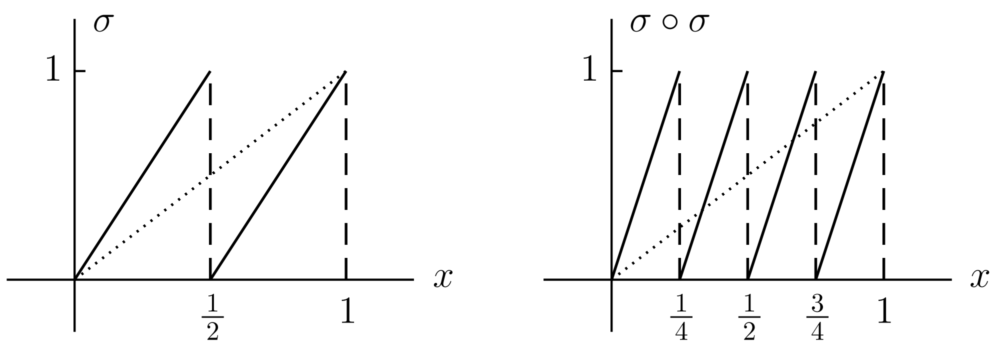
1. x_0 is a fixed point
2. x_n is a fixed point for some $n \geq 1$: eventually fixed
3. $x_n \to$ fixed point as $n \to \infty$: asymptotically fixed
4. $x_0 \in p$-cycle : periodic
5. $x_n \in p$-cycle for some $n \geq 1$: eventually periodic
6. $x_n \to p$-cycle as $n \to \infty$: asymptotically periodic
7. none of the above : $\{x_n\}$ is aperiodic or chaotic

---

example : sawtooth map

$x_{n+1} = \sigma(x_n)\,,\ \sigma : [0,1) \to [0,1)$

$$\sigma(x) = 2x \bmod 1 = \text{fractional part of } 2x = \begin{cases} 2x & \text{if } 0 \leq x < 1/2 \\ 2x - 1 & \text{if } 1/2 \leq x < 1 \end{cases}$$



If $2\pi x_n$ is an angle, then σ is the doubling map on a circle.

$x_n = [b_1, b_2, \ldots]_2 = b_1/2 + b_2/2^2 + \cdots$: binary notation

$$\begin{aligned} x_{n+1} &= 2b_1/2 + 2b_2/2^2 + \cdots \bmod 1 \\ &= b_2/2 + b_3/2^2 + \cdots = [b_2, b_3, \ldots]_2 : \text{Bernoulli shift} \end{aligned}$$

$X = 0$: unique fixed point , $\sigma'(0) = 2$: unstable , are there any p-cycles?

$\sigma \circ \sigma(x) = 4x \bmod 1$

	$\sigma(x)$		$\sigma(\sigma(x))$
$0 \leq x < 1/4$	$2x$	$0 \leq \sigma(x) < 1/2$	$2\sigma(x) = 4x$
$1/4 \leq x < 1/2$	$2x$	$1/2 \leq \sigma(x) < 1$	$2\sigma(x) - 1 = 4x - 1$
$1/2 \leq x < 3/4$	$2x - 1$	$0 \leq \sigma(x) < 1/2$	$2\sigma(x) = 4x - 2$
$3/4 \leq x < 1$	$2x - 1$	$1/2 \leq \sigma(x) < 1$	$2\sigma(x) - 1 = 4x - 3$

$\sigma \circ \sigma$ has 2 nonzero fixed points : $X = 1/3\,, 2/3$

$\Rightarrow \sigma$ has a 2-cycle : $\{1/3\,, 2/3\}$, $(\sigma \circ \sigma)'(1/3) = 4 \Rightarrow$ unstable

x_0	$\{x_n\}$	type
1/3	1/3 , 2/3 , 1/3 , 2/3 , ...	2-cycle
1/4	1/4 , 1/2 , 0 , 0 , ...	eventually fixed
1/5	1/5 , 2/5 , 4/5 , 3/5 , 1/5 , ...	4-cycle
1/6	1/6 , 1/3 , 2/3 , 1/3 , 2/3 , ...	eventually periodic
1/7	1/7 , 2/7 , 4/7 , 1/7 , ...	3-cycle

1. p-cycles exist for all $p \geq 1$, all unstable
2. x_0 : irrational $\Rightarrow \{x_n\}$ is chaotic , pf : binary expansion of x_0 is not repeating
3. $x_n = 2^n x_0 \bmod 1$, $y_n = 2^n y_0 \bmod 1 \Rightarrow x_n - y_n = 2^n(x_0 - y_0) \bmod 1$

$\Rightarrow$ sensitive dependence on initial data

3.3, 3.4 attractors

definition : A point x is a limit point of an orbit $\{x_n\}$ if there is a subsequence of $\{x_n\}$ that converges to x; the ω-limit set of $\{x_n\} = \{$all limit points of $\{x_n\}\}$.

example : $x_{n+1} = (-1)^n x_n^2$

$|x_0| < 1 \Rightarrow x_n \to 0 \Rightarrow$ the ω-limit set is $\{0\}$

$x_0 = 1 \Rightarrow x_n = (-1)^n \Rightarrow$ the ω-limit set is $\{\pm 1\}$

note : An ω-limit set is a closed invariant set. , pf : omit

definition : An attractor is an ω-limit set which is the same for all nearby orbits.

example above : $\{0\}$ is an attractor, $\{\pm 1\}$ is not an attractor.

example : sawtooth map

x_0 : rational $\Rightarrow \omega$-limit set is a p-cycle

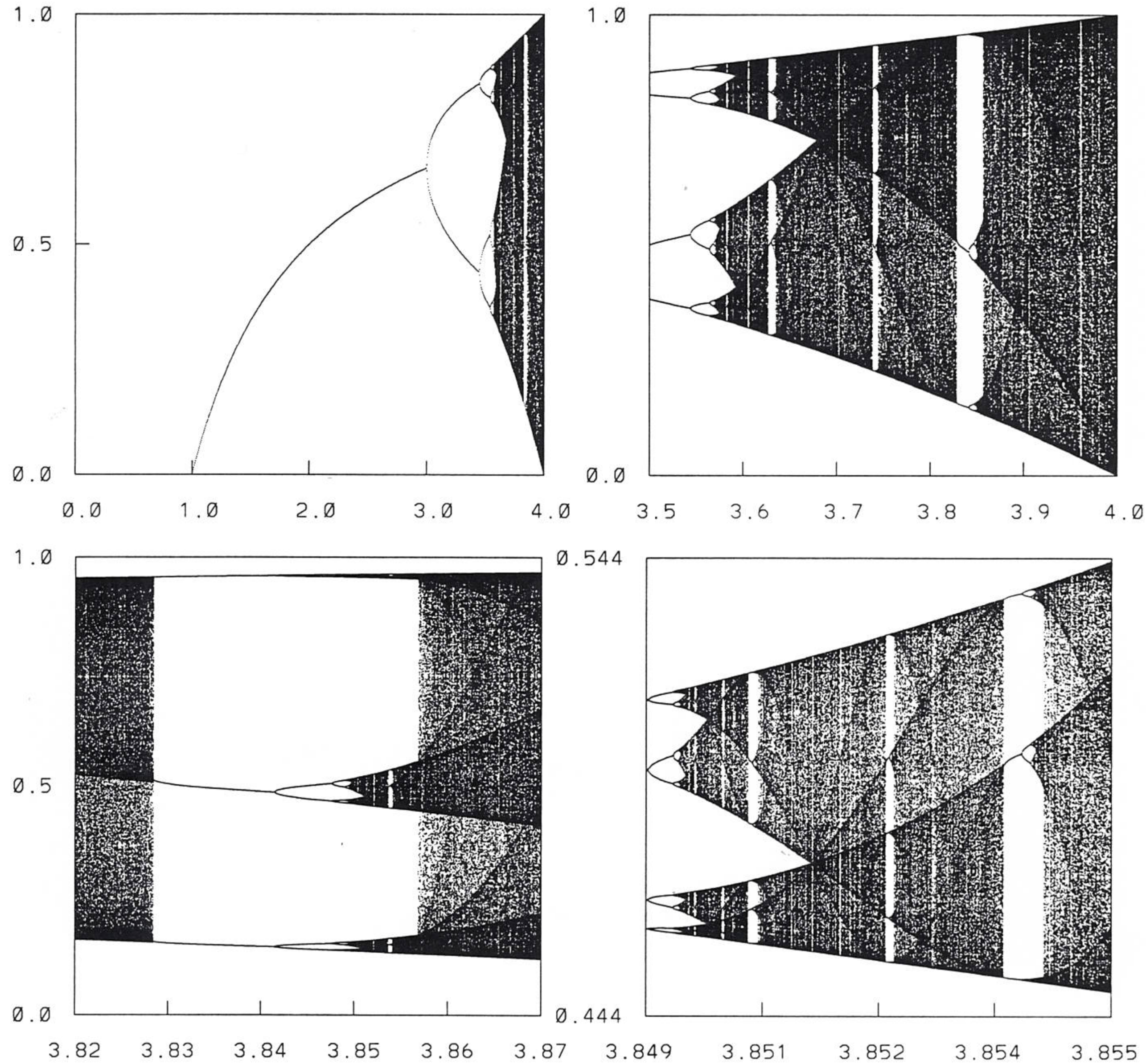
x_0 : irrational $\Rightarrow \omega$-limit set is $[0, 1)$, pf : ...

$\Rightarrow$ the sawtooth map has no attractor

example : logistic map

	$0 < a < 1$	$1 < a < 3$	$3 < a < a_2$	$a_2 < a < a_3$	$\cdots$
attractor	$\{0\}$	$\{(a-1)/a\}$	2-cycle	4-cycle	$\cdots$

<u>attractors of the logistic map</u>



1. For $1 < a < a_\infty = 3.57$ the attractors are 2^r-cycles. A 2^{r+1}-cycle forms due to a flip bifurcation in a 2^r-cycle (<u>period-doubling</u>). The bifurcation points form a Feigenbaum sequence $a_r \sim a_\infty - A\delta^r$ with $\delta = 4.6692$.

2. For $a > a_\infty$ the attractor is a Cantor set for most values of a. A <u>Cantor set</u> is uncountable, totally disconnected, closed, and every point is a limit point (more later). Such an attractor is called a <u>strange attractor</u> or <u>chaotic attractor</u>.

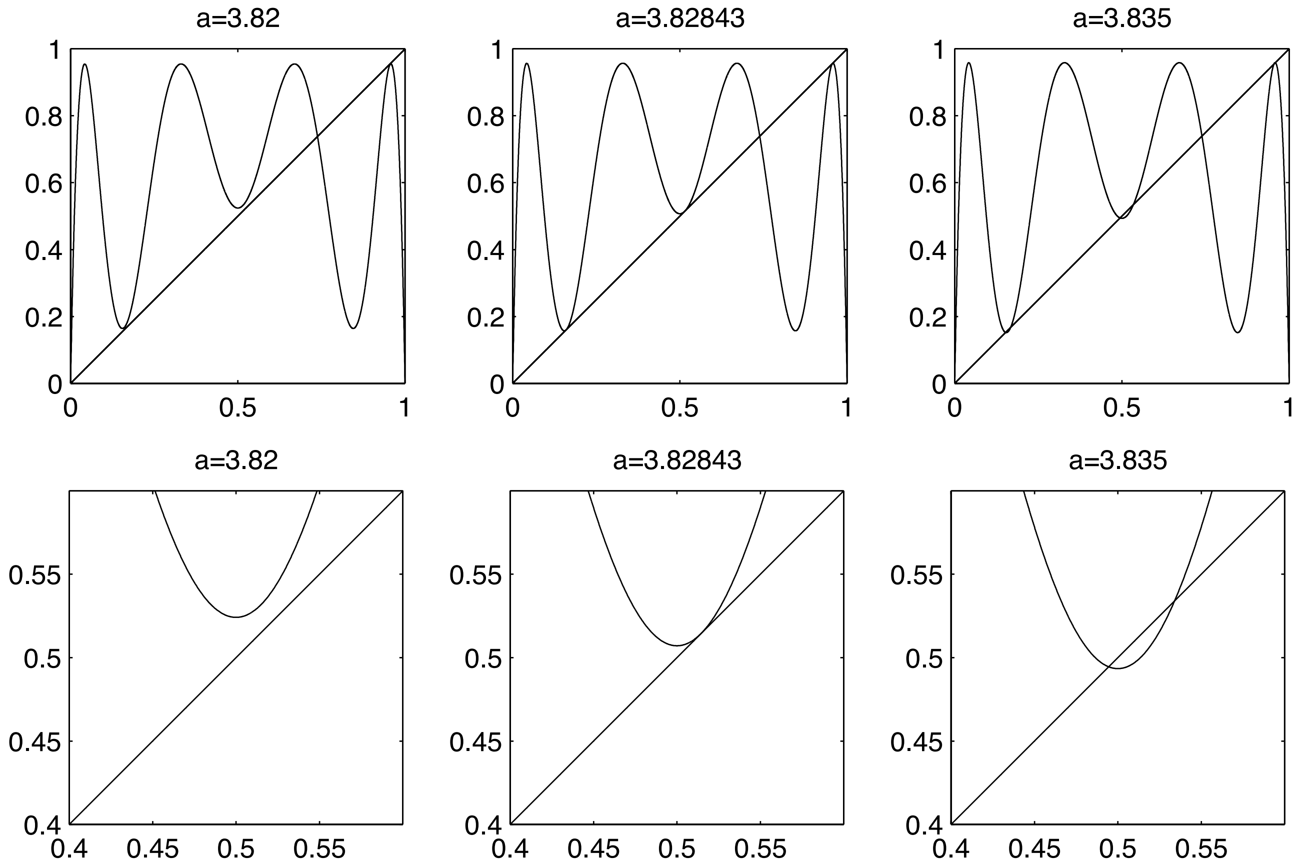
3. The 2^r-cycles that form for $a < a_\infty$ exist for $a > a_\infty$, but they are unstable.

4. For any $p \geq 2$ there is an interval where the attractor is a p-cycle; for example, a stable 3-cycle forms for $a = 1 + \sqrt{8} = 3.82843$, and for $3.82843 < a < 3.8495$ there is a period-doubling sequence in which stable $3 \cdot 2^r$-cycles form.

5. The set of attractors is self-similar.

<u>creation of a 3-cycle</u> for the logistic map

The 3rd generation map is shown below.

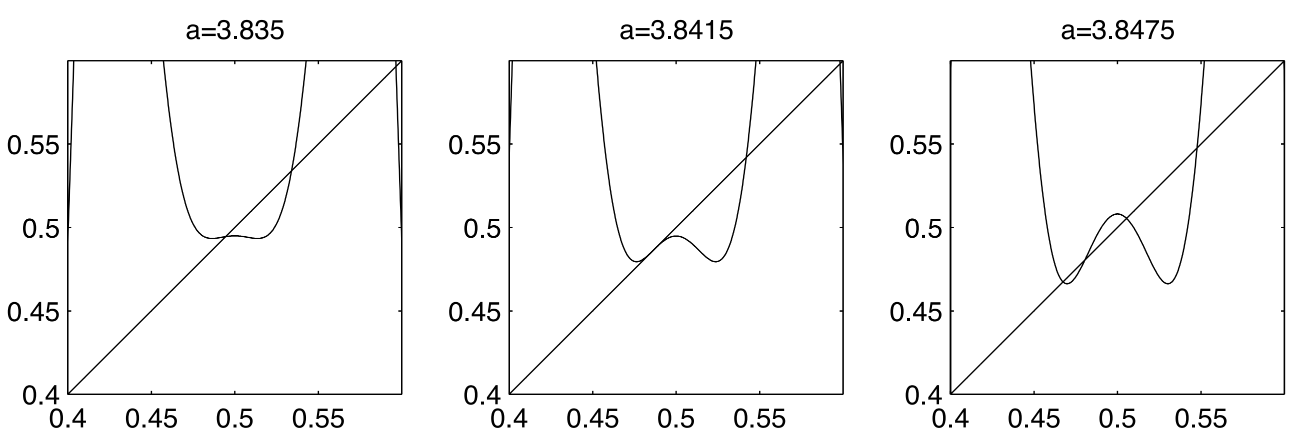


$a = 3.82$: three lobes approach $y = x$

$a = 3.82843$: a 3-cycle forms, tangent to $y = x$

$a = 3.835$: two 3-cycles, one stable and one unstable

The 6th generation map is shown below.



$a = 3.8415$: a 6-cycle forms by a flip bifurcation of the stable 3-cycle

15
Mon
3/6

definition

$F : \mathbb{R}^m \to \mathbb{R}^m$, $\mu_n = \mu(F^n(S))$: measure $\sim$ volume, area, length

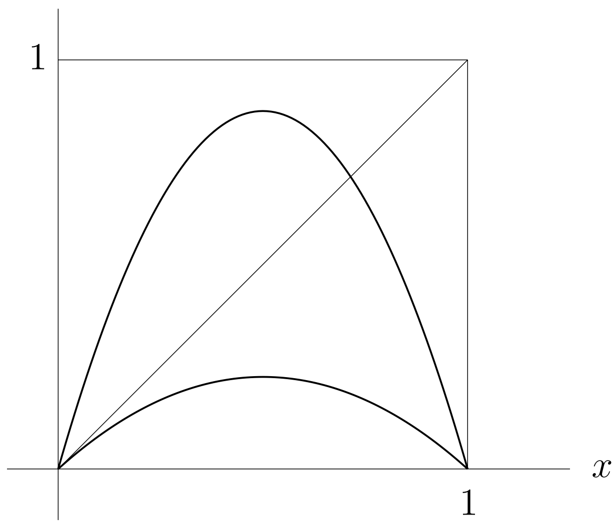
$|\det J| = 1 \Rightarrow \mu_n = \mu_0$: F is measure-preserving

$|\det J| > 1 \Rightarrow \mu_{n+1} > \mu_n$: F is expanding

$|\det J| < 1 \Rightarrow \mu_{n+1} < \mu_n$: F is contracting

An attractor can exist only if F is contracting.

example : $F(x) = ax(1-x)$



$a < 1 \Rightarrow F$ is contracting for $0 \le x \le 1$

$a > 1 \Rightarrow F$ is $\begin{cases} \text{contracting for } x \sim 1/2 \\ \text{expanding for } x \sim 0\,,1 \end{cases}$

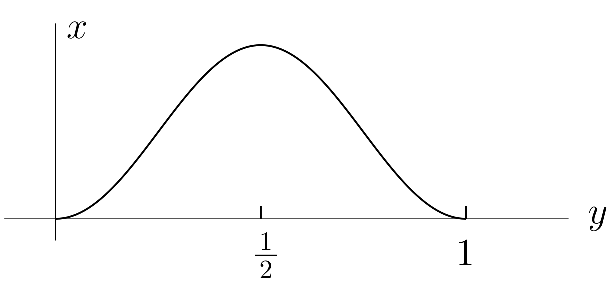
definition : Let A be an attractor of $x_{n+1} = F(x_n)$. The domain of attraction of A is $\{x_0 : \text{the } \omega\text{-limit set of } \{x_n\} \text{ is } A\} = D(A)$.

example : $F(x) = ax(1-x)$

	A	$D(A)$
$0 < a < 1$	$\{0\}$	$[0,1]$
$1 < a < 3$	$\{(a-1)/a\}$	$(0,1)$
$3 < a < 1+\sqrt{6}$	2-cycle	$(0,(a-1)/a) \cup ((a-1)/a,1)$

example : $y_{n+1} = \sigma(y_n) = 2y_n \bmod 1$: sawtooth map

let $x_n = \sin^2 \pi y_n$



$x_{n+1} = \sin^2 \pi y_{n+1} = \sin^2 2\pi y_n = (2 \sin \pi y_n \cos \pi y_n)^2 = 4x_n(1-x_n)$

so $\{x_n\} = \{\sin^2 \pi y_n\}$ is an orbit of the logistic map with $a = 4$

$\{0\}$: fixed point of $\sigma \Rightarrow \{0\}$: fixed point of F

$\{\frac{1}{3}, \frac{2}{3}\}$: 2-cycle of $\sigma \Rightarrow \{\frac{3}{4}\}$: fixed point of F 16 Wed 3/8

$\{\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5}\}$: 4-cycle of $\sigma \Rightarrow \{\frac{1}{8}(5 \pm \sqrt{5})\}$: 2-cycle of F

check : $\sin^2 \frac{\pi}{5} = \frac{1}{8}(5 - \sqrt{5})\,, \ldots$

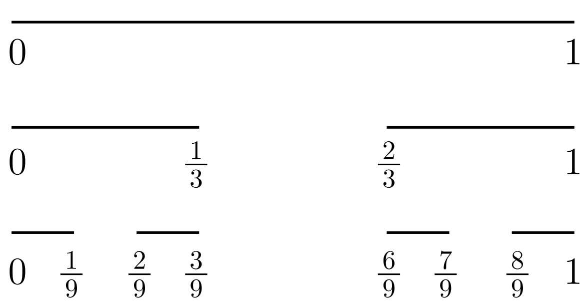
every p-cycle of σ is unstable , σ has no attractor $\Rightarrow$ same is true for F

y_0 : irrational $\Rightarrow \{y_n\}$ is a chaotic orbit $\Rightarrow \{x_n\}$ is a chaotic orbit

In this case $\{y_n\}$ is uniformly distributed on $[0,1]$, but $\{x_n\}$ is not uniformly distributed on $[0,1]$, it spends more time near $x = 0\,, 1$ than near $x = 1/2$.

