Math 571 Numerical Linear Algebra Winter 2021

hw7 , due: Monday, April 26 at 5pm

1. Gershgorin's theorem
If λ is an eigenvalue of $A = (a_{ij})$, then there exists an index i such that $|\lambda - a_{ii}| \le \sum_{\substack{j=1\\ j\neq i}}^{m} |a_{ij}|$.
a) Prove the theorem.
b) Gershgorin's theorem shows that the eigenvalues of a matrix lie in circles in the complex plane. Plot the eigenvalues and circles for the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}$ using Matlab.

2. Bauer-Fike theorem
Assume A is diagonalizable with $A = XDX^{-1}$, where $D = \text{diag}(\lambda_i)$, and let α be an eigenvalue of a perturbed matrix $A + \delta A$. Then there exists an index i such that $|\alpha - \lambda_i| \le ||\delta A||_2 \cdot \kappa_2(X)$.
a) Prove the theorem.
b) Let $A = \begin{pmatrix} 1 & \epsilon^{-1} \\ \epsilon & 1 \end{pmatrix}$, $A + \delta A = \begin{pmatrix} 1 & \epsilon^{-1} \\ 0 & 1 \end{pmatrix}$, where $\epsilon = 10^{-2}$.
Find $\lambda_1, \lambda_2, \alpha_1, \alpha_2, ||\delta A||_2, \kappa_2(X)$, and verify the theorem. You may use Matlab.

3. Which of the following matrices are normal? For those that are normal find a unitary diagonalization, and for those that are not normal find a Schur factorization. Do the calculations by hand. (You may check your results using the Matlab functions `eig, schur`.)
a) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ b) $\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ c) $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$ d) $\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$

Problems 4a and 5 refer to the matrix $A = \begin{pmatrix} 8 & 4 & 2 & 1 \\ 4 & 8 & 4 & 2 \\ 2 & 4 & 8 & 4 \\ 1 & 2 & 4 & 8 \end{pmatrix}$.

4. (a) In class we derived an algorithm for reducing a general $m \times m$ matrix A to upper Hessenberg form by unitary similarity transformations with operation count $\frac{10}{3}m^3$. It was stated that if A is real symmetric, then the reduced matrix is tridiagonal and the algorithm can be modified so that the operation count is $\frac{4}{3}m^3$. Derive the latter version of the algorithm (see pages 199-200 of the textbook for a hint) and indicate which steps in the algorithm lead to the $\frac{4}{3}m^3$ operation count. Code both versions in Matlab (general, real symmetric), apply the codes to the matrix A, and print the final result in each case. Discuss the results.
(b) A random symmetric matrix can be created by the commands `A = randn(m); A = A'*A;`. Run the 2 codes from (a) for a sequence of random symmetric matrices with $m = 100:100:1000$, and present a table with the following; column 1: m, column 2: code 1 cpu time, column 3: column $2/m^3$, column 4: code 2 cpu time, column 5: column $4/m^3$, column 6: column 3/column 5. Are the cpu time results consistent with the theoretical operation counts? Can you propose an explanation for any differences between the theory and the actual results?

5. Here we will compute the e-values of A using various methods. Omit phase 1, i.e. do not reduce A to tridiagonal form. Present the results in a table with `format long`. Discuss and explain the results using relevant theorems from class.
a) Apply the power method, inverse iteration (unshifted, i.e. $\mu = 0$), and Rayleigh quotient iteration to A. You may use backslash to solve the linear systems. For each method, start with $v^{(0)} = (1,1,1,1)^T/2$, take 10 steps, and present the iterates $\lambda^{(k)}$, $k = 0:9$ in a table. Do the methods converge to the expected e-value?

b) Take 10 steps of the QR algorithm starting with $A^{(0)} = A$. You may use the Matlab `qr` command. Print the maximum magnitude of the off-diagonal elements at each step and also print the final matrix.

6. Let A be a 4×4 real symmetric tridiagonal matrix and consider the QR factorization $A = QR$,

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{12} & a_{22} & a_{23} & 0 \\ 0 & a_{23} & a_{33} & a_{34} \\ 0 & 0 & a_{34} & a_{44} \end{pmatrix} = \begin{pmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{21} & q_{22} & q_{23} & q_{24} \\ q_{31} & q_{32} & q_{33} & q_{34} \\ q_{41} & q_{42} & q_{43} & q_{44} \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ 0 & r_{22} & r_{23} & r_{24} \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & 0 & r_{44} \end{pmatrix}.$$

a) Copy the matrix equation, and draw a circle around the elements of Q and the upper triangular elements of R which must be zero. You may experiment with Matlab, but when you claim that an element is zero, give a reason to justify your answer.

b) When the QR algorithm is used to compute the e-values and e-vectors of a matrix $A = QR$, the first iteration produces the matrix $B = RQ$. Assuming that A is symmetric and tridiagonal, show that B is also symmetric and tridiagonal.

Note: This example can be generalized to the $m \times m$ case, and it justifies the claim made in class that the QR algorithm preserves the structure of a symmetric tridiagonal matrix. This implies that the work required to compute the QR factorization in each step can be reduced. This justifies using phase 1 in which a real symmetric matrix is reduced to tridiagonal form by unitary similarity transformations.

<u>announcement</u> : The final exam is on Friday, April 30 at 10:30am-12:30pm on Zoom. It will cover the entire course up to the material presented on Tuesday April 20. Calculators are not allowed. You may use one page of notes (both sides) handwritten or typed, but do not photocopy the lecture notes.