Math 571 Numerical Linear Algebra Winter 2021

hw1 , due: Tuesday, February 9, 10am

Please write neatly. Upload the solutions as a PDF into Gradescope.

0. (optional) Give a brief description of your academic background and research interests. If you work in a lab or research group, please give your supervisor's name and describe your project. One paragraph is fine.

1. page 15 (orthogonal vectors and matrices) 2.2a , 2.3 , 2.6

2. page 24 (norms) 3.1 , 3.2

3. a) Let λ be an eigenvalue of an orthogonal matrix. Show that $|\lambda| = 1$.

b) A permutation matrix is a square matrix whose elements are either 0 or 1, such that every row and column has exactly one nonzero element. Such matrices arise by permuting the rows or columns of the identity matrix. It is easy to see that the rows and columns of a permutation matrix are orthonormal vectors. This implies that a permutation matrix is orthogonal and hence from part (a) its eigenvalues satisfy $|\lambda| = 1$. Give an example of a 4×4 permutation matrix whose eigenvalues are $\lambda = \{\pm 1, \ \pm i\}$.

4. Consider the rotation matrix, $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Show that R_θ is orthogonal and find its eigenvalues.

5. Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Solve $Ax = b$ by the spectral method discussed in class.

6. Prove the following statements.

a) Given a vector norm $||x||$, the formula $||A|| = \sup_{x \neq 0} \frac{||Ax||}{||x||}$ defines a matrix norm; this is the induced matrix norm corresponding to the given vector norm.

b) $||A||_\infty = \max_i \sum_j |a_{ij}|$

hint: make sure you understand the derivation of the formula for $||A||_1$ given in class

7. Consider the 2-point BVP, $-y'' + (4x^2 + 2)y = 2x(1 + 2x^2)$, $0 \le x \le 1$, $y(0) = 1$, $y(1) = 1 + e$. Show that $y(x) = x + \exp(x^2)$ is the exact solution. Write a Matlab code to solve the problem using the 2nd order finite-difference scheme discussed in class, $-D_+D_-u_i + d_iu_i = f_i$, $u_0 = y(0), u_{n+1} = y(1)$. Use mesh size $h = 1/2^p = 1/(n+1)$, where p is a positive integer. Solve the linear system using the tridiagonal LU method derived in class (you get partial credit if you use a full matrix; to get full credit you should use only vectors). Turn in the program listing. For $p = 1:4$, plot the exact solution ($y(x)$ vs. x) and the numerical solution (u_i vs. x_i, including the boundary points). For $p = 1:25$, present a table with the following data. column 1: p, column 2: h, column 3: $||u_h - y_h||_\infty$, column 4: $||u_h - y_h||_\infty/h^2$, column 5: cpu time, column 6: (cpu time)$/n$. Discuss and explain the trends in each column.

hints:

a) Debug your code using small values of p, e.g. $p \le 10$; otherwise the cpu time is too large.

b) Type `help cputime` in Matlab to learn how to find the cpu time.

c) Try different output formats, e.g. `format long`, `format short e`, `format short g`.