Math 572 Numerical Methods for Differential Equations Winter 2019

additional problems

1. Consider the scalar wave equation, $v_t + cv_x = 0$, discretized by the Lax-Friedrichs scheme,
$\dfrac{u_j^{n+1} - \frac{1}{2}(u_{j+1}^n + u_{j-1}^n)}{k} + cD_0 u_j^n = 0$, where c may be positive or negative.

a) Find the amplification factor.

b) Show that the scheme is stable in the 2-norm if $|c|\lambda \le 1$. Is this also true for the ∞-norm?

c) Recall that the truncation error τ_j^n is defined by $\dfrac{v_j^{n+1} - \frac{1}{2}(v_{j+1}^n + v_{j-1}^n)}{k} + cD_0 v_j^n = \tau_j^n$, where v_j^n is the exact solution. Show that $\tau_j^n = \frac{h}{2\lambda}(c^2\lambda^2 - 1)(v_{xx})_j^n + O(k^2)$.

d) Show that the scheme is 1st order accurate in the 2-norm, i.e. $||v^n - u^n||_2 = O(k)$.

e) Show that the model equation is $\phi_t + c\phi_x = \frac{h}{2\lambda}(1 - c^2\lambda^2)\phi_{xx}$, i.e. $||\phi^n - u^n||_2 = O(k^2)$.

2. Consider the scalar wave equation $v_t + cv_x = 0$ with $c > 0$. For all the schemes considered in class, the numerical wave speed has an expansion of the form $\widetilde{c} = c(1 + \gamma_1 \xi h + \gamma_2 (\xi h)^2) + \cdots)$ in the long wave limit $\xi h \to 0$. Find γ_1, γ_2 for the upwind and Lax-Friedrichs schemes (Lax-Wendroff was done in class). Which scheme has the smallest error in the wave speed?

Announcement

The final exam is on Thursday, May 2, 10:30am-12:30pm in 3866 East Hall. The exam will cover the entire course, but PDEs will be emphasized. You may use two pages of notes (i.e. $2 \times 8.5\,\text{in} \times 11\,\text{in}$). Calculators are not allowed.