The orbits $\{x_n\}, \{y_n\}$ depend sensitively on x_0, y_0, but their statistical properties do not.

4.1 Cantor sets



$K_0 = [0,1]$

$K_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$K_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

K_n = union of 2^n intervals of length $1/3^n$, $K_n \subset K_{n-1} \subset \cdots \subset K_0$

$K = \bigcap_{n=0}^{\infty}$: Cantor set , some points in K : $\{0, 1, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}\}$

ternary expansion : $x \in [0,1] \Rightarrow x = 0.t_1 t_2 t_3 = \dfrac{t_1}{3} + \dfrac{t_2}{3^2} + \dfrac{t_3}{3^3} + \cdots$, $t_n \in \{0,1,2\}$

for example, $1/3 = 0.1 = 0.02222\cdots$, note : $x \in K \Leftrightarrow t_n \in \{0,2\}$ for all n

properties of K

1. totally disconnected (contains no interval)

in fact the length of all the intervals removed is

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots = \frac{1}{3}\Big(1 + \frac{2}{3} + \frac{4}{9} + \cdots\Big) = \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} = 1$$

2. uncountable, 3. closed, 4. every point is a limit point, 5. self-similar

proof

1. length of $K_n = \left(\frac{2}{3}\right)^n \to 0$ as $n \to \infty$

2. assume K is countable , $x_1 = 0.t_{11}t_{12}t_{13}\cdots$, $x_2 = 0.t_{21}t_{22}t_{23}\cdots$, and so on

define $y = 0.s_1 s_2 s_3 \cdots$, $s_j = \begin{cases} 0 \text{ if } t_{jj} = 2 \\ 2 \text{ if } t_{jj} = 0 \end{cases}$

Then $y \in K$, but $y \neq x_n$ for all n; this contradicts the assumption; in fact, K is in 1-1 correspondence with $[0,1]$ (replace 2 by 1 in ternary expansion).

3. intersection of closed sets is closed

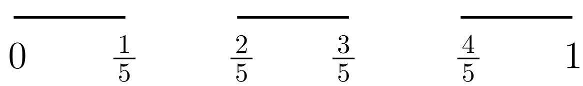
4. let $x = 0.t_1 t_2 t_3 \cdots \in K$

$$\left.\begin{array}{l} \text{if expansion is infinite, set } x_n = 0.t_1 \cdots t_n \\ \text{if expansion is finite, set } x_n = x + 0.0\cdots \underset{\substack{\uparrow \\ n\text{th place}}}{0}20\cdots \end{array}\right\} \text{then } x_n \in K, x_n \neq x, x_n \to x$$

5. ok

note

1. Other Cantor sets are constructed similarly.



2. A Cantor set is defined by properties 1-4 (need not be self-similar).

3. There are Cantor sets in $\mathbb{R}^m$, $m > 1$.

4. For $a > a_\infty$, most attractors of the logistic map are Cantor sets.

4,2 dimension

d : topological dimension , an integer

finite set of points : $d = 0$, curve : $d = 1$, area : $d = 2$, volume : $d = 3$

What about more general sets? Is the dimension always an integer?

definition : Consider a set $E \subset \mathbb{R}^m$, and for any $\epsilon > 0$, let $N(\epsilon)$ be the number of m-dimensional boxes of side ϵ required to cover E; if $N(\epsilon) \sim c\epsilon^{-D}$ as $\epsilon \to 0$, where c, D are independent of ϵ, then D is the box dimension of E.

If D is not an integer, then E is a fractal set. (Mandelbrot)

examples

1. $E = [a,b] \subset \mathbb{R}$, $N(\epsilon) = (b-a)/\epsilon = (b-a)\epsilon^{-1} \Rightarrow D = 1 = d$

2. $E = \{x_1, \ldots, x_n\} \subset \mathbb{R}$, $N(\epsilon) = n = n\epsilon^0 \Rightarrow D = 0 = d$

3. K = Cantor set = $\bigcap_{n=0}^\infty K_n$, K_n = union of 2^n intervals of length $1/3^n$

$\epsilon = 1/3^n$, $N(\epsilon) = 2^n$, $\ln \epsilon = -n \ln 3$, $\ln N(\epsilon) = n \ln 2$

$$D = \lim_{\epsilon \to o} \frac{\ln N(\epsilon)}{\ln(\epsilon^{-1})} = \lim_{\epsilon \to 0} \frac{n \ln 2}{n \ln 3} = \frac{\ln 2}{\ln 3} = 0.63 \text{ , but } d = 0$$

17 Mon 3/13

4.4 Lyapunov exponents

$x_{n+1} = F(x_n)\,,\ F : \mathbb{R} \to \mathbb{R}$

consider $F^N(x_0 + \epsilon) - F^N(x_0) \sim \epsilon F^N_x(x_0)$

$F^N_x(x_0) = F_x(x_{N-1})F_x(x_{N-2}) \cdots F_x(x_0) \Rightarrow \ln|F^N_x(x_0)| = \sum_{n=0}^{N-1} \ln|F_x(x_n)|$

$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \ln|F_x(x_n)| = \lambda$: Lyapunov exponent of $\{x_n\}$

interpretation

$N\lambda \sim \ln|F^N_x(x_0)| \Rightarrow |F^N(x_0+\epsilon) - F^N(x_0)| \sim \epsilon|F^N_x(x_0)| \sim \epsilon e^{N\lambda}$

$\Rightarrow \lambda$ is the exponential rate of separation of nearby orbits averaged over $\{x_n\}$

$\lambda < 0$: stable , $\lambda > 0$: unstable $\begin{cases} \text{sensitive dependence on initial data,} \\ \text{necessary condition for chaos} \end{cases}$

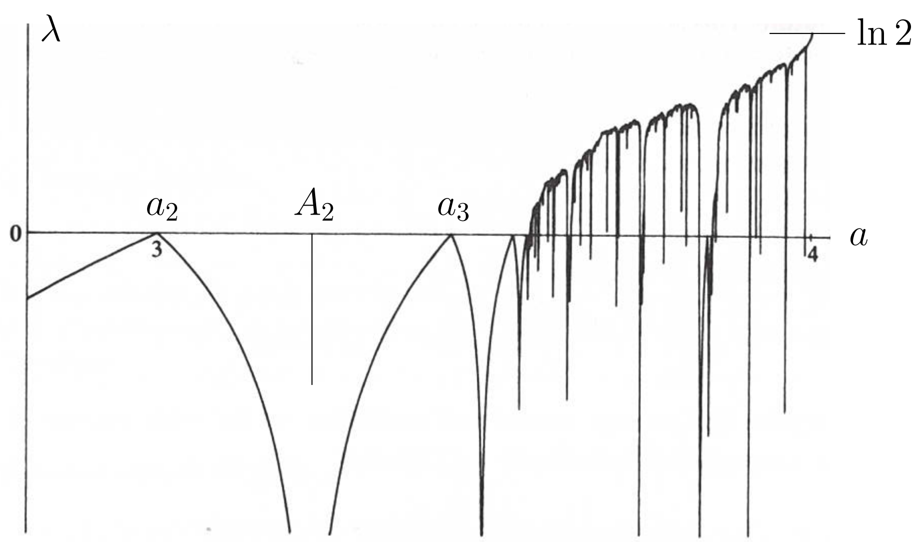
examples

1. $F(x) = ax \Rightarrow |F^N_x(x)| = |a^N| = e^{N\ln|a|} \Rightarrow \lambda = \ln|a|$

$|a| < 1 \Rightarrow \lambda < 0$: stable , $|a| > 1 \Rightarrow \lambda > 0$: unstable, but no chaos

2. $\sigma(x)$: sawtooth map , $\sigma^N_x(x_0) \sim 2^N = e^{N\ln 2} \Rightarrow \lambda = \ln 2 = 0.69$: chaos

3. logistic map , plot λ for each attractor



definition : F is chaotic on an invariant set S if the following conditions hold.

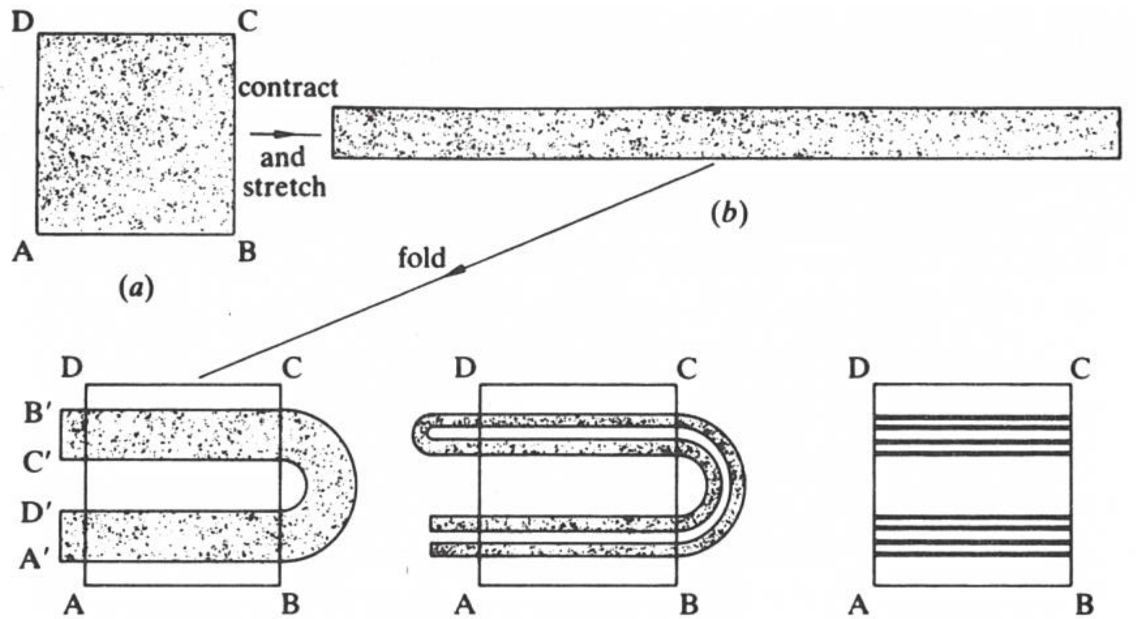
1. F has sensitive dependence on initial data ($\lambda > 0$)

2. F is topologically transitive, i.e. for any two open sets $A, B \subset S$, there exists $N \geq 0$ such that $F^N(A) \bigcap B \neq \emptyset$ (mixing)

3. F has a dense set of p-cycles

3.6 two-dimensional maps

example (Smale, 1967) , $F : S \to \mathbb{R}^2$, S = square , $F(S)$ = horseshoe



attractor : $\bigcap_{n=0}^{\infty} F^n(S) \sim [0,1] \times K$, K : Cantor set

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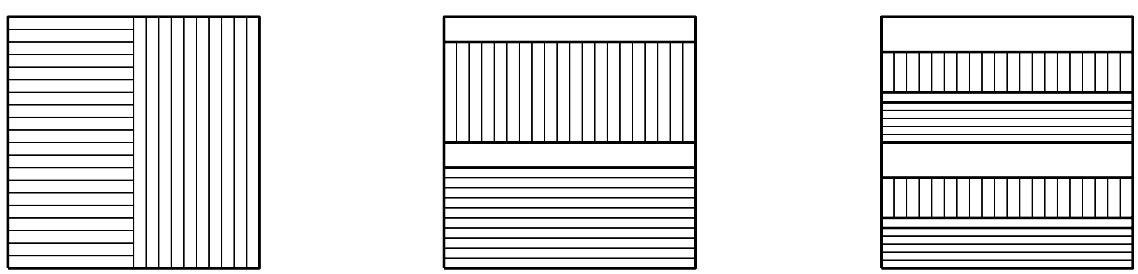
example : baker's map , $F : S \to S$, $S = [0,1)^2$

$$F(x,y) = \begin{cases} (2x, ay/2) & \text{if } 0 \le x < \frac{1}{2} \\ (2x-1, ay/2 + 1/2) & \text{if } \frac{1}{2} < x < 1 \end{cases}$$

x-coordinate is sawtooth map

assume $0 < a < 1$, map is expanding in x and contracting in y



attractor : $[0,1) \times K$, $K = \bigcap_{n=0}^{\infty} K_n$: Cantor set , what is the dimension?

K_n = union of 2^n intervals of length $(a/2)^n$

$\epsilon = (a/2)^n$, $N(\epsilon) = 2^n \Rightarrow \ln \epsilon = n(\ln a - \ln 2)$, $\ln N(\epsilon) = n \ln 2$

box dimension of $K = \lim_{\epsilon \to 0} \dfrac{\ln N(\epsilon)}{\ln(\epsilon^{-1})} = \dfrac{\ln 2}{\ln 2 - \ln a}$

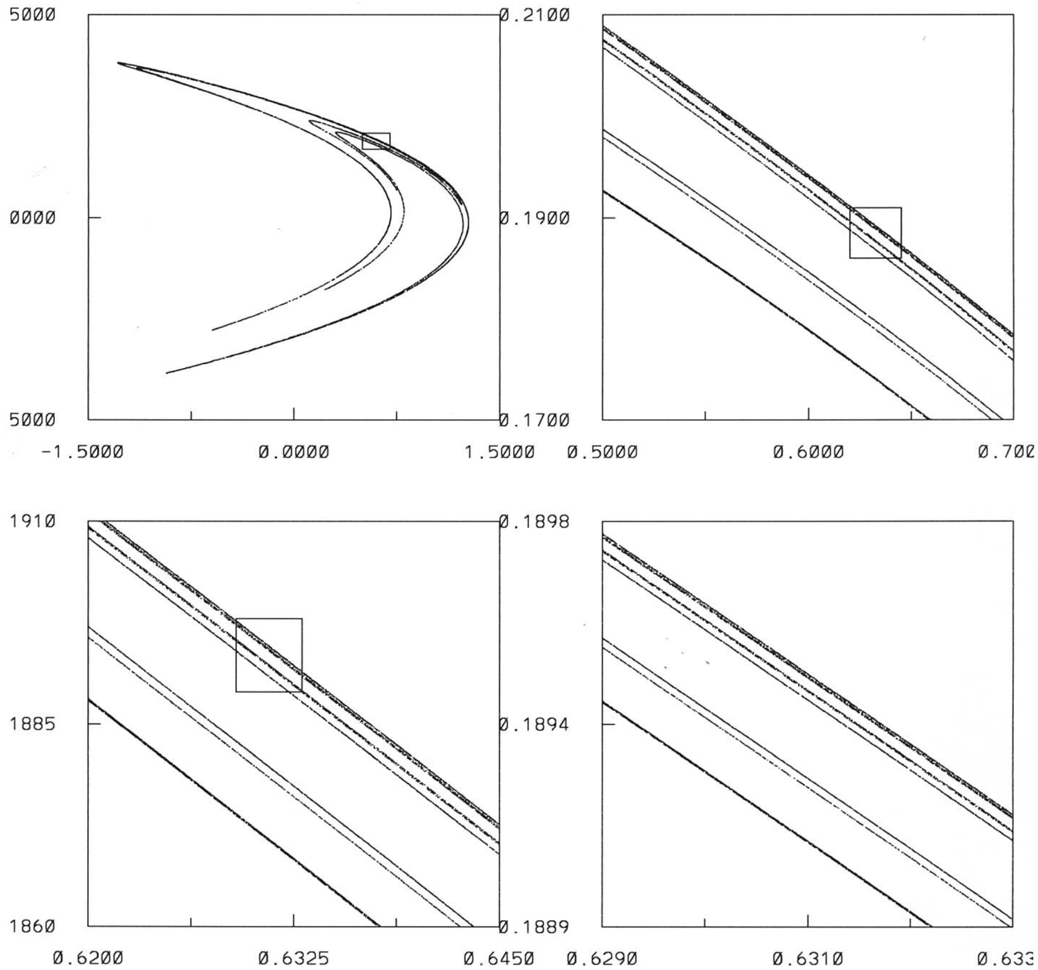
box dimension of attractor of baker's map : $D = 1 + \dfrac{\ln 2}{\ln 2 - \ln a}$, $1 < D < 2$

<u>example</u> : Hénon map

$$\left.\begin{aligned} x_{n+1} &= 1 + y_n - ax_n^2 \\ y_{n+1} &= bx_n \end{aligned}\right\} \Rightarrow J = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix} \text{ , assume } 0 < b < 1 : \text{ contracting}$$

fixed points , period-doubling , Feigenbaum sequence , chaos

$a = 1.4$, $b = 0.3$: strange attractor , self-similar



<u>Lyapunov exponents in higher dimension</u>

$F : \mathbb{R}^m \to \mathbb{R}^m$, q_j : eigenvalues of $F_x^N(x_0)$, $j = 1 : m$

$\lambda_j = \lim\limits_{N\to\infty} N^{-1} \ln |q_j|$: jth Lyapunov exponent , assume $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$

<u>conjecture</u> (Kaplan-Yorke, 1979)

If k is the largest integer such that $\lambda_1 + \cdots + \lambda_k > 0$, then the attractor has Hausdorff dimension $D = k + (\lambda_1 + \cdots + \lambda_k)/|\lambda_{k+1}|$.

