MATH 572 Numerical Methods for Differential Equations Winter 2024

hw1 , due: Tuesday, January 30 at 4pm

Scan the solutions using a scanning app; upload as a PDF into Gradescope through Canvas.

0. (optional) Give a brief description of your academic background and scientific interests. What is your major and what year are you in? If you work in a lab or research group, please give your supervisor's name and describe your project.

1. Let $f(x) = \sin x$ and take $x = \frac{\pi}{4}$, so that $f'(x) = f'(\frac{\pi}{4}) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} = 0.70710678$.

a) The forward difference approximation for $f'(x)$ is $D_+f(x) = \dfrac{f(x+h)-f(x)}{h}$.

Present a table in the format below; take $h = 0.1, 0.05, 0.025, 0.0125$; the first line for $h = 0.1$ is given and you must fill in the remaining entries. Indicate the limit of each column, as we did in class. You may prepare the table by hand or use a word processor.

h	$D_+f(x)$	$\lvert f'(x) - D_+f(x)\rvert$	$\dfrac{\lvert f'(x) - D_+f(x)\rvert}{h}$	$\dfrac{\lvert f'(x) - D_+f(x)\rvert}{h^2}$	$\dfrac{\lvert f'(x) - D_+f(x)\rvert}{h^3}$
0.1	0.67060297	0.03650381	0.3650381	3.650381	36.50381

In class we showed that $D_+f(x) = f'(x) + \frac{1}{2}f''(x)h + \cdots$, which implies that $D_+f(x)$ is 1st order accurate (i.e. the truncation error is $O(h)$). Is your table consistent with this?

b) The central difference approximation for $f'(x)$ is $D_0f(x) = \dfrac{f(x+h)-f(x-h)}{2h}$.

Show that $D_0f(x) = f'(x) + ch^2 + \cdots$ for some contant c which is independent of h.

c) Present a table for $D_0f(x)$ in the same format as part (a). What is the order of accuracy of $D_0f(x)$? For a given h, which approximation is more accurate, $D_+f(x)$ or $D_0f(x)$?

d) Modify the Matlab code given in class to plot the error in $D_+f(x)$ and $D_0f(x)$ for step size $h = 1/2^{(j-1)}$ with $j = 1 : 65$. Use log scales for the error $|f'(x) - Df(x)|$ and the step size h. Plot both cases on the same graph. Explain the results.

2. Consider the problem $y' = ay + b$, $y(0) = y_0$, and let u_n be the numerical solution at time $t = nh$ given by Euler's method. Find an expression for u_n, compute $\lim_{n\to\infty} u_n$, and show that the limit satisfies the differential equation and initial condition.

3. Consider the problem $y' = -y^2$, $y(0) = 1$.

a) Compute $y(1)$ using Euler's method with time step $h = 0.1$, 0.05, 0.025, 0.0125. Present the results in a table with the following format.

column 1: h, column 2: u_n, column 3: $u_n - y_n$, column 4: $\dfrac{u_n - y_n}{h}$

b) Find the principal error function $E(t)$ and evaluate $E(1)$. Compare with column 4.

c) Perform Richardson extrapolation on the values in column 2.

4. The numerical solution of the problem $y' = f(y)$, $y(0) = y_0$ by Euler's method has an asymptotic expansion of the form $u_n = y_n + hE_n + h^2D_n + O(h^3)$ as $h \to 0$, where $E_n = E(t_n)$, $D_n = D(t_n)$ for certain functions $E(t)$, $D(t)$.

a) The differential equation for $E(t)$ was derived in class; now find the equation for $D(t)$.

b) Consider the problem $y' = y$, $y_0 = 1$. In class we showed that $E(t) = -\frac{1}{2}te^t$. Find $D(t)$.

c) Compute $y(1)$ using Euler's method with time step $h = 0.1$, 0.05, 0.025, 0.0125. Present the results in a table with the format below. Evaluate $E(1), D(1)$ and compare with the table.

column 1: h, column 2: u_n, column 3: $u_n - y_n$, column 4: $\dfrac{u_n - y_n}{h}$, column 5: $u_n - (y_n + hE_n)$, column 6: $\dfrac{u_n - (y_n + hE_n)}{h^2}$