Math 571 Numerical Linear Algebra Winter 2021

hw6 , due: Tuesday, April 13

1. Let $A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$. Apply Gaussian elimination with partial pivoting to find P, L, U such that $PA = LU$. Write out all the steps, and in each step, pivot on the largest element below the diagonal.

2. Let $A \in \mathbb{C}^{m\times m}$ be hermitian. Show that the following statements are equivalent by proving $a \Rightarrow b \Rightarrow c \Rightarrow d \Rightarrow e \Rightarrow a$.

a) The eigenvalues of A are positive.

b) A is positive definite.

c) Δ_k is positive definite for $k = 1, \ldots, m$, where $\Delta_k = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}$.

d) $\det \Delta_k > 0$ for $k = 1, \ldots, m$

e) A has a Cholesky factorization.

3. Let $A = \begin{pmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{pmatrix}$.

a) Show that A is positive definite using one of the criteria in problem 2.

b) Find the Cholesky factorization of A. Write out all the steps.

4. page 177 (Cholesky factorization) 23.3
Repeat the computations using $m = 5000$ instead of $m = 200$, and record the elapsed times. For each part (a) to (g), answer questions (i) and (ii) in the problem statement.

5. Let A be the 5-point discrete Laplacian on the unit square with $h = \frac{1}{4}$, so that A is the 9×9 matrix on the bottom of page 53 in the notes. Note that A is a band matrix, and furthermore it is sparse within the band. Suppose that Gaussian elimination without pivoting is carried out, and assume that the L, U factors are stored in the original matrix A. Write down the matrix A and draw a box around each zero element that gets filled in during the elimination. (Hint: this requires no computation.) In class we saw that Gaussian elimination preserves the bandwidth of a band matrix; this exercise shows that sparsity within the band may be lost; in this case we say that fill-in occurs within the band.

6. Let $u(x, y)$ be the temperature in a square plate that is heated on one side and cooled on the other three sides. The temperature satisfies the Laplace equation $u_{xx} + u_{yy} = 0$ for $(x, y) \in (0, 1) \times (0, 1)$, with Dirichlet boundary conditions $u(x, 1) = 1, u(0, y) = u(1, y) = u(x, 0) = 0$.

a) A file called bvp2d on the 571 Canvas site contains part of a Matlab m-file to solve for $u(x, y)$ using the 5-point discrete Laplacian with mesh size $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$. The code creates the matrix and right-hand side; your assignment is to write a function that solves the linear system by Cholesky factorization using the algorithm on page 55 of the notes; after performing the Cholesky factorization, you may use any method to solve the triangular systems. The numerical solution will be displayed as contour and surface plots. Submit a printout of the completed m-file and the contour and surface plots.

b) Print out the numerical solution $u_h(\frac{3}{4}, \frac{3}{4})$, i.e. the temperature at the point $x = \frac{3}{4}, y = \frac{3}{4}$, for each value of h. Print the results using `format long`. On the basis of these results, does the numerical solution converge as $h \to 0$? What is the order of convergence? Justify your answer.