<u>example</u> : baker's map , $J = \begin{pmatrix} 2 & 0 \\ 0 & a/2 \end{pmatrix} \Rightarrow \lambda_1 = \ln 2$, $\lambda_2 = \ln a - \ln 2$

$\lambda_1 > 0$, $\lambda_1 + \lambda_2 = \ln a < 0 \Rightarrow k = 1 \Rightarrow D = 1 + \lambda_1/|\lambda_2| = 1 + \dfrac{\ln 2}{\ln 2 - \ln a}$ <u>ok</u>

3.7 complex maps

$z_{n+1} = F(z_n)$, $F : S^2 \to S^2$, where $S^2 = \mathbb{C} \cup \{\infty\}$

example : $z_{n+1} = z_n^2$, fixed points : $X = \{0, 1, \infty\}$: stable, unstable, stable

domain of attraction : $D(0) = \{|z| < 1\}$, $D(\infty) = \{|z| > 1\}$

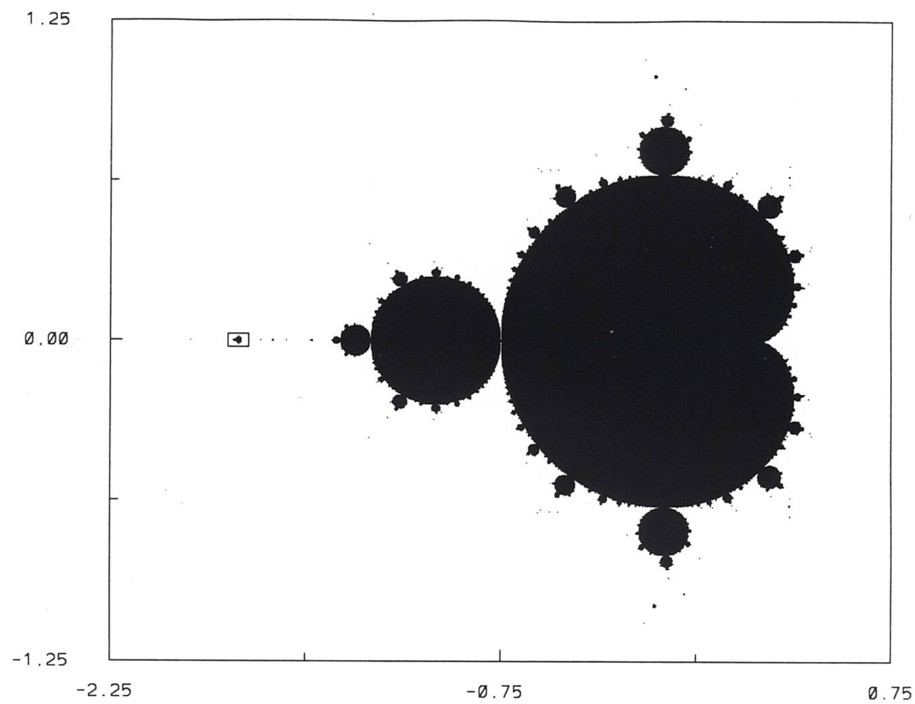
definition : If F has two attractors A_1, A_2 such that $\partial D(A_1) = \partial D(A_2) = J$, then J is called the Julia set of F.

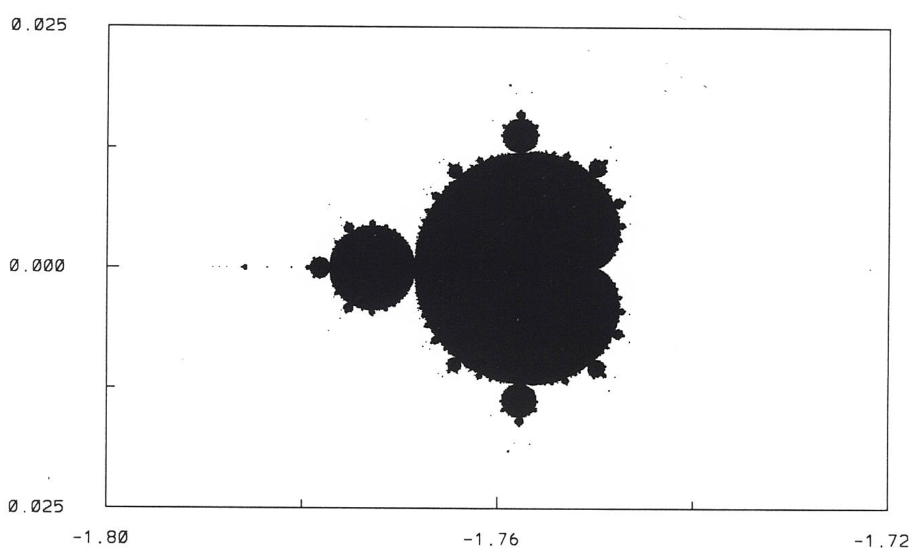
example : $F(z) = z^2$, $A_1 = \{0\}$, $A_2 = \{\infty\}$, $\partial D(0) = \{|z| = 1\} = \partial D(\infty) = J$

more generally , $F(a, z) = z^2 + a$, $a \in \mathbb{C}$

definition : $\{a \in \mathbb{C} : 0 \notin D(\infty)\} = M$: Mandelbrot set

For example, if $a = 0$, then $0 \notin D(\infty)$, so $0 \in M$.





18 Mon 3/20

5.1 ordinary differential equations

$\dfrac{dx}{dt} = F(x)\,,\ x(0) = x_0\,,\ x \in \mathbb{R}^m\,,\ x(t) : \text{orbit}$

$x_0 \to \phi(x_0,t)$: flow map , defined by $\phi_t(x_0,t) = F(\phi(x_0,t))\,,\ \phi(x_0,0) = x_0$

Hence $\phi(x_0,t)$ is the location at time t of the orbit initially located at x_0.

definition

ϕ is conservative or measure-preserving if $\mu(\phi(S,t)) = \mu(S)$ for all $S \subset \mathbb{R}^m, t > 0$

theorem

1) $J_t = (\nabla \cdot F)J$, where $J = \det(\phi_x)$, note : previously $J = F_x$

2) ϕ is conservative $\Leftrightarrow \nabla \cdot F = 0$

proof

1) $\phi(x,t+h) = \phi(x,t) + h\phi_t(x,t) + \cdots = \phi(x,t) + hF(\phi(x,t)) + \cdots$

$$\phi_x(x,t+h) = \phi_x(x,t) + hF_x(\phi(x,t))\phi_x(x,t) + \cdots$$

$$= \big(I + hF_x(\phi(x,t)) + \cdots\big)\phi_x(x,t) = \begin{pmatrix} 1 + h\partial_1 F_1 + \cdots & h\partial_2 F_1 + \cdots \\ h\partial_1 F_2 + \cdots & 1 + h\partial_2 F_2 + \cdots \end{pmatrix}\phi_x(x,t)$$

$$J(x,t+h) = (1 + h\nabla \cdot F + \cdots)J(x,t)$$

$$J_t(x,t) = \lim_{h\to 0} \frac{J(x,t+h) - J(x,t)}{h} = (\nabla \cdot F)J(x,t) \quad \text{ok}$$

2) $\phi(x,0) = x \Rightarrow \phi_x(x,0) = I \Rightarrow J(x,0) = \det I = 1 \Rightarrow J(x,t) = 1$ by part (1)

$$\mu(\phi(S,t)) = \int_{\phi(S,t)} dx = \int_S J(x,t)dx = \int_S dx = \mu(S) \quad \text{ok}$$

5.2 Hamiltonian systems

$x \in \mathbb{R}^2\,,\ x = (q,p)\,,\ q$: position , p : momentum , $H(q,p)$: Hamiltonian

Hamilton's equations : $\dfrac{dq}{dt} = \dfrac{\partial H}{\partial p}\,,\ \dfrac{dp}{dt} = -\dfrac{\partial H}{\partial q}$

example : $H(q,p) = \dfrac{1}{2}p^2 + V(q)$: kinetic energy + potential energy

$\dfrac{dq}{dt} = p\,,\ \dfrac{dp}{dt} = -V'(q) = f(q) \Rightarrow \dfrac{d^2q}{dt^2} = f(q)$: nonlinear oscillator

general case : $x \in \mathbb{R}^{2m}$, $H(q_1,\ldots,q_m,p_1,\ldots,p_m)$, $\dfrac{dq_j}{dt} = \dfrac{\partial H}{\partial p_j}\,,\ \dfrac{dp_j}{dt} = -\dfrac{\partial H}{\partial q_j}$

example : coupled nonlinear oscillators

$$H(q_1,q_2,,p_1,p_2) = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1) + V(q_2) + f(q_1,q_2)$$

claim

1. A Hamiltonian system preserves volume in the $2m$-dimensional phase space. (Liouville's theorem)

2. $H(q(t), p(t))$ is a constant of motion. (conservation of energy)

proof

1. $x = (q_1, \ldots, q_m, p_1, \ldots, p_m) \in \mathbb{R}^{2m}$, $dx/dt = F(x)$, will show $\nabla \cdot F = 0$

$F = \big(\partial p_1 H, \ldots, \partial p_m H, -\partial q_1 H, \ldots, -\partial q_m H\big)$

$\nabla \cdot F = \sum_{j=1}^{2m} \partial x_j F_j = \sum_{j=1}^{m} \partial q_j (\partial p_j H) + \sum_{j=1}^{m} \partial p_j (-\partial q_j H) = 0$

2. $H' = \sum_{j=1}^{m} \big(\partial_{q_j} H \cdot q_j' + \partial_{p_j} H \cdot p_j'\big) = \sum_{j=1}^{m} \big(\partial_{q_j} H \cdot \partial_{p_j} H + \partial_{p_j} H \cdot -\partial_{q_j} H\big) = 0$

Hence, Hamiltonian systems have no attractors, and the orbit $(q(t), p(t))$ lies on a level set of the energy $H(q, p)$.

Poincaré recurrence theorem : Consider a Hamiltonian system $dx/dt = F(x)$ and suppose $S \subset \mathbb{R}^{2m}$ is a bounded invariant set of positive measure. Then for any $\epsilon > 0$ and almost all $x_0 \in S$, there exists $t > 0$ such that $|x(t) - x_0| < \epsilon$.

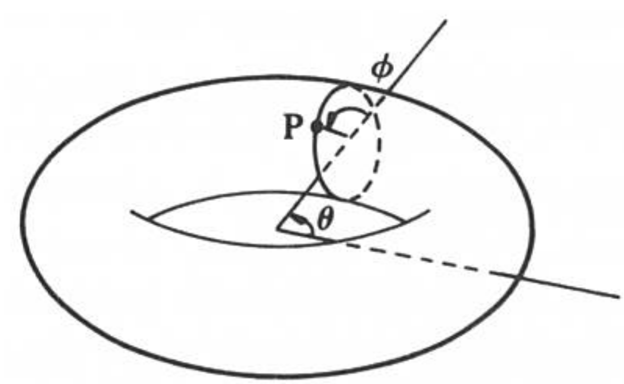
5.3 geometry of orbits

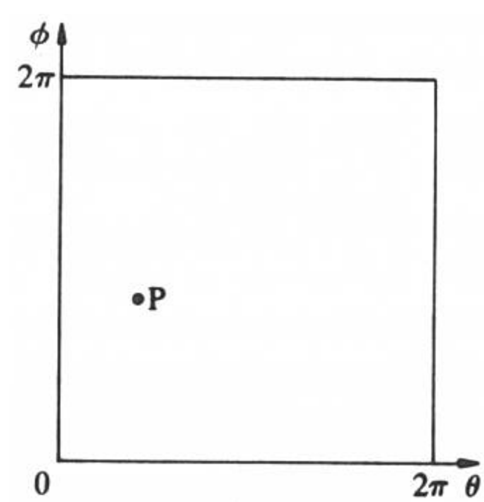
$dx/dt = F(x)$, $x \in \mathbb{R}^m$: autonomous system , RHS is independent of time 19 Wed 3/22

1. $x(t)$ is either a fixed point or a non-self-intersecting curve

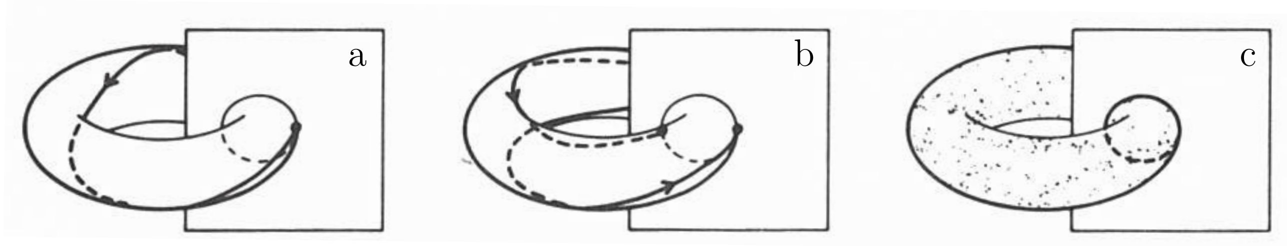
2. What kind of bounded orbits are possible? It depends on the dimension m; $x(t)$ can be a fixed point when $m \geq 1$, or a planar curve when $m \geq 2$, or a curve that winds around a torus when $m \geq 3$.

Let $(\theta(t), \phi(t))$ be the polar angles of an orbit on the torus $T^2 = [0, 2\pi) \times [0, 2\pi)$.





The plane $\theta = 0$ is the Poincaré section and this gives rise to the Poincaré map.



a,b) periodic orbits on torus $\Rightarrow$ fixed point, 2-cycle of Poincaré map

c) quasiperiodic orbit on torus $\Rightarrow$ dense orbit of Poincaré map

special case : $x(t) = (\theta(t), \phi(t)) = (\omega_1 t, \omega_2 t) \bmod 2\pi$

The orbit on the torus cuts the Poincaré section at time t_n with angles (θ_n, ϕ_n).

$$\theta_{n+1} = \theta_n + 2\pi \Rightarrow \omega_1 t_{n+1} = \omega_1 t_n + 2\pi \Rightarrow t_{n+1} = t_n + \frac{2\pi}{\omega_1} \Rightarrow \phi_{n+1} = \phi_n + \frac{2\pi\omega_2}{\omega_1}$$

define $a = \dfrac{\omega_2}{\omega_1}$, $x_n = \dfrac{\phi_n}{2\pi} \bmod 1 \Rightarrow \begin{cases} 0 \le x_n < 1\,,\ x_n \text{ is a proxy for } \phi(t) \\ x_{n+1} = x_n + a \bmod 1 : \text{rotation map} \end{cases}$

case 1 : a is rational

$a = \dfrac{\omega_2}{\omega_1} = \dfrac{q}{p}$, where p, q are relatively prime integers , $p\omega_2 = q\omega_1$

$x_{n+1} = x_n + q/p \bmod 1$

$\{x_n\} = \{x_0, x_0 + q/p, x_0 + 2q/p, \ldots, x_0 + pq/p\} \bmod 1$: p-cycle for all x_0

$x(t)$ is periodic and in each period it winds around p times in the θ-direction and q times in the ϕ-direction; in pictures above, $q/p = 1/1, q/p = 1/2$

case 2 : a is irrational

$p\omega_2 \neq q\omega_1$ for any relatively prime integers p, q

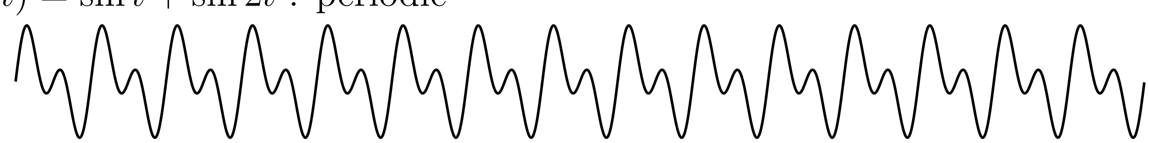
$\{x_n\}$ is dense in $[0, 1)$ for all x_0

$x(t)$ is quasiperiodic and winds around densely on the torus

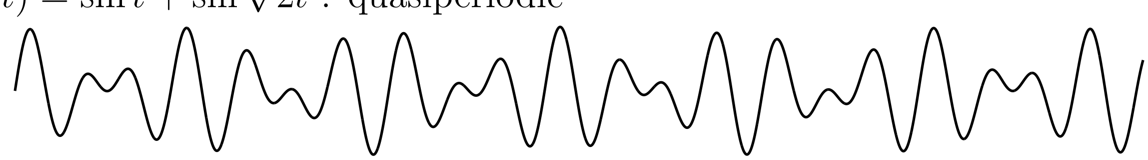
definition : In the general case $dx/dt = F(x)$, $x \in \mathbb{R}^m$, a quasiperiodic orbit has the form $x(t) = f(\omega_1 t, \ldots, \omega_r t)$, where f is 2π-periodic in each argument, no frequency ω_j is a rational multiple of another frequency, and $r \geq 2$.

examples with $m = 1, r = 2$

$x(t) = \sin t + \sin 2t$: periodic



$x(t) = \sin t + \sin \sqrt{2}t$: quasiperiodic



5.4 stability of periodic orbits

20 Mon 3/27

$$\frac{dx}{dt} = F(x)\,,\; F : \mathbb{R}^m \to \mathbb{R}^m$$

suppose $X(t)$ is a T-periodic orbit : $X(t+T) = X(t)$, is it stable or unstable?

linear stability : $x' = x - X\,,\; \dfrac{dx'}{dt} = A(t)x'\,,\; A(t) = F_x(X(t))\,,\; A(t+T) = A(t)$

Hence the perturbation satisfies a linear ODE with periodic coefficients.

Floquet theory

There are solutions of the form $x'(t) = p(t)e^{st}$, where $s \in \mathbb{C},\; p(t+T) = p(t)$.

note : this generalizes the case $A(t) =$ constant

$x'(0) = p(0)\,,\; x'(T) = p(T)e^{sT} = p(0)e^{sT} = x'(0)e^{sT}$

define s : Floquet exponent , $q = e^{sT}$: Floquet multiplier , $x'(nT) = q^n x'(0)$

$|q| < 1 \Rightarrow$ stable , $|q| > 1 \Rightarrow$ unstable

$q = 1 \Rightarrow x'(T) = x'(0)$

$$q = -1 \Rightarrow x'(T) = -x'(0) \Rightarrow \begin{cases} x'(2T) = x'(0) : x'(t) \text{ has period } 2T \\ \text{period-doubling} \\ \text{subharmonic instability} : \omega \to \omega/2 \end{cases}$$

$$|q| = 1\,,\; q \neq \pm 1 \Rightarrow \operatorname{Im} s \neq 0\,,\pi \Rightarrow \begin{cases} x'(t) \text{ has frequencies } \omega_1 = T/2\pi\,,\; \omega_2 = \operatorname{Im} s \\ \text{quasiperiodic in general} \end{cases}$$

example : $\dfrac{dx}{dt} = A(t)x\,,\; A(t) = \begin{pmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\sin t\cos t \\ -1 - \frac{3}{2}\sin t\cos t & -1 + \frac{3}{2}\sin^2 t \end{pmatrix}\,,\; T = \pi$

The eigenvalues of $A(t)$ have negative real part for all t, but $x(t) = \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix} e^{t/2}$ is a solution, so the equation has a positive Floquet exponent. (hw5)

note : In general there are m Floquet multipliers $q_1, \ldots, q_m$, and when the system depends on a parameter a, there can be bifurcations.

6.1 2D autonomous systems of ODEs

$$\frac{dx}{dt} = F(x,y)\,,\; \frac{dy}{dt} = G(x,y) \Rightarrow \frac{dy}{dx} = \frac{G(x,y)}{F(x,y)} \Rightarrow \text{curves in } (x,y)\text{-phase plane}$$

6.2 linear systems

Let $(0,0)$ be an equilibrium point, how do the neighboring orbits behave?

$$\frac{dx}{dt} = ax + by\,,\ \frac{dy}{dt} = cx + dy\,,\ a = F_{x0}\,,\ b = F_{y0}\,,\ c = G_{x0}\,,\ d = G_{y0}$$

$$\frac{d\vec{x}}{dt} = J\vec{x}\ ,\ J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$, look for $\vec{x}(t) = e^{st}\vec{u}$: <u>normal mode</u> , $J\vec{u} = s\vec{u}$, $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$

$$\det(J - sI) = (a - s)(d - s) - bc = s^2 - ps + q = (s - s_1)(s - s_2)$$

$$p = a + d = s_1 + s_2 = \text{trace}J\ ,\ q = ad - bc = s_1 s_2 = \det J\ ,\ s_{1,2} = \frac{p \pm \sqrt{p^2 - 4q}}{2}$$

if $p^2 \neq 4q$, then $s_1 \neq s_2$, $\vec{x}(t) = c_1 e^{s_1 t}\vec{u}_1 + c_2 e^{s_2 t}\vec{u}_2$

if $p^2 = 4q$, then $s_1 = s_2 = s$, $\vec{x}(t) = c_1 e^{st}\vec{u}_1 + c_2 e^{st}(t\vec{u}_1 + \vec{u}_2)$, $(J - sI)\vec{u}_2 = \vec{u}_1$

case			eigenvalues s_1, s_2
1	node	$p^2 > 4q\,, q > 0$	real, distinct, same sign
2	saddle	$p^2 > 4q\,, q < 0$	real, distinct, opposite sign
3	focus	$p^2 < 4q\,, p \neq 0$	complex conjugate pair, nonzero real part

<u>case 1</u> : node, $(0,0)$ is stable if $p < 0$, unstable if $p > 0$

if $p < 0$, then $s_2 < s_1 < 0 \Rightarrow \vec{x}(t) \sim \begin{cases} c_1 e^{s_1 t}\vec{u}_1 \text{ as } t \to \infty & \text{if } c_1 \neq 0 \\ c_2 e^{s_2 t}\vec{u}_2 \text{ as } t \to -\infty & \text{if } c_2 \neq 0 \end{cases}$

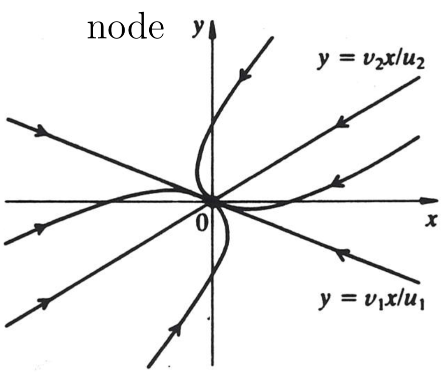
if $p > 0$, then $s_1 > s_2 > 0$ and some features of the sketch change ...

