Math 571 Numerical Linear Algebra Winter 2021

hw3 , due: Tuesday, March 2, 10am

1. page 47 (projectors) 6.4 , 6.5

2. page 55 (QR factorization) 7.3 , also draw in picture in 2D to illustrate this result

3. page 68 (Matlab) 9.2 , 9.3

note : in 9.2, the bound you obtain should imply that $\sigma_m \to 0$ as $m \to \infty$

extra instructions for 9.3:

i. in part (b), when you give the mathematically exact rank, justify your answer mathematically, i.e. give a reason that does not rely on using Matlab

ii. in part (c), plot the results on one page in a 3×4 matrix format using `subplot`

iii. use the following commands to display an image of a matrix B

```
map = [1.  1.  1.; .75 .75 .75; .5 .5 .5; .25 .25 .25; 0 0 0];
colormap(map)
pcolor(B)
```

iv. `pcolor` flips the matrix; you need to unflip it

4. On hw1 (problem 3.1, page 24) it was shown that if $||x||$ is a vector norm and W is a nonsingular matrix, then $||x||_W = ||Wx||$ is a new vector norm. Let $||A||$ be the matrix norm induced by the original vector norm $||x||$, and let $||A||_W$ be the matrix norm induced by the new vector norm $||x||_W$. Show that $||A||_W = ||WAW^{-1}||$.

5. Let $A \in \mathbb{C}^{m \times n},\ m \geq n$, rank $A = n$.

a) Show that A^*A is invertible.

b) Let $P = A(A^*A)^{-1}A^*$. Show that $P^2 = P$, $P^* = P$, range P = range A (you must show that range $P \subseteq$ range A and range $A \subseteq$ range P); hence P is the orthogonal projector onto range A.

c) Show that $P = \hat{Q}\hat{Q}^*$, where $A = \hat{Q}\hat{R}$ is the reduced QR factorization of A.

6. In class we used the classical Gram-Schmidt method to compute the reduced QR factorization

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{6} \\ 0 & 2/\sqrt{6} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} \end{pmatrix}.$$

a) Find the orthogonal projector P onto range A.

b) Apply Householder's method to compute a full QR factorization of A. Carry out the steps by hand; do not use decimal approximations; express all results exactly using square roots.

7. Let $A \in \mathbb{C}^{m \times n},\ m \geq n$, rank $A = n$.

a) Show that the operation count for mGS is $2mn^2$, the same as for cGS.

b) Let $m = n = 3$ and $A = QR$. In class we showed that mGS can be expressed in matrix form, $AR_1R_2R_3 = Q$, with $(R_1R_2R_3)^{-1} = R$, where R_i, R are upper triangular matrices with positive diagonal elements. Show that cGS can also be expressed in matrix form for some other choice of R_1, R_2, R_3.

announcement. The midterm exam is on Thursday March 4 and attendance in the Zoom class meeting is mandatory. The exam sheet will be distributed in the chat box and your solutions should be uploaded into Gradescope at the end of the exam. The exam will cover everything up to and including the Tuesday March 2 class. Calculators are not allowed. You may use one page (one side) of handwritten or typed notes, but do not photocopy the lecture notes.