Math 571 Numerical Linear Algebra Winter 2021

hw2 , due: Tuesday, February 16, 10am

1. page 30 (singular value decomposition) 4.1 , 4.4

note : in problem 4.1 the s-values should appear in decreasing magnitude on the diagonal of Σ; in problem 4.4 you need to check two statements, (1) unitarily equivalent $\Rightarrow$ same s-values, and (2) same s-values $\Rightarrow$ unitarily equivalent

2. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Find $||A||_p$ for $p = 1,\ 2,\ \infty$.

3. Prove.

a) $||x||_\infty \le ||x||_2 \le ||x||_1$

b) If A is hermitian with eigenvalues $\lambda_1 \ge \cdots \ge \lambda_n$, then $\lambda_n = \min\limits_{x \ne 0} R_A(x)$.

c) If U is unitary, then $||Ux||_2 = ||x||_2$, $||U||_2 = 1$, $||UA||_2 = ||A||_2 = ||AU||_2$.

d) If $||A|| < 1$ for some induced matrix norm, then $I + A$ is invertible.

4. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$.

a) Use Matlab's command `svd` to find the SVD of A. State U, Σ, V (4-digit format is fine).

b) In one plot draw the unit circle C and indicate the vectors v_1, v_2, and in another plot draw the ellipse AC (i.e. the image of the circle under the map $x \to Ax$), and indicate the vectors $Av_1 = \sigma_1 u_1$, $Av_2 = \sigma_2 u_2$. Use the `axis('equal')` command in Matlab to ensure that the horizontal and vertical axes have the same scale.

c) Find a vector x such that $\dfrac{||Ax||_2}{||x||_2} = ||A||_2$.

d) Find A_1, the best rank-1 approximation of A in the 2-norm. Find $||A - A_1||_2$.

5. Let $f(x) = \sin x$ and take $x = \frac{\pi}{4}$, so that $f'(x) = f'(\frac{\pi}{4}) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} = 0.70710678$.

a) The forward difference approximation for $f'(x)$ is $D_+f(x) = \dfrac{f(x+h) - f(x)}{h}$.

Present a table in the format below; take $h = 0.1, 0.05, 0.025, 0.0125$; the first line for $h = 0.1$ is shown and you must fill in the entries for the remaining values of h. Indicate the limit of each column, as we did in class. You may prepare the table by hand or use a word processor.

h	$D_+f(x)$	$\lvert f'(x) - D_+f(x)\rvert$	$\dfrac{\lvert f'(x) - D_+f(x)\rvert}{h}$	$\dfrac{\lvert f'(x) - D_+f(x)\rvert}{h^2}$	$\dfrac{\lvert f'(x) - D_+f(x)\rvert}{h^3}$
0.1	0.67060297	0.03650381	0.3650381	3.650381	36.50381

In class we showed that $D_+f(x) = f'(x) + \frac{1}{2}f''(x)h + \cdots$, which implies that $D_+f(x)$ is 1st order accurate (i.e. the truncation error is $O(h)$). Is the table consistent with this?

b) The centered difference approximation for $f'(x)$ is $D_0f(x) = \dfrac{f(x+h) - f(x-h)}{2h}$.

Show that $D_0f(x) = f'(x) + ch^2 + \cdots$ for some contant c, which is independent of h.

c) Present a table for $D_0f(x)$ in the same format as part (a). What is the order of accuracy of $D_0f(x)$? For a given value of h, which approximation is more accurate, $D_+f(x)$ or $D_0f(x)$?

d) Modify the Matlab code given in class so that it plots the error in $D_+f(x)$ and $D_0f(x)$ for step size $h = 1/2^{(j-1)}$ with $j = 1 : 65$. Use log scales for the error $|f'(x) - Df(x)|$ and the step size h. Plot both cases on the same graph (to do this in Matlab, type `hold on` after the first `loglog` command). Explain the results.