<u>case 2</u> : saddle, 1 stable direction and 1 unstable direction, $(0,0)$ is unstable

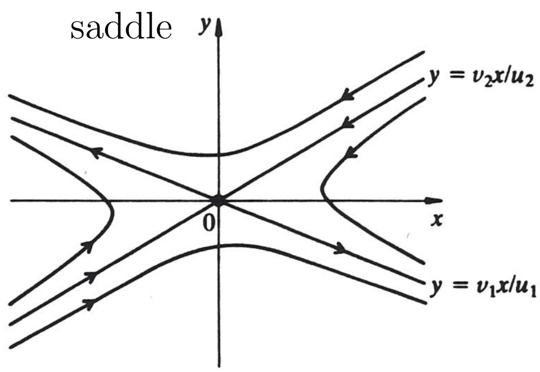
<u>case 3</u> : focus, $(0,0)$ is stable if $p < 0$, unstable if $p > 0$, orbits are spirals

$$s = \frac{p \pm i\sqrt{4q - p^2}}{2}\ ,\ e^{st} = e^{pt/2}e^{\pm i\beta t}\ ,\ \beta = \sqrt{q - p^2/4} > 0$$

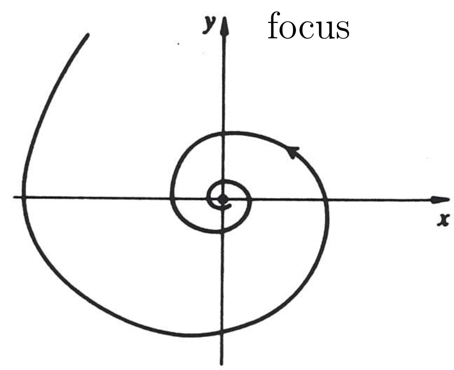
node



saddle



focus



In these cases the real part of each eigenvalue is nonzero; the equilibrium point is said to be <u>hyperbolic</u>, and the phase portrait is structurally stable with respect to perturbations of a, b, c, d and weak nonlinearity.

21 Wed 3/29

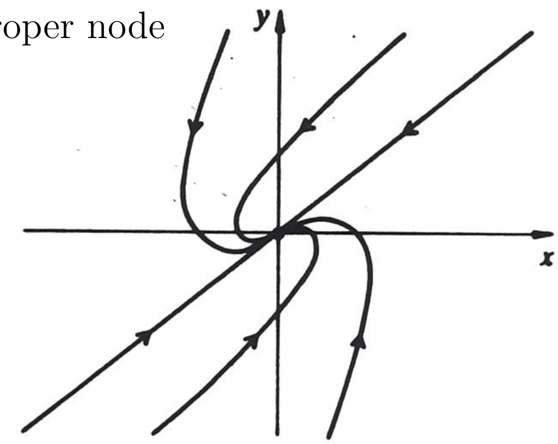
case			eigenvalues s_1, s_2
4	improper node	$p^2 = 4q\,, q > 0$	$s_1 = s_2 = p/2$ is real
5	center	$p^2 < 4q\,, p = 0 \Rightarrow q > 0$	$s_{1,2} = \pm i\sqrt{q}$ are imaginary

<u>case 4</u> : improper node, $(0,0)$ is stable if $p < 0$, unstable if $p > 0$

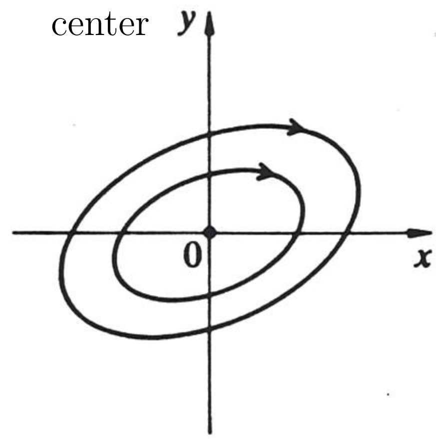
this is a limit of case 1 (node) when $\vec{u}_1 \to \vec{u}_2$

<u>case 5</u> : center, $\vec{x}(t) = c_1 e^{i\sqrt{q}t}\vec{u}_1 + c_2 e^{-i\sqrt{q}t}\vec{u}_2$, orbits are periodic, ellipses

improper node



center

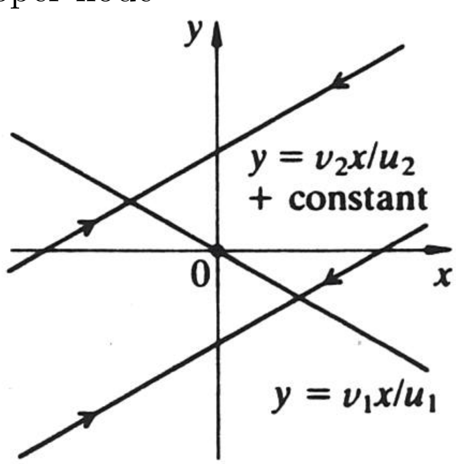


case			eigenvalues s_1, s_2
6	improper node	$p^2 > 4q\,, q = 0 \Rightarrow p \neq 0$	$s_1 = p$ is real, $s_2 = 0$
7	source/sink	$p^2 = 4q\,, q > 0\,, b = c = 0$	$s_1 = s_2 = a = d$ is real

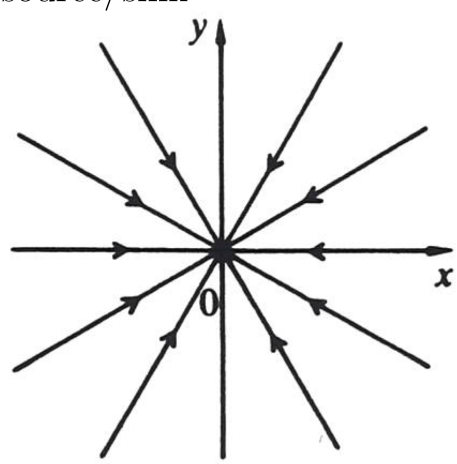
<u>case 6</u> : improper node, $\vec{x}(t) = c_1\vec{u}_1 + c_2 e^{pt}\vec{u}_2$, orbits are straight lines

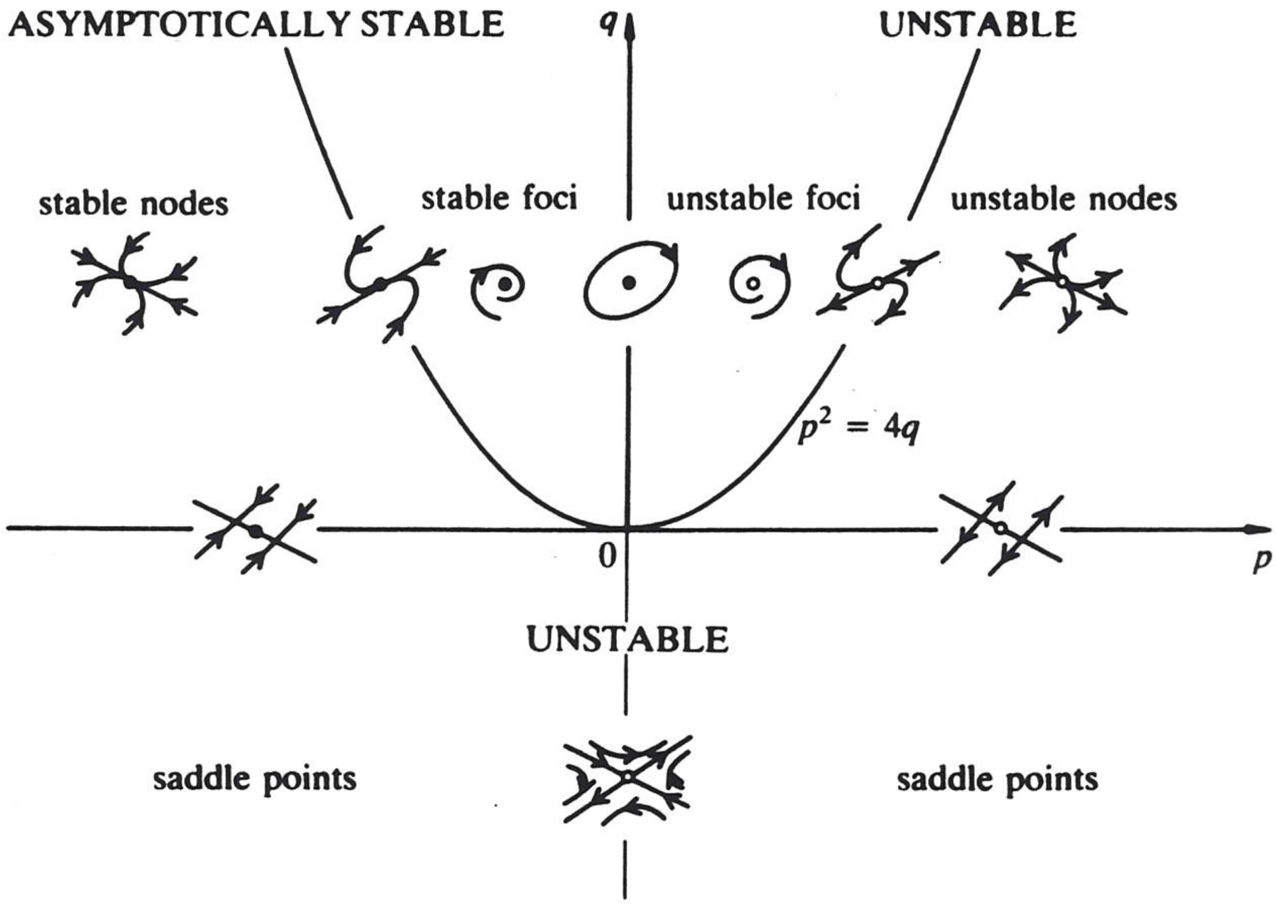
<u>case 7</u> : source/sink, $\dfrac{d\vec{x}}{dt} = a\vec{x} \Rightarrow \vec{x}(t) = \vec{x}_0 e^{at}$, orbits are radial straight lines

improper node



source/sink





example : $\dfrac{dx}{dt} = ax - cy$, $\dfrac{dy}{dt} = cx + ay$, assume $a < 0$, $c \neq 0$

$J = \begin{pmatrix} a & -c \\ c & a \end{pmatrix} \Rightarrow p = 2a$, $q = a^2 + c^2$, $p^2 - 4q = 4a^2 - 4(a^2 + c^2) = -4c^2 < 0$

$\Rightarrow (0,0)$ is a stable focus (more precisely, globally asymptotically stable)

note : $\det(J - sI) = (a - s)^2 + c^2 = 0 \Rightarrow s = a \pm ic \Rightarrow e^{st} = e^{at}e^{\pm ic}$

6.3 Lyapunov's direct method (energy method)

example (continued)

define $H(x,y) = \frac{1}{2}(x^2 + y^2)$, consider $H(x(t), y(t))$

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} = x(ax - cy) + y(cx + ay) = a(x^2 + y^2) = 2aH$$

$$\Rightarrow \frac{dH}{dt} = 2aH \Rightarrow H(x(t), y(t)) = H(x_0, y_0)\, e^{2at}$$

$$\Rightarrow \tfrac{1}{2}(x(t)^2 + y(t)^2) = \tfrac{1}{2}(x(0)^2 + y(0)^2)e^{2at} \to 0 \text{ as } t \to \infty$$

Hence $(0,0)$ is globally asymptotically stable.

note : In some cases the energy method can also handle nonlinear terms.

example (continued)

$$\frac{dx}{dt} = ax - cy + f(x,y) \ , \ \frac{dy}{dt} = cx + ay + g(x,y)$$

assume $f, g = O(x^2 + y^2)$ as $x, y \to 0$

$$\frac{dH}{dt} = 2aH + xf(x,y) + yg(x,y) = aH + \underbrace{aH + xf(x,y) + yg(x,y)}_{\leq 0 \text{ for } x,y \text{ sufficiently small}}$$

$\Rightarrow \dfrac{dH}{dt} \leq aH$ for x, y sufficiently small

$\Rightarrow H(x(t), y(t)) \leq H(x_0, y_0)\, e^{at} \Rightarrow (0,0)$ is asymptotically stable

This argument does not imply global stability with respect to initial data, but it does show that a stable focus is structurally stable with respect to sufficiently small nonlinear perturbations.

definition

Let $dx/dt = F(x)\,, x \in \mathbb{R}^m$, $F(0) = 0$, so $X = 0$ is an equilibrium point.

Then $H(x)$ is a Lyapunov function or energy function of the ODE if ...

1. $H(x)$ is positive definite, i.e. $H(x) > 0$ for $x \neq 0$ and $H(0) = 0$,
2. the directional derivative $F \cdot \nabla H$ is negative definite.

theorem

If the ODE has a Lyapunov function, then $X = 0$ is asymptotically stable.

proof : $\dfrac{dH}{dt} = \nabla H \cdot \dfrac{dx}{dt} = F \cdot \nabla H \leq 0 \ldots$ ok

This says that if the system dissipates energy, then the equilibrium is stable.

example

$\dfrac{dx}{dt} = -x - 2y^2 \, , \ \dfrac{dy}{dt} = xy - y^3 \ : \ (0,0)$ is an equilibrium

$\dfrac{dx'}{dt} = -x' \, , \ \dfrac{dy'}{dt} = 0 \Rightarrow p = -1 \, , \ q = 0 \ :$ case 6 , improper node

cannot conclude that $(0,0)$ is asymptotically stable

let $H(x,y) = \frac{1}{2}x^2 + y^2 \Rightarrow F \cdot \nabla H = (-x - 2y^2)x + (xy - y^3)2y = -(x^2 + 2y^4)$

Then H is a Lyapunov function and $(0,0)$ is asymptotically stable.

6.4 Lindstedt-Poincaré method for periodic orbits

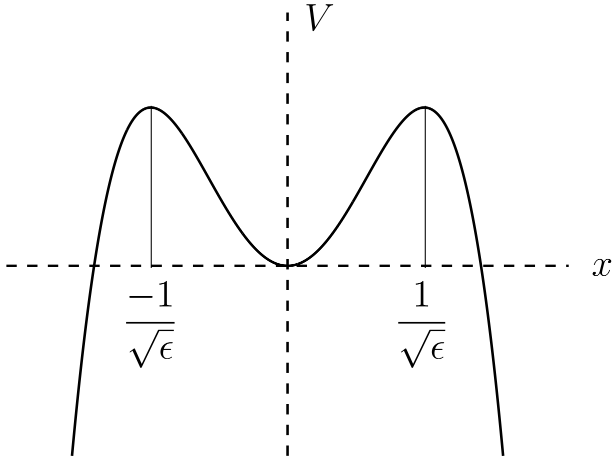
21 Mon 4/3

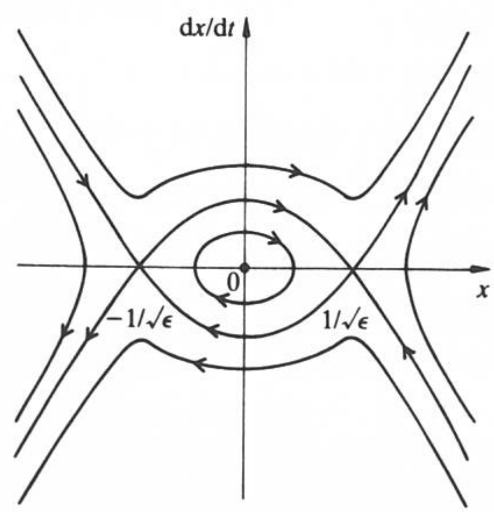
example Q1.17 , hw2

$$\frac{d^2x}{dt^2} + x - \epsilon x^3 = 0 \;,\; \epsilon > 0 \;:\; \text{nonlinear oscillation}$$

equilibrium points : $x(1-\epsilon x^2) = 0 \Rightarrow X = 0\,,\, \pm 1/\sqrt{\epsilon}$, linear stability ...

energy : $E = \frac{1}{2}\Big(\frac{dx}{dt}\Big)^2 + V(x)\,,\, V(x) = \frac{1}{2}x^2 - \frac{1}{4}\epsilon x^4$





Let $x(0) = a$, $|a| < \frac{1}{\sqrt{\epsilon}}$, $\frac{dx}{dt}(0) = 0$, a : amplitude of orbit

$$T = 2\int_{x_1}^{x_2} \frac{dx}{2\sqrt{E - V(x)}} = \frac{4}{\sqrt{1-(\epsilon/2)a^2}}\, K\Big(\frac{\epsilon a^2}{2-\epsilon a^2}\Big)\,,\; K(m) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - m\sin^2\phi}}$$

$= 2\pi\Big(1 + \frac{3}{8}\epsilon a^2 + \cdots\Big)$ as $\epsilon \to 0$, we also want to express $x(t)$ as a Fourier series

regular perturbation series

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots$$

$$\epsilon^0 : \frac{d^2x_0}{dt^2} + x_0 = 0\,,\; x_0(0) = a\,,\; \frac{dx_0}{dt}(0) = 0 \;\Rightarrow x_0(t) = a\cos t$$

$$\epsilon^1 : \frac{d^2x_1}{dt^2} + x_1 - x_0^3 = 0\,,\; x_1(0) = 0\,,\; \frac{dx_1}{dt}(0) = 0 \;\Rightarrow \frac{d^2x_1}{dt^2} + x_1 = a^3\cos^3 t$$

$$\mathrm{Re}\big(e^{3it}\big) = \cos 3t = \mathrm{Re}(\cos t + i\sin t)^3 = \cos^3 t - 3\cos t\sin^2 t$$

$$= \cos^3 t - 3\cos t(1-\cos^2 t) = 4\cos^3 t - 3\cos t \Rightarrow \cos^3 t = \frac{3}{4}\cos t + \frac{1}{4}\cos 3t$$

$$\frac{d^2x_1}{dt^2} + x_1 = a^3\Big(\frac{3}{4}\cos t + \frac{1}{4}\cos 3t\Big)$$

particular solution : $x_1(t) = At\sin t + Bt\cos t + C\cos 3t$

$$\frac{d^2x_1}{dt^2} = A(t\cdot -\sin t + 2\cos t) + B(t\cdot -\cos t - 2\sin t) - 9C\cos 3t$$

$$\frac{d^2x_1}{dt^2} + x_1 = 2A\cos t - 2B\sin t - 8C\cos 3t$$

$$\Rightarrow 2A = \frac{3}{4}a^3\,,\ B = 0\,,\ -8C = \frac{1}{4}a^3 \Rightarrow A = \frac{3}{8}a^3\,,\ C = -\frac{1}{32}a^3$$

$$x_1(t) = \frac{3}{8}a^3 t\sin t - \frac{1}{32}a^3\cos 3t + D\sin t + E\cos t$$

$$x_1(0) = 0 \Rightarrow -\frac{1}{32}a^3 + E = 0 \Rightarrow E = \frac{1}{32}a^3\ ,\ \frac{dx_1}{dt}(0) = 0 \Rightarrow D = 0$$

$$x_1(t) = \frac{3}{8}a^3 t\sin t - \frac{1}{32}a^3\cos 3t + \frac{1}{32}a^3\cos t$$

$$x(t) = x_0(t) + \epsilon x_1(t) + \cdots = a\cos t + \epsilon a^3\Big(\frac{3}{8}\underset{\substack{\uparrow\\ \underline{\text{secular term}}}}{t\sin t} - \frac{1}{32}\cos 3t + \frac{1}{32}\cos t\Big) + \cdots$$

1. For any fixed $t > 0$, the approximation converges as $\epsilon \to 0$.
$|x(t) - x_0(t)| = O(\epsilon)\ ,\ |x(t) - (x_0(t) + \epsilon x_1(t))| = O(\epsilon^2)\,,\ \ldots$

2. For any $\epsilon > 0$, the approximation is accurate if $\epsilon t << 1 \Leftrightarrow t << \epsilon^{-1}$, but it is not uniformly valid for all $t > 0$, the method fails to provide the period T.

Lindstedt-Poincaré method

$T = 2\pi/\omega$, define $s = \omega t$: scaled time , $\widetilde{x}(s) = x(t)$

$$x(t+T) = x(t) \Rightarrow \widetilde{x}(s+2\pi) = \widetilde{x}(s)$$

$\dfrac{d^2x}{dt^2} + x - \epsilon x^3 = 0\,,\ x(0) = a\,,\ \dfrac{dx}{dt}(0) = 0$, replace $\dfrac{d}{dt} = \omega\dfrac{d}{ds}$, drop tilde

$$\omega^2\frac{d^2x}{ds^2} + x - \epsilon x^3 = 0\ ,\ x(0) = a\ ,\ \frac{dx}{ds}(0) = 0\ ,\ x(s+2\pi) = x(s)$$

$$x(s) = x_0(s) + \epsilon x_1(s) + \epsilon^2 x_2(s) + \cdots\,,\ \omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots$$

$$\omega^2\frac{d^2x}{ds^2} = (\omega_0^2 + 2\epsilon\omega_0\omega_1 + \cdots)\Big(\frac{d^2x_0}{ds^2} + \epsilon\frac{d^2x_1}{ds^2} + \cdots\Big)$$

$$= \omega_0^2\frac{d^2x_0}{ds^2} + \epsilon\Big(\omega_0^2\frac{d^2x_1}{ds^2} + 2\omega_0\omega_1\frac{d^2x_0}{ds^2}\Big) + \cdots$$

$$\epsilon^0 : \omega_0^2\frac{d^2x_0}{ds^2} + x_0 = 0\ ,\ x_0(0) = a\ ,\ \frac{dx_0}{ds}(0) = 0$$

2π-periodicity $\Rightarrow \omega_0 = 1\,,\ x_0(s) = a\cos s$

$\epsilon^1 : \omega_0^2 \frac{d^2x_1}{ds^2} + 2\omega_0\omega_1 \frac{d^2x_0}{ds^2} + x_1 - x_0^3 = 0$

$\frac{d^2x_1}{ds^2} + x_1 = 2\omega_1 a \cos s + a^3 \cos^3 s = \left(2\omega_1 a + \frac{3}{4}a^3\right) \cos s + \frac{1}{4}a^3 \cos 3s$

periodicity $\Rightarrow 2\omega_1 a + \frac{3}{4}a^3 = 0 \Rightarrow \omega_1 = -\frac{3}{8}a^2$, eliminates secular term

$x_1(s) = \frac{1}{32}a^3(\cos s - \cos 3s)$, check : $x_1(0) = 0$, $\frac{dx_1}{ds}(0) = 0$, ...

$\widetilde{x}(s) = \widetilde{x}_0(s) + \epsilon\widetilde{x}_1(s) + \cdots = a\cos s + \frac{\epsilon}{32}a^3(\cos s - \cos 3s) + \cdots$

$\omega = \omega_0 + \epsilon\omega_1 + \cdots = 1 - \frac{3}{8}\epsilon a^2 + \cdots$

$x(t) = a\cos\omega t + \frac{\epsilon}{32}a^3(\cos\omega t - \cos 3\omega t) + \cdots$: Fourier series

1. $T = \frac{2\pi}{\omega} = \frac{2\pi}{1 - \frac{3}{8}\epsilon a^2 + \cdots} = 2\pi\left(1 + \frac{3}{8}\epsilon a^2 + \cdots\right)$

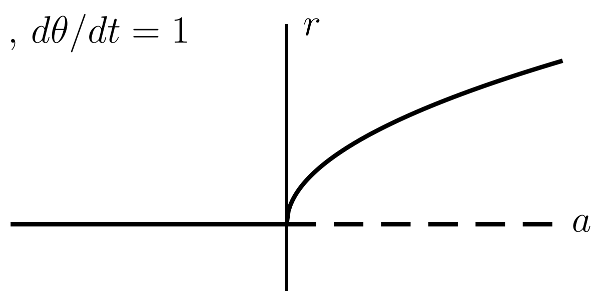
2. The expansion is uniformly valid for all $t > 0$, the error is $O(\epsilon)$ or $O(\epsilon^2)$ or ... for all $t > 0$.

6.5 limit cycles 22 Wed 4/5

recall : $dx/dt = -y + (a - x^2 - y^2)x$, $dy/dt = x + (a - x^2 - y^2)y$

$(0,0)$: equilibrium point , $J = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix} \Rightarrow s = a \pm i$: focus $\begin{cases} \text{stable if } a < 0 \\ \text{unstable if } a > 0 \end{cases}$

$dr/dt = r(a - r^2)$, $d\theta/dt = 1$



There is a supercritical Hopf bifurcation at $a = 0$, for $a > 0$ there is a limit cycle (periodic attracting orbit) given by $x(t) = \sqrt{a}\cos(t + \theta_0)$, $y(t) = \sqrt{a}\sin(t + \theta_0)$.

example : $dx/dt = -y + ax + xy^2$, $dy/dt = x + ay - x^2$

$(0,0)$: equilibrium point , $J = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}$: same as above

Hence there is a Hopf bifurcation at $a = 0$ here too, but the polar coordinate form of the system is hard to analyze.

Lindstedt-Poincaré method

Look for a periodic orbit of amplitude ϵ near $(0,0)$.

previous example suggests $\epsilon = \sqrt{a}\,,\ a = \epsilon^2$.

$s = \omega t\,,\ \widetilde{x}(s) = x(t)\,,\ \widetilde{y}(s) = y(t)\,,\ \widetilde{x}(s)\,,\ \widetilde{y}(s)$: 2π-periodic

$d/dt = \omega d/ds$, drop tilde , $\omega \dfrac{dx}{ds} = -y + \epsilon^2 x + xy^2,\ \omega \dfrac{dy}{ds} = x + \epsilon^2 y - x^2$

assume $x(s)$ has a maximum at $s = 0$, so $\dfrac{dx}{ds}(0) = 0$

$x(s) = \cancel{x_0(s)} + \epsilon x_1(s) + \epsilon^2 x_2(s) + \epsilon^3 x_3(s) + \cdots$

$y(s) = \cancel{y_0(s)} + \epsilon y_1(s) + \epsilon^2 y_2(s) + \epsilon^3 y_3(s) + \cdots$

$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots$

$$\omega\frac{dx}{ds} = \epsilon\omega_0\frac{dx_1}{ds} + \epsilon^2\Big(\omega_0\frac{dx_2}{ds} + \omega_1\frac{dx_1}{ds}\Big) + \epsilon^3\Big(\omega_0\frac{dx_3}{ds} + \omega_1\frac{dx_2}{ds} + \omega_2\frac{dx_1}{ds}\Big) + \cdots$$

$$\omega\frac{dy}{ds} = \cdots$$

$x^2 = \epsilon^2 x_1^2 + \epsilon^3\cdot 2x_1x_2 + \cdots\ ,\ y^2 = \epsilon^2 y_1^2 + \epsilon^3\cdot 2y_1y_2 + \cdots\ ,\ xy^2 = \epsilon^3 x_1 y_1^2 + \cdots$

$\epsilon^1 : \omega_0\dfrac{dx_1}{ds} = -y_1\,,\ \omega_0\dfrac{dy_1}{ds} = x_1 \Rightarrow \omega_0^2\dfrac{d^2x_1}{ds^2} + x_1 = 0$, 2π-periodicity $\Rightarrow \omega_0 = 1$

$\dfrac{dx_1}{ds}(0) = 0 \Rightarrow x_1(s) = A_1\cos s\,,\ y_1(s) = A_1\sin s$

$\epsilon^2 : \omega_0\dfrac{dx_2}{ds} + \omega_1\dfrac{dx_1}{ds} = -y_2\,,\ \omega_0\dfrac{dy_2}{ds} + \omega_1\dfrac{dy_1}{ds} = x_2 - x_1^2$

$\dfrac{dx_2}{ds} - \omega_1 A_1\sin s = -y_2\,,\ \dfrac{dy_2}{ds} + \omega_1 A_1\cos s = x_2 - A_1^2\cos^2 s$

$\dfrac{d^2x_2}{ds} - \omega_1 A_1\cos s = -\dfrac{dy_2}{ds} = -(-\omega_1 A_1\cos s + x_2 - A_1^2\cos^2 s)$

$\dfrac{d^2x_2}{ds^2} + x_2 = 2\cancel{\omega_1}A_1\cos s + A_1^2\cdot\dfrac{1}{2}(1+\cos 2s)$, periodicity $\Rightarrow \omega_1 = 0$

$x_2(s) = \alpha + \beta\cos 2s$: particular solution

$-4\beta\cos 2s + \alpha + \beta\cos 2s = \dfrac{1}{2}A_1^2(1+\cos 2s) \Rightarrow \alpha = \dfrac{1}{2}A_1^2\,,\ \beta = -\dfrac{1}{6}A_1^2$

$x_2(s) = A_2\cos s + A_1^2(\dfrac{1}{2} - \dfrac{1}{6}\cos 2s)$ using $\dfrac{dx_2}{ds}(0) = 0$

$y_2(s) = A_2\sin s - \dfrac{1}{3}A_1^2\sin 2s$

$$\epsilon^3 : \omega_0 \frac{dx_3}{ds} + \cancel{\omega_1} \frac{dx_2}{ds} + \omega_2 \frac{dx_1}{ds} = -y_3 + x_1 + x_1 y_1^2$$

$$\omega_0 \frac{dy_3}{ds} + \cancel{\omega_1} \frac{dy_2}{ds} + \omega_2 \frac{dy_1}{ds} = x_3 + y_1 - 2x_1 x_2$$

$$\Rightarrow \frac{dx_3}{ds} - \omega_2 A_1 \sin s = -y_3 + A_1 \cos s + A_1^3 \cos s \sin^2 s$$

$$\frac{dy_3}{ds} + \omega_2 A_1 \cos s = x_3 + A_1 \sin s - 2A_1 \cos s (A_2 \cos s + A_1^2 (\frac{1}{2} - \frac{1}{6} \cos 2s))$$

$$\Rightarrow \frac{d^2 x_3}{ds^2} - \omega_2 A_1 \cos s = -\frac{dy_3}{ds} - A_1 \sin s + A_1^3 (2 \cos^2 s \sin s - \sin^3 s)$$

$$= -(-\omega_2 A_1 \cos s + x_3 + A_1 \sin s - 2A_1 \cos s (A_2 \cos s + A_1^2 (\frac{1}{2} - \frac{1}{6}(2\cos^2 s - 1))))$$

$$-A_1 \sin s + A_1^3 \sin s (2(1 - \sin^2 s) - \sin^2 s)$$

$$\frac{d^2 x_3}{ds^2} + x_3 = A_1 (2\omega_2 \cos s - 2 \sin s) + 2A_1 A_2 \cdot \frac{1}{2}(1 + \cos 2s)$$

$$+ A_1^3 (\cos s (\frac{4}{3} - \frac{2}{3} \cos^2 s) + \sin s (2 - 3 \sin^2 s))$$

$$\text{recall}: \cos^3 s = \frac{3}{4} \cos s + \frac{1}{4} \cos 3s \Rightarrow \cos^3 (s + \frac{\pi}{2}) = \frac{3}{4} \cos (s + \frac{\pi}{2}) + \frac{1}{4} \cos 3(s + \frac{\pi}{2})$$

$$\Rightarrow \sin^3 s = \frac{3}{4} \sin s - \frac{1}{4} \sin 3s$$

$$\frac{d^2 x_3}{ds^2} + x_3 = 2A_1 (\omega_2 \cos s - \sin s) + A_1 A_2 (1 + \cos 2s)$$

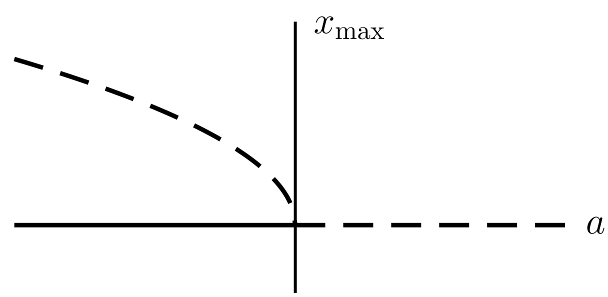
$$+ A_1^3 (\frac{5}{6} \cos s - \frac{1}{6} \cos 3s - \frac{1}{4} \sin s + \frac{3}{4} \sin 3s)$$

$$\text{periodicity} \Rightarrow 2A_1 \omega_2 + \frac{5}{6} A_1^3 = 0\,,\; -2A_1 - \frac{1}{4} A_1^3 = 0 \Rightarrow A_1^2 = -8\,,\; \omega_2 = \frac{10}{3}$$

$$\widetilde{x}(s) = \epsilon A_1 \cos s + O(\epsilon^2)\,,\; \widetilde{y}(s) = \epsilon A_1 \sin s + O(\epsilon^2)$$

$$x(t) = \sqrt{-8a} \cos \omega t + O(a)\,,\; y(t) = \sqrt{-8a} \sin \omega t + O(a)\,,\; \omega = 1 + \frac{10}{3} a + O(a^{3/2})$$

A periodic orbit of amplitude $\sqrt{-8a}$ exists for $a < 0$ (presumably unstable), so $a = 0$ is a subcritical Hopf bifurcation point.



23 Mon 4/10

Poincaré-Bendixson theorem

$\frac{dx}{dt} = F(x,y)\,,\ \frac{dy}{dt} = G(x,y)\,,\ (x,y) \in \mathbb{R}^2$

If $D \subset \mathbb{R}^2$ is a compact invariant set which contains no equilibrium points, then any orbit starting in D is either periodic or converges to a limit cycle.

Hence an attractor of a 2D autonomous system consists of equilibrium points or limit cycles, and nothing more complex like a quasiperiodic orbit on a torus or a chaotic attractor.

example : $\frac{dx}{dt} = x - y - x(x^2+2y^2)\ ,\ \frac{dy}{dt} = x + y - y(x^2+y^2)$

$(0,0)$: equilibrium point , $J = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Rightarrow s = 1 \pm i$: unstable focus

$r^2 = x^2 + y^2 \Rightarrow \cancel{2}r\frac{dr}{dt} = \cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt}$

$$r\frac{dr}{dt} = x(x - \cancel{y} - x(x^2+2y^2)) + y(\cancel{x} + y - y(x^2+y^2))$$

$$= x^2 + y^2 - (x^2+y^2)^2 - x^2y^2 = r^2 - r^4(1 + \cos^2\theta\sin^2\theta)$$

$$\left.\begin{array}{l} r = 1 \Rightarrow \frac{dr}{dt} = -\cos^2\theta\sin^2\theta \le 0 \\ r = \epsilon \Rightarrow \frac{dr}{dt} = \epsilon - O(\epsilon^3) > 0 \text{ for } \epsilon \to 0^+ \end{array}\right\} \Rightarrow \begin{array}{l} D = \{(x,y) : \epsilon \le r \le 1\} \\ \text{is a compact invariant set} \end{array}$$

$$\tan\theta = \frac{y}{x} \Rightarrow \sec^2\theta\,\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2} \Rightarrow r^2\frac{d\theta}{dt} = x\frac{dy}{dt} - y\frac{dx}{dt}$$

$$r^2\frac{d\theta}{dt} = x(x + \cancel{y} - y(\cancel{x^2} + y^2)) - y(\cancel{x} - y - x(\cancel{x^2} + 2y^2)) = r^2 + \frac{1}{2}r^4\sin^2\theta\sin 2\theta$$

$\Rightarrow \frac{d\theta}{dt} \neq 0$ in $D \Rightarrow$ no equilibrium points in D

PB theorem $\Rightarrow$ there exists a periodic orbit or a limit cycle in D

6.6 damped oscillations

example : $\frac{d^2x}{dt^2} + 2\epsilon\frac{dx}{dt} + x = 0\ ,\ 0 < \epsilon < 1$

look for $x(t) = e^{st} \Rightarrow s^2 + 2\epsilon s + 1 = 0 \Rightarrow s = \frac{-2\epsilon \pm \sqrt{4\epsilon^2 - 4}}{2} = -\epsilon \pm i\sqrt{1-\epsilon^2}$

$x(t) = e^{-\epsilon t}(A\cos\sqrt{1-\epsilon^2}t + B\sin\sqrt{1-\epsilon^2}t)$: damped oscillations if we expand in powers of ϵ , then ...

$$e^{-\epsilon t}\cos\sqrt{1-\epsilon^2}t = (1-\epsilon t+\tfrac{1}{2}\epsilon^2t^2+\cdots)(\cos t-\tfrac{1}{2}\epsilon^2\sin t+\cdots)$$

$$= \cos t-\epsilon t\cos t+\tfrac{1}{2}\epsilon^2(t^2\cos t-\sin t)+\cdots \ : \text{ cannot see correct behavior}$$

<u>method of averaging</u>

$$\frac{dx}{dt}=y\,,\ \frac{dy}{dt}=-x-2\epsilon y \ : \text{ 2D autonomous linear system}$$

$$r\frac{dr}{dt}=x\frac{dx}{dt}+y\frac{dy}{dt}=xy+y(-x-2\epsilon y)=-2\epsilon y^2 \Rightarrow \frac{dr}{dt}=-2\epsilon r\sin^2\theta$$

$a(t)$ = average radius at time t for one revolution

$$\frac{da}{dt}\sim -2\epsilon a\cdot\frac{1}{2}=-\epsilon a \Rightarrow a(t)\sim a_0e^{-\epsilon t} \ : \text{ correct behavior}$$

<u>van der Pol equation</u>

$$\frac{d^2x}{dt^2}+\epsilon(x^2-1)\frac{dx}{dt}+x=0 \begin{cases}\epsilon=0: & \text{simple harmonic motion}\\ \epsilon>0: & \text{growing for } x^2<1, \text{decaying for } x^2>1\end{cases}$$

$$\frac{dx}{dt}=y\ ,\ \frac{dy}{dt}=-x-\epsilon(x^2-1)y \ : \text{ 2D autonomous nonlinear system}$$

$$(0,0) : \text{ equilibrium point , } J=\begin{pmatrix}0 & 1\\ -1 & \epsilon\end{pmatrix} \Rightarrow s=\frac{\epsilon\pm i\sqrt{4-\epsilon^2}}{2} \ : \text{ unstable focus}$$

$$r\frac{dr}{dt}=x\frac{dx}{dt}+y\frac{dy}{dt}=\cancel{xy}+y(\cancel{-x}-\epsilon(x^2-1)y)=-\epsilon(r^2\cos^2\theta-1)r^2\sin^2\theta$$

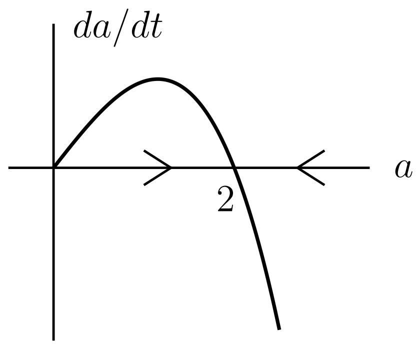
$$\frac{dr}{dt}=-\epsilon r(r^2\cos^2\theta-1)\sin^2\theta$$

<u>case 1</u> : method of averaging

$$(\cos^2\theta\sin^2\theta)_{\text{avg}}=(\frac{1}{4}\sin^2 2\theta)_{\text{avg}}=\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{8}$$

$$(\cos^2\theta)_{\text{avg}}\cdot(\sin^2\theta)_{\text{avg}}=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$$

$$\frac{da}{dt}=-\epsilon a\Big(a^2\cdot\frac{1}{8}-\frac{1}{2}\Big)=\frac{\epsilon}{8}a(4-a^2)\Rightarrow a(t)=2 \text{ is a stable equilibrium}$$



Hence the vdP equation has a limit cycle given by $x(t)\sim 2\cos(t+\theta_0)$ as $\epsilon\to 0$.

case 2 : $\epsilon \to \infty$, ad-hoc method

let $u(t) = \int^t x(s)ds$, so $\dfrac{du}{dt} = x$

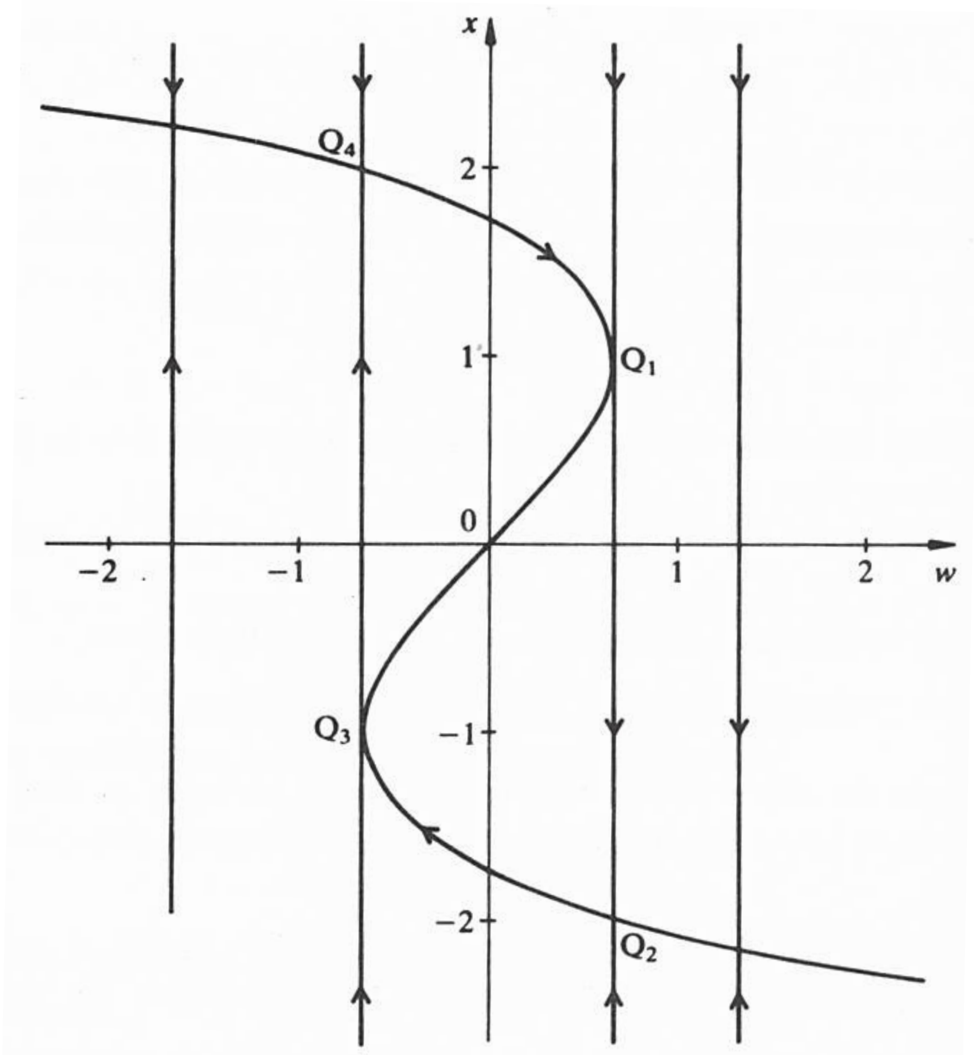
$$\frac{d^2x}{dt^2} + \epsilon(x^2-1)\frac{dx}{dt} + x = 0 \Rightarrow \frac{d^3u}{dt^3} + \epsilon\Big(\Big(\frac{du}{dt}\Big)^2 - 1\Big)\frac{d^2u}{dt^2} + \frac{du}{dt} = 0$$

$$\frac{d^2u}{dt^2} + \epsilon\Big(\frac{1}{3}\Big(\frac{du}{dt}\Big)^3 - \frac{du}{dt}\Big) + u = 0 :$$ Rayleigh equation

$$\frac{du}{dt} = x \,,\ \frac{dx}{dt} = -\epsilon\Big(\frac{1}{3}x^3 - x\Big) - u = -u + \epsilon f(x) = -\epsilon(w - f(x))$$

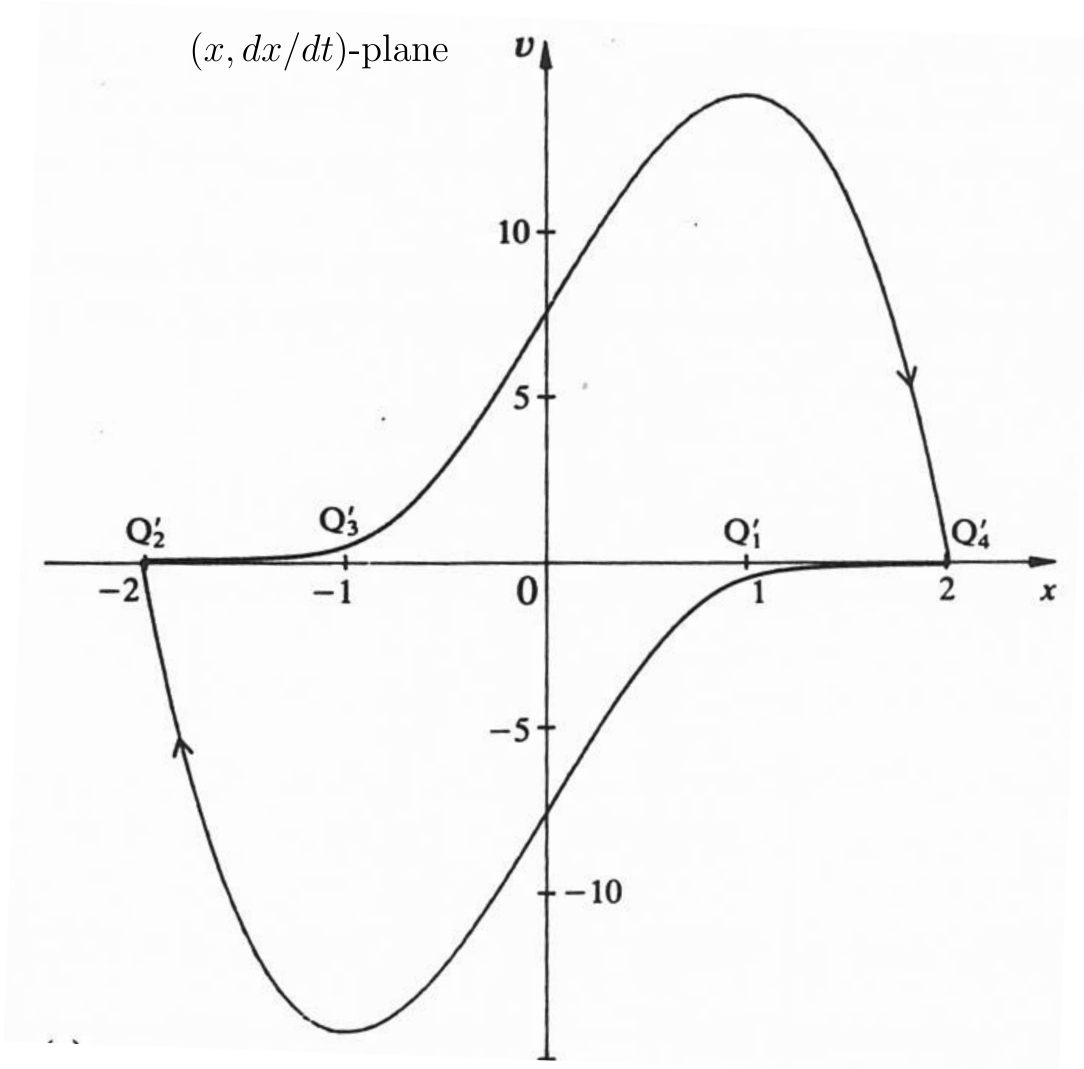
where $f(x) = x - \dfrac{1}{3}x^3$, $u = \epsilon w \Rightarrow \dfrac{dx}{dw} = \dfrac{dx/dt}{dw/dt} = -\epsilon^2\Big(\dfrac{w - f(x)}{x}\Big)$

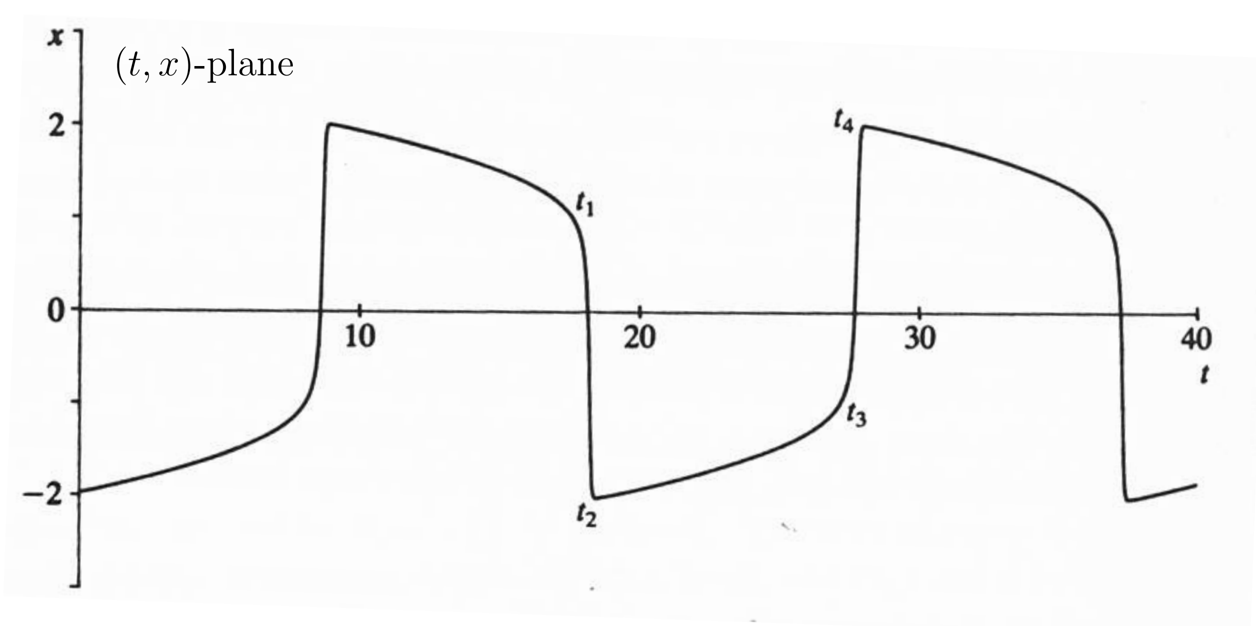
consider the orbits in the (w, x)-plane for $\epsilon \to \infty$



The limit cycle is $Q_1Q_2Q_3Q_4Q_1$, where $x(t)$ changes rapidly on Q_1Q_2 and Q_3Q_4, and slowly on Q_2Q_3 and Q_4Q_1; this is a relaxation oscillation; it did not occur in previous limit cycle examples.

limit cycle for van der Pol's equation : $\epsilon = 10\,, T = 19.08$

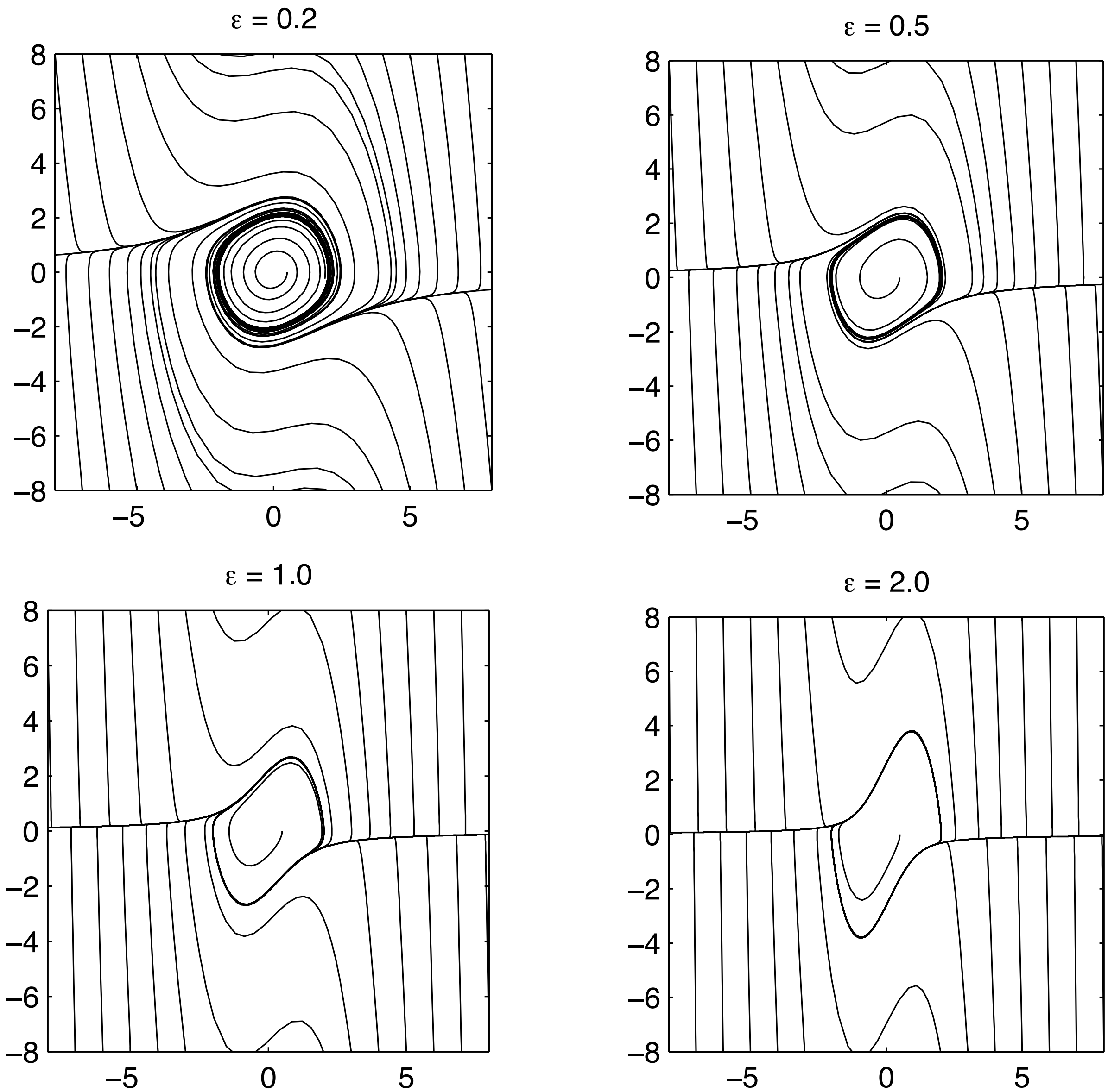




van der Pol equation

$$\frac{d^2x}{dt^2} - \epsilon(x^2 - 1)\frac{dx}{dt} + x = 0$$

$$\frac{dx}{dt} = y \ , \ \frac{dv}{dt} = -\epsilon(x^2 - 1)y - x$$



24 Wed 4/12

7.1 forced oscillations

$$\frac{d^2x}{dt^2} = f(x, \frac{dx}{dt}) + F(t) = \text{restoring/damping force} + \text{external force}$$

$F = 0$: 2D autonomous system $\Rightarrow$ no chaos

$F \neq 0$: 2D non-autonomous system $\Rightarrow$ chaos is possible

example

$$\frac{d^2x}{dt^2} + \Omega^2 x = \Gamma \cos \omega t \ : \ \text{simple harmonic motion + forcing}$$

Ω : natural frequency , ω : forcing frequency

case 1 : $\omega^2 \neq \Omega^2$: non-resonance

$$x(t) = A \cos \Omega t + B \sin \Omega t + \frac{\Gamma \cos \omega t}{\Omega^2 - \omega^2} \ , \ \text{check} \ldots$$

$x(t)$ is quasiperiodic if ω is not a rational multiple of Ω

$$\frac{\Gamma \cos \omega t}{\Omega^2 - \omega^2} \ : \ \text{response} \begin{cases} \text{same frequency and phase as forcing (synchronous)} \\ \text{bounded amplitude for } t > 0 \\ \text{unbounded amplitude for } \omega^2 \to \Omega^2 \end{cases}$$

case 2 : $\omega^2 = \Omega^2$: resonance

$$x(t) = A \cos \Omega t + B \sin \Omega t + \frac{\Gamma t \sin \omega t}{2\omega} \ , \ \text{check} \ldots$$

The response is asynchronous, the amplitude grows linearly in time.

7.2 weakly nonlinear forced oscillations away from resonance

example

$$\frac{d^2x}{dt^2} + \Omega^2 x - \epsilon x^3 = \Gamma \cos \omega t \ : \ \text{Duffing equation} \ , \ \text{assume } \epsilon > 0$$

recall : $\dfrac{d^2x}{dt^2} + \Omega^2 x - \epsilon x^3 = 0$, page 57 , 1 center , 2 saddles

Lindstedt-Poincaré $\Rightarrow$ there are periodic orbits of the form

$$x(t) = a \cos \widetilde{\omega} t + \frac{1}{32} \epsilon a^3 (\cos \widetilde{\omega} t - \cos 3\widetilde{\omega} t) + \cdots \ , \ \widetilde{\omega} = \Omega(1 - \frac{3}{8} \epsilon a^2 + \cdots)$$

where $x(0) = a \, , dx/dt(0) = 0$

We want to find a synchronous response for $\Gamma \neq 0$; for $\epsilon = 0$ this is only possible if $\omega^2 \neq \Omega^2$ (non-resonance), for $\epsilon > 0$ the non-resonance condition is different.

$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots$: regular perturbation series

$$\epsilon^0 : \frac{d^2x_0}{dt^2} + \Omega^2 x_0 = \Gamma \cos \omega t \Rightarrow x_0(t) = \frac{\Gamma \cos \omega t}{\Omega^2 - \omega^2}$$

$$\epsilon^1 : \frac{d^2x_1}{dt^2} + \Omega^2 x_1 = x_0^3 = \frac{\Gamma^3}{(\Omega^2 - \omega^2)^3}\Big(\frac{3}{4}\cos \omega t + \frac{1}{4}\cos 3\omega t\Big)$$

$$x_1(t) = a\cos \omega t + b \cos 3\omega t$$

$$\cos \omega t \ : \ -a\omega^2 + a\Omega^2 = \frac{\frac{3}{4}\Gamma^3}{(\Omega^2 - \omega^2)^3}$$

$$\cos 3\omega t \ : \ -9b\omega^2 + b\Omega^2 = \frac{\frac{1}{4}\Gamma^3}{(\Omega^2 - \omega^2)^3}$$

$$x_1(t) = \frac{\frac{3}{4}\Gamma^3}{(\Omega^2 - \omega^2)^4}\cos \omega t + \frac{\frac{1}{4}\Gamma^3}{(\Omega^2 - \omega^2)^3(\Omega^2 - 9\omega^2)}\cos 3\omega t$$

$$\epsilon^2 : \frac{d^2x_2}{dt^2} + \Omega^2 x_2 = 3x_0^2 x_1 \sim \cos^2 \omega t\{\cos \omega t, \cos 3\omega t\}$$

$x_2(t) = a\cos \omega t + b\cos 3\omega t + c \cos 5\omega t$, solve for $a, b, c \ldots$

expansion is valid for $\omega^2 \neq \Omega^2 \, , \, \frac{\Omega^2}{9} \, , \, \frac{\Omega^2}{25} \, , \, \ldots$: non-resonance condition

---

7.3 weakly nonlinear forced oscillations near resonance

example

$$\frac{d^2x}{dt^2} + \Omega^2 x - \delta x^3 = \Gamma \cos \omega t$$

$s = \omega t$, $\frac{d}{dt} = \omega \frac{d}{ds}$, $\widetilde{x}(s) = x(t)$, drop tilde

$$\frac{d^2x}{ds^2} + \frac{\Omega^2}{\omega^2}x - \frac{\delta}{\omega^2}x^3 = \frac{\Gamma}{\omega^2}\cos s$$

We will show that for certain parameter values near resonance, there is a synchronous response of bounded amplitude.

$\delta/\omega^2 = \epsilon$: weak nonlinearity

$\Gamma/\omega^2 = \epsilon\gamma$: small-amplitude forcing

$\Omega^2/\omega^2 = 1 + \epsilon\beta$: forcing frequency close to natural frequency

$$\frac{d^2x}{ds^2} + x = \epsilon(\gamma \cos s - \beta x + x^3) \ , \ x(s + 2\pi) = x(s)$$

$x(s) = x_0(s) + \epsilon x_1(s) + \cdots$

$\epsilon^0 : \dfrac{d^2x_0}{ds^2} + x_0 = 0 \Rightarrow x_0(s) = a_0 \cos s + b_0 \sin s$

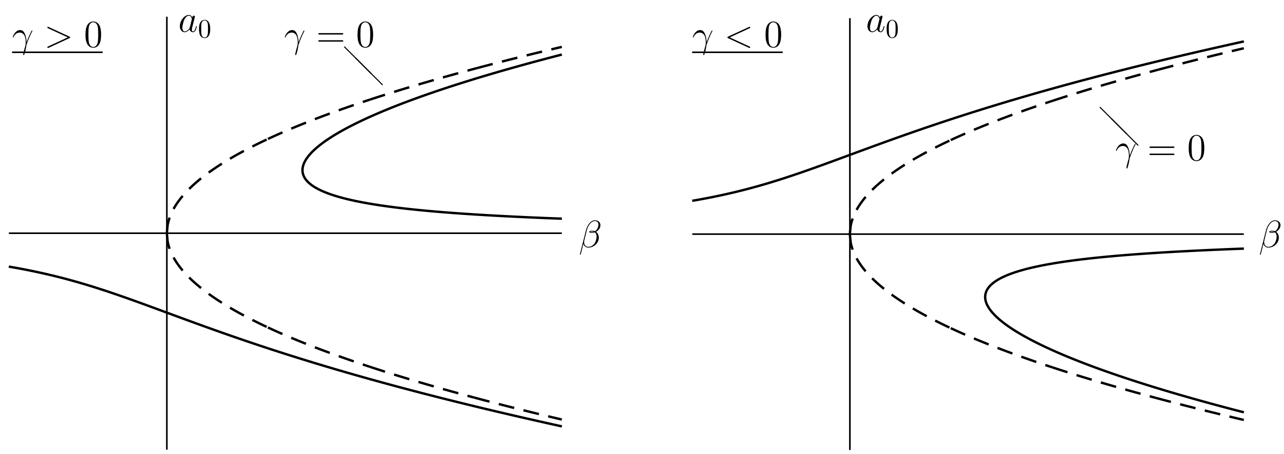
$\epsilon^1 : \dfrac{d^2x_1}{ds^2} + x_1 = \gamma \cos -\beta x_0 + x_0^3$

$$
\begin{aligned}
x_0^3 &= (a_0 \cos s + b_0 \sin s)^3 \\
&= a_0^3 \cos^3 s + 3a_0^2 b_0 \cos^2 \sin s + 3a_0 b_0^2 \cos s \sin^2 s + b_0^3 \sin^3 s \\
&= a_0^3 \cos^3 s + 3a_0^2 b_0 (1 - \sin^2 s) \sin s + 3a_0 b_0^2 \cos s (1 - \cos^2 s) + b_0^3 \sin^3 s \\
&= 3a_0 b_0^2 \cos s + 3a_0^2 b_0 \sin s \\
&+ (a_0^3 - 3a_0 b_0^2)(\tfrac{3}{4} \cos s + \tfrac{1}{4} \cos 3s) + (b_0^3 - 3a_0^2 b_0)(\tfrac{3}{4} \sin s - \tfrac{1}{4} \sin 3s) \\
&= \tfrac{3}{4}(a_0^3 + a_0 b_0^2) \cos s + \tfrac{3}{4}(b_0^3 + a_0^2 b_0) \sin s \\
&+ \tfrac{1}{4}(a_0^3 - 3a_0 b_0^2) \cos 3s - \tfrac{1}{4}(b_0^3 - 3a_0^2 b_0) \sin 3s
\end{aligned}
$$

$$
\begin{aligned}
\frac{d^2x_1}{ds^2} + x_1 &= (\gamma - \beta a_0 + \tfrac{3}{4}(a_0^3 + a_0 b_0^2)) \cos s + (-\beta b_0 + \tfrac{3}{4}(b_0^3 + a_0^2 b_0)) \sin s \\
&+ \tfrac{1}{4}(a_0^3 - 3a_0 b_0^2) \cos 3s - \tfrac{1}{4}(b_0^3 - 3a_0^2 b_0) \sin 3s
\end{aligned}
$$

2π-periodicity $\Rightarrow \begin{cases} a_0(\beta - \frac{3}{4}(a_0^2 + b_0^2)) = \gamma \\ b_0(\beta - \frac{3}{4}(a_0^2 + b_0^2)) = 0 \end{cases}$: <u>amplitude equations</u> 25 Mon 4/17

$\gamma \neq 0 \Rightarrow \beta - \dfrac{3}{4}(a_0^2 + b_0^2) \neq 0 \Rightarrow b_0 = 0\,,\ \beta = \dfrac{3}{4} a_0^2 + \dfrac{\gamma}{a_0}$



$\dfrac{d^2x_1}{ds^2} + x_1 = \dfrac{1}{4} a_0^3 \cos 3s$

$x(s) = a_0 \cos s + \epsilon(a_1 \cos s + b_1 \sin s - \dfrac{1}{32} a_0^3 \cos 3s) + \cdots$

The response is in phase with the forcing (or π radians out of phase); the response amplitude a_0 can take on 1, 2, or 3 possible values depending on β (how close the forcing frequency is to the natural frequency).

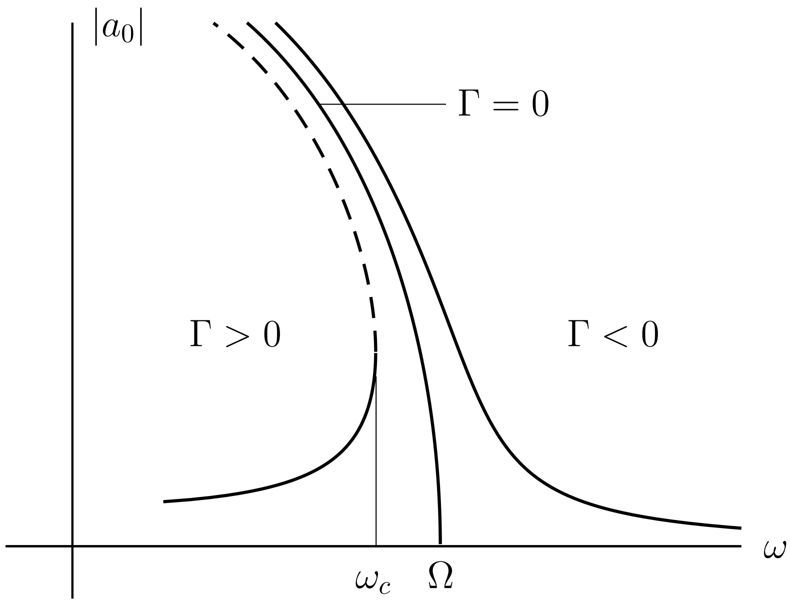
return to original parameters

$$\beta = \frac{3}{4}a_0^2 + \frac{\gamma}{a_0} \Rightarrow \left(\frac{\Omega^2}{\omega^2} - 1\right)\frac{1}{\epsilon} = \frac{3}{4}a_0^2 \cdot \frac{\delta}{\epsilon\omega^2} + \frac{\Gamma}{\epsilon\omega^2} \cdot \frac{1}{a_0}$$

$$\Omega^2 - \omega^2 = \frac{3}{4}\delta a_0^2 + \frac{\Gamma}{a_0} \Rightarrow a_0 = a_0(\Omega, \omega, \delta, \Gamma)$$

note : $\delta = 0 \Rightarrow a_0 = \dfrac{\Gamma}{\Omega^2 - \omega^2}$ <u>ok</u>

<u>$\delta > 0$</u>



$\omega > \omega_c$: 1 response , stable π radians out of phase

$\omega < \omega_c$: 3 responses , only 1 is stable and in phase

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<u>8.1 Lorenz system</u>

<u>8.2 Duffing equation with negative stiffness</u>

8.3 homoclinic chaos

Consider a planar Hamiltonian system with a time-periodic perturbation,

$dx/dt = \vec{F}(x) + \epsilon f(x,t)\,,\; x = (x,y) \in \mathbb{R}^2\,,\; x_0 = x(0)$,

where $F = (H_y, -H_x), H = H(x,y)$ and $f(x, t+T) = f(x,t)$.

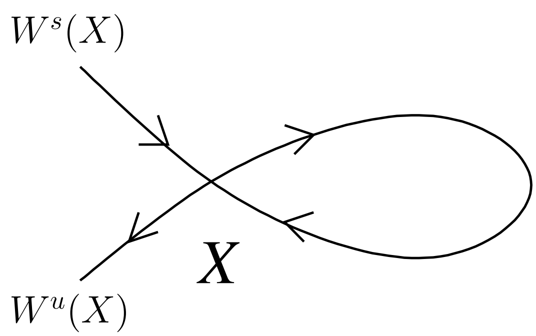
For $\epsilon = 0$ an orbit $x(t)$ cannot cross itself, but for $\epsilon \neq 0$ it can.

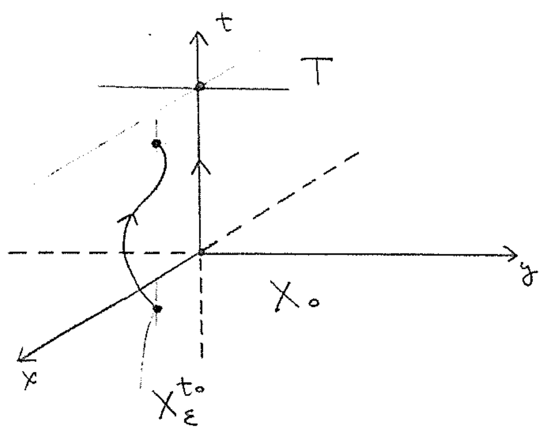
Let X_0 be a saddle point for the unperturbed system.

$W^s(X_0) = \big\{x_0 : x(t) \to X_0 \text{ as } t \to \infty\big\}$: stable manifold of X_0

$W^u(X_0) = \big\{x_0 : x(t) \to X_0 \text{ as } t \to -\infty\big\}$: unstable manifold of X_0

Assume the unperturbed system has a homoclinic orbit $q_0(t)$ such that $q_0(t) \to X_0$ as $t \to \pm\infty$; what happens for $\epsilon \neq 0$?





Consider the Poincaré map for $\epsilon \geq 0$ defined by $x_0 \to x(t_0+T) = P_\epsilon^{t_0}(x_0)$, where $dx/dt = F(x) + \epsilon f(x,t), x(t_0) = x_0$, and assume $P_\epsilon^{t_0}$ has a saddle point $X_\epsilon^{t_0}$, such that $X_\epsilon^{t_0} \to X_0$ as $\epsilon \to 0$. Next consider the iterated map, $x_{n+1} = P_\epsilon^{t_0}(x_n)$, and define analogues of $W^s(X_0), W^u(X_0)$ for $\epsilon > 0$.

$M_\epsilon^s(X_\epsilon^{t_0}) = \{x : (P_\epsilon^{t_0})^n(x) \to X_\epsilon^{t_0} \text{ as } n \to \infty\}$

$M_\epsilon^u(X_\epsilon^{t_0}) = \{x : (P_\epsilon^{t_0})^n(x) \to X_\epsilon^{t_0} \text{ as } n \to -\infty\}$

1. For $\epsilon = 0$ the manifolds W^s, W^u coincide, but for $\epsilon > 0$ the manifolds $M_\epsilon^s, M_\epsilon^u$ may be disjoint or may intersect transversely.

2. Since M^s, M^u are invariant sets, if they intersect once, then they will intersect an infinite number of times.

3. A lobe between M^s and M^u is mapped onto the 2nd lobe ahead. If the system is area-preserving, then as the lobes approach $X_\epsilon^{t_0}$ for $n \to \infty$, they contract along M^s and expand along M^u, and conversely for $n \to -\infty$.

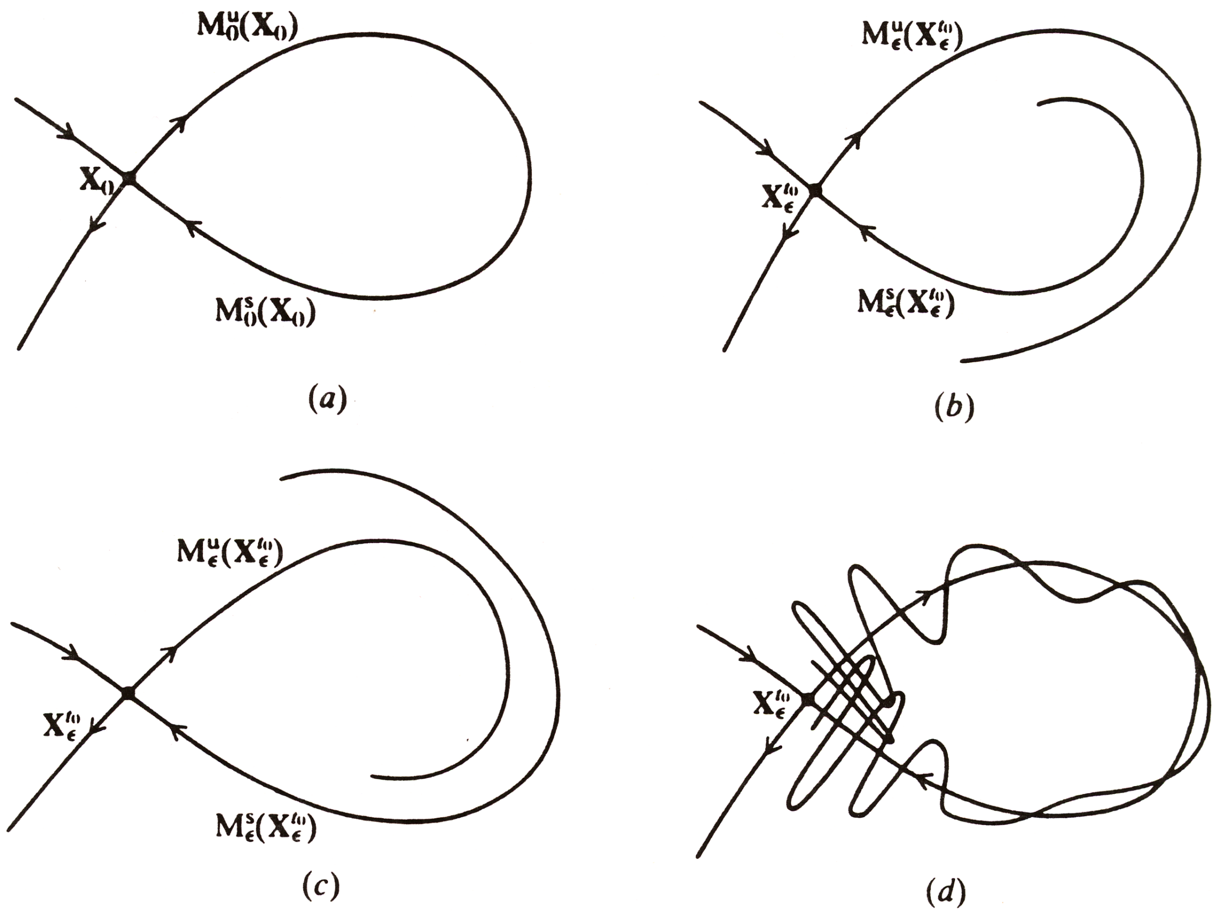
4. The resulting homoclinic tangle contains a Smale horseshoe with ...

a) a countable set of periodic orbits of arbitrarily high period,

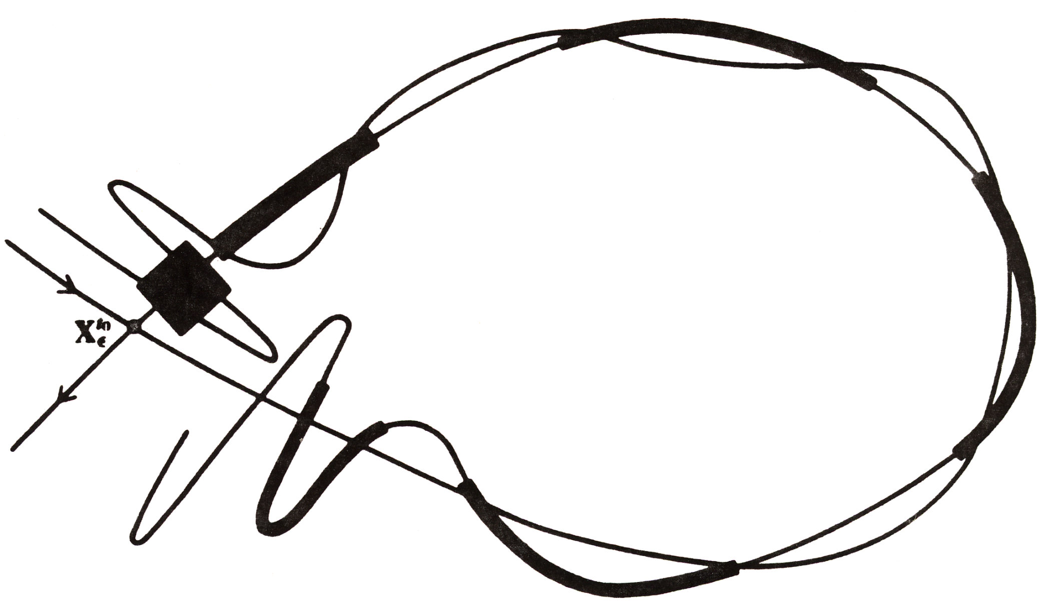
b) an uncountable set of non-periodic orbits,

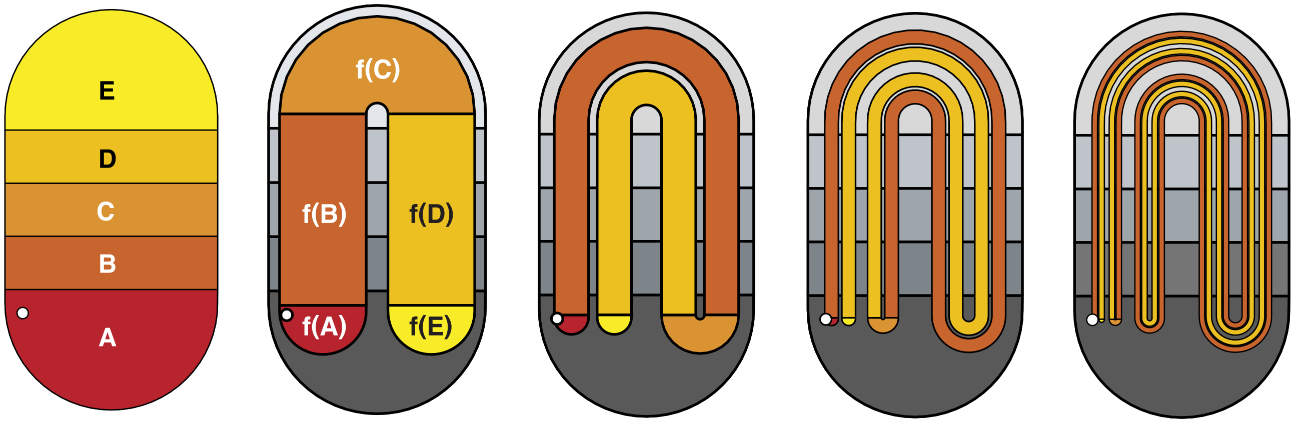
c) a dense chaotic orbit.

Some possible configurations of stable and unstable manifolds of a saddle point of a planar map. (a) homoclinic connection of X_0, (b,c) non-intersecting manifolds of $X_\epsilon^{t_0}$, (d) transversely intersecting manifolds of $X_\epsilon^{t_0} \Rightarrow$ homoclinic tangle.



In case (d) a square starting near $X_\epsilon^{t_0}$ is contracted, stretched, and folded as it enters the homoclinc tangle; contrast that to what happens in the other cases.





from *Picturing the Horseshoe Map*, by W. Casselman (2005) AMS Notices
see also *What is … a Horseshoe?*, by M. Shub (2005) AMS Notices

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We can determine whether or not M^s, M^u intersect transversely without calculating them explicitly using Melnikov's method.

$dx/dt = F(x) + \epsilon f(x,t)$

If $q_0(t)$ is a homoclinic orbit for $\epsilon = 0$, then so is $q_0(t - t_0)$ for any t_0.

Let $q_\epsilon^s(t,t_0), q_\epsilon^u(t,t_0)$ be orbits for $\epsilon \neq 0$ such that

$q_\epsilon^s(t,t_0) \to X_\epsilon^{t_0}$ as $t \to \infty$, $q_\epsilon^s(t_0,t_0) \to q_0(0)$ as $\epsilon \to 0$

$q_\epsilon^u(t,t_0) \to X_\epsilon^{t_0}$ as $t \to -\infty$, $q_\epsilon^u(t_0,t_0) \to q_0(0)$ as $\epsilon \to 0$

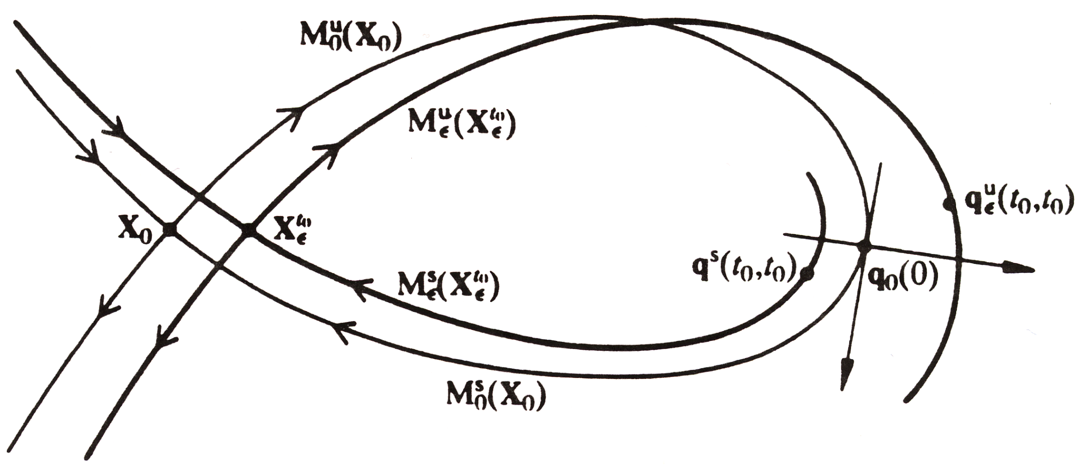


Fig. 8.12 The distance between the orbits on the stable and unstable manifolds near $q_0(0)$ at time t_0.

now define $d(t_0) = q_\epsilon^u(t_0,t_0) - q_\epsilon^s(t_0,t_0)$: hard to compute

consider instead $D(t_0) = d(t_0) \cdot n(t_0)$: Melnikov distance between M_ϵ^u and M_ϵ^s

$n(t_0) = \dfrac{F(q_0(0))^\perp}{|F(q_0(0))|}$, where $a^\perp = (a_1, a_2)^\perp = (-a_2, a_1)$: unit normal to $q_0(0)$

Expand $D(t_0)$ for $\epsilon \to 0$.

$q_\epsilon^s(t, t_0) = q_0(t - t_0) + \epsilon q_1^s(t, t_0) + \cdots$ for $t \geq t_0$

$q_\epsilon^u(t, t_0) = q_0(t - t_0) + \epsilon q_1^u(t, t_0) + \cdots$ for $t \leq t_0$

$\Rightarrow\ d(t_0) = q_\epsilon^u(t_0, t_0) - q_\epsilon^s(t_0, t_0) = \epsilon(q_1^u(t_0, t_0) - q_1^s(t_0, t_0)) + \cdots$

Derive differential equations for $q_1^s(t, t_0), q_1^u(t, t_0)$.

$$\frac{d}{dt} q_\epsilon^s(t, t_0) = F(q_\epsilon^s(t, t_0)) + \epsilon f(q_\epsilon^s(t, t_0), t)$$

$$\cancel{\frac{d}{dt} q_0(t - t_0)} + \epsilon \frac{d}{dt} q_1^s(t, t_0) + \cdots$$

$$= F(q_0(t - t_0) + \epsilon q_1^s(t, t_0) + \cdots) + \epsilon f(q_0(t - t_0) + \cdots, t)$$

$$= \cancel{F(q_0(t - t_0))} + J(q_0(t - t_0)) \cdot \epsilon q_1^s(t, t_0) + \cdots + \epsilon f(q_0(t - t_0), t) + \cdots$$

$$\frac{d}{dt} q_1^s(t, t_0) = J(q_0(t - t_0)) q_1^s(t, t_0) + f(q_0(t - t_0), t) \quad , \quad \text{similarly for } q_1^u$$

$$D(t_0) = d(t_0) \cdot n(t_0)$$

$$= \epsilon(q_1^u(t_0, t_0) - q_1^s(t_0, t_0)) \cdot \frac{F(q_0(0))^\perp}{|F(q_0(0))|} + \cdots$$

$$= \epsilon \frac{F(q_0(0)) \wedge (q_1^u(t_0, t_0) - q_1^s(t_0, t_0))}{|F(q_0(0))|} + \cdots$$

$a \wedge b = (a_1, a_2) \wedge (b_1, b_2) = a_1 b_2 - a_2 b_1 = (b_1, b_2) \cdot (-a_2, a_1) = b \cdot a^\perp$

define $\Delta^s(t, t_0) = F(q_0(t - t_0)) \wedge q_1^s(t, t_0)$

$$\frac{d}{dt} \Delta^s(t, t_0) = J(q_0(t - t_0)) F(q_0(t - t_0)) \wedge q_1^s(t, t_0)$$
$$+ F(q_0(t - t_0)) \wedge (J(q_0(t - t_0)) q_1^s(t, t_0) + f(q_0(t - t_0), t))$$

$(Ja) \wedge b + a \wedge (Jb) = \operatorname{tr} J \cdot a \wedge b$

<u>proof</u> : $Ja = (J_{11}a_1 + J_{12}a_2, J_{21}a_1 + J_{22}a_2)$

$$(J_{11}a_1 + \cancel{J_{12}a_2}) b_2 - (\cancel{J_{21}a_1} + J_{22}a_2) b_1 + a_1(\cancel{J_{21}b_1} + J_{22}b_2) - a_2(J_{11}b_1 + \cancel{J_{12}b_2})$$

$$= J_{11}(a_1 b_2 - a_2 b_1) + J_{22}(a_1 b_2 - a_2 b_1) \quad \underline{\text{ok}}$$

$$F = (H_y, -H_x) \ \Rightarrow\ J = \begin{pmatrix} H_{xy} & H_{yy} \\ -H_{xx} & -H_{xy} \end{pmatrix} \ \Rightarrow\ \operatorname{tr} J = 0$$

$$\frac{d}{dt}\Delta^s(t,t_0) = F(q_0(t-t_0)) \wedge f(q_0(t-t_0),t)$$

$$\Delta^s(\infty,t_0) - \Delta^s(t,t_0) = \int_t^\infty F(q_0(r-t_0)) \wedge f(q_0(r-t_0),r)dr$$

$$\Delta^s(t_0,t_0) = \cancel{\Delta^s(\infty,t_0)} - \int_{t_0}^\infty F(q_0(t-t_0)) \wedge f(q_0(t-t_0),t)dt$$

$$\Delta^s(\infty,t_0) = \lim_{t\to\infty} F(q_0(t-t_0)) \wedge q_1^s(t,t_0) = F(X_0) \wedge q_1^s(\infty,t_0) = 0$$

similarly $\Delta^u(t_0,t_0) = \int_{-\infty}^{t_0} F(q_0(t-t_0)) \wedge f(q_0(t-t_0),t)dt$ - hw

$$D(t_0) = \frac{\epsilon(\Delta^u(t_0,t_0) - \Delta^s(t_0.t_0))}{|F(q_0(0))|} + \cdots = \frac{\epsilon M(t_0)}{|F(q_0(0))|} + \cdots$$

$M(t_0) = \int_{-\infty}^{\infty} F(q_0(t-t_0)) \wedge f(q_0(t-t_0),t)dt$: <u>Melnikov function</u>

$M(t_0)$ has a simple root

$\Rightarrow D(t_0)$ has a simple root nearby

$\Rightarrow M^u, M^s$ intersect transversely

$\Rightarrow$ homoclinic tangle, Smale horseshoe, chaos

<u>example</u> : Duffing equation

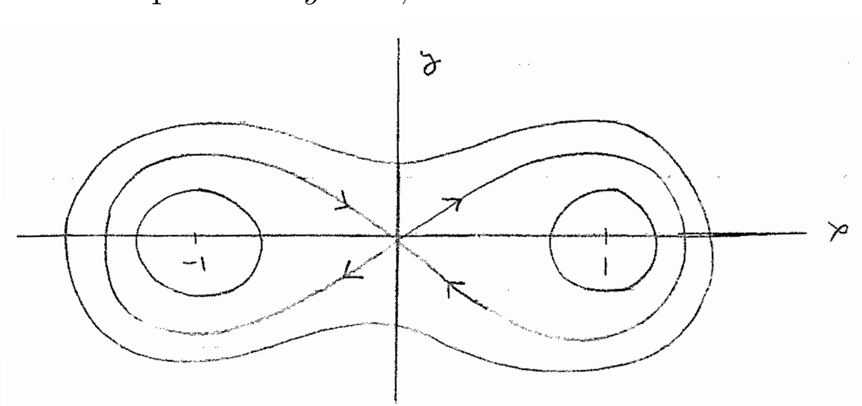
$$\frac{d^2x}{dt^2} + \epsilon\delta\frac{dx}{dt} - x + x^3 = \epsilon\gamma\cos\omega t : \begin{cases}\text{negative stiffness}\\ \text{weak damping}\\ \text{weak periodic forcing}\end{cases}$$

first-order system : $\dfrac{dx}{dt} = y$, $\dfrac{dy}{dt} = x - x^3 + \epsilon(\gamma\cos\omega t - \delta y)$

$H(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$: Hamiltonian for $\epsilon = 0$

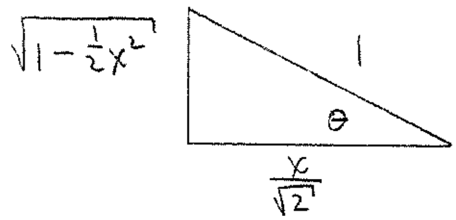
equilibrium points : $y = 0$, $x = 0$ or $x = \pm 1$

pha



homoclinic orbit : $H(x,y)=0 \Rightarrow y^2 = x^2\left(1-\frac{1}{2}x^2\right)$, we need $x(t), y(t)$

$\frac{dx}{dt} = x\sqrt{1-\frac{1}{2}x^2} \Rightarrow \frac{dx}{x\sqrt{1-\frac{1}{2}x^2}} = dt$, note : $y=0 \Rightarrow x = 0\,,\ \pm\sqrt{2}$



$\cos\theta = \frac{x}{\sqrt{2}}\,,\ -\sin\theta d\theta = \frac{dx}{\sqrt{2}}\,, \sin\theta = \sqrt{1-\frac{1}{2}x^2}$

$$\frac{-\sqrt{2}\sin\theta d\theta}{\sqrt{2}\cos\theta\sin\theta} = dt \Rightarrow \frac{d\theta}{\cos\theta} = -dt$$

$$\frac{d\theta}{\cos\theta} = \frac{\cos\theta d\theta}{\cos^2\theta} = \frac{\cos\theta d\theta}{1-\sin^2\theta} \quad , \text{ set } u = \sin\theta\,,\ du = \cos\theta d\theta$$

$$= \frac{du}{1-u^2} = \left(\frac{1/2}{1-u} + \frac{1/2}{1+u}\right) du = -dt$$

$$\Rightarrow \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right) = -t + C \Rightarrow \frac{1+u}{1-u} = Ce^{-2t}$$

choose $x=\sqrt{2}$ at $t=0 \Rightarrow \theta = 0 \Rightarrow u = 0 \Rightarrow C = 1$

$$1+u = (1-u)e^{-2t} \Rightarrow u(1+e^{-2t}) = e^{-2t} - 1$$

$$u = \frac{e^{-2t}-1}{e^{-2t}+1} = \frac{e^{-t}-e^{t}}{e^{-t}+e^{t}} = -\tanh t = \sin\theta = \sqrt{1-\tfrac{1}{2}x^2}$$

$$\tanh^2 t = 1 - \frac{1}{2}x^2 \Rightarrow x^2 = 2(1-\tanh^2 t) = 2\,\mathrm{sech}^2 t$$

$$x = \sqrt{2}\,\mathrm{sech}\, t\ ,\ y = \frac{dx}{dt} = -\sqrt{2}\,\mathrm{sech}\, t\cdot\tanh t$$

$$q_0(t) = (x,y) = \sqrt{2}\,\mathrm{sech}\, t(1, -\tanh t)$$

$$F(q_0(t)) = (y, x-x^3)\ ,\ f(q_0(t)) = (0, \gamma\cos\omega t - \delta y)$$

$$F(q_0(t-t_0)) \wedge f(q_0(t-t_0), t) = y(t-t_0)(\gamma\cos\omega t - \delta y(t-t_0))$$

$$M(t_0) = \int_{-\infty}^{\infty} y(t-t_0)(\gamma\cos\omega t - \delta y(t-t_0))dt \quad , \text{ set } s = t - t_0$$

$$= \int_{-\infty}^{\infty} y(s)(\gamma\cos\omega(s+t_0) - \delta y(s))ds$$

$$= \int_{-\infty}^{\infty} y(s)(\gamma(\cos \omega s \cos \omega t_0 - \sin \omega s \sin \omega t_0) - \delta y(s))ds$$

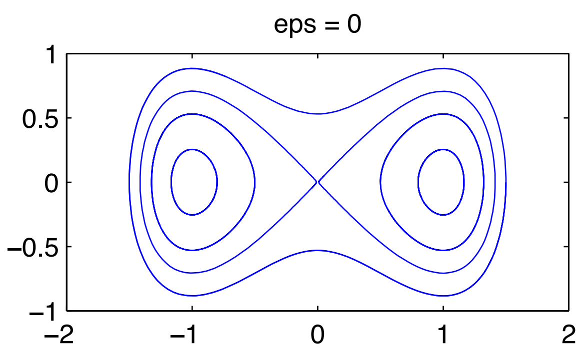
$$= -\sin \omega t_0 \cdot -\sqrt{2}\gamma \int_{-\infty}^{\infty} \operatorname{sech} s \tanh s \cdot \sin \omega s ds - \delta \cdot 2 \int_{-\infty}^{\infty} \operatorname{sech}^2 s \tanh^2 s \, ds$$

$$= \sqrt{2}\pi\gamma\omega \operatorname{sech}\frac{\pi\omega}{2} \sin \omega t_0 - \frac{4}{3}\delta$$

$M(t_0) = 0 \Leftrightarrow \sqrt{2}\pi\gamma\omega \operatorname{sech}\frac{\pi\omega}{2} > \frac{4}{3}\delta \Leftrightarrow$ forcing dominates damping

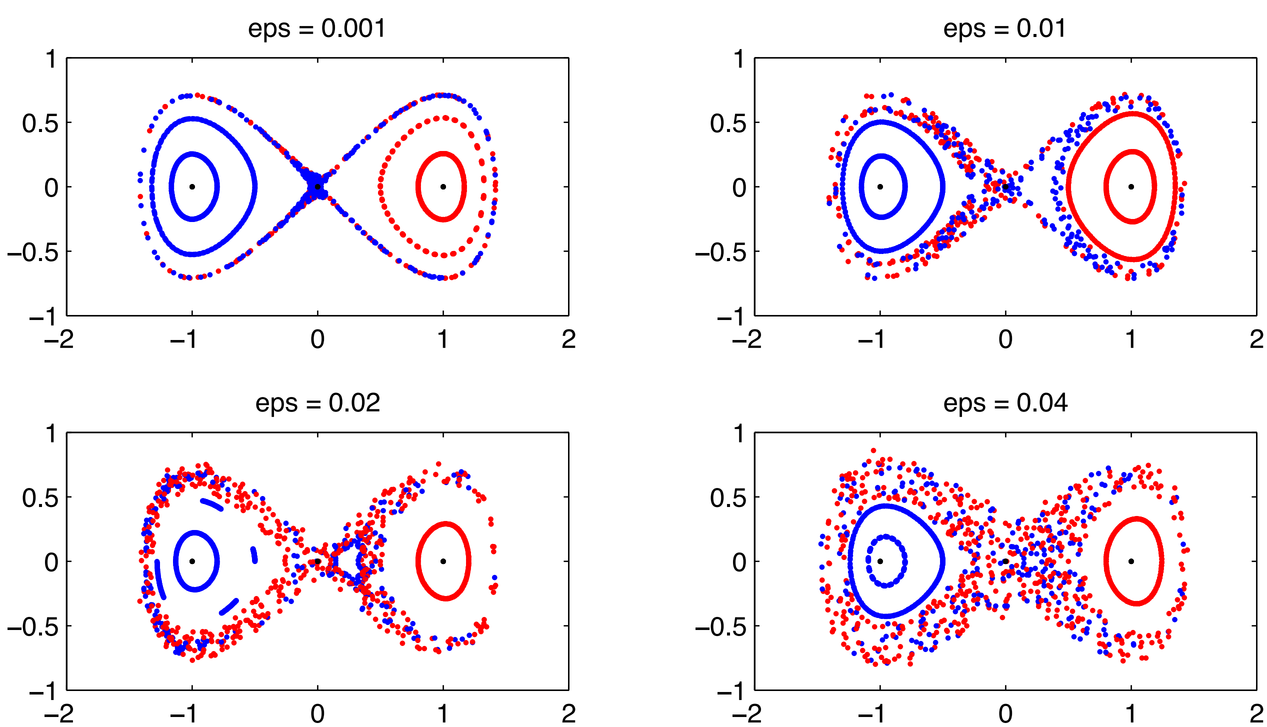
<u>example</u>

$\frac{d^2x}{dt^2} - x + x^3 = \epsilon \cos t$: Duffing equation, zero damping, weak periodic forcing



set $\epsilon = 0.001, 0.01, 0.02, 0.04$, take 250 iterations of Poincaré map

$x_0 = \pm 0.01, \pm 0.5, \pm 0.8, \pm 1.5, y_0 = 0$, $x_0 < 0$ is blue, $x_0 > 0$ is red



conclusion : particle transport, mixing, chaos