Math 572 Numerical Methods for Differential Equations Winter 2024

hw2 , due : Tuesday , February 13 at 4pm

1. Let $y' = f(y)$, where $f(y)$ satisfies a Lipschitz condition, $|f(u) - f(v)| \le L|u - v|$, and consider the midpoint method, $u_{n+1} = u_n + hf\Big(u_n + \frac{h}{2}f(u_n)\Big)$.

a) In class it was shown that an explicit 1-step method of the form $u_{n+1} = u_n + hF(u_n, h)$ converges if $F(u, h)$ satisfies two conditions: (a) $F(u, 0) = f(u)$, (b) $|F(u, h) - F(v, h)| \le \widetilde{L}|u-v|$. Find $F(u, h)$ for the midpoint method and show that it satisfies these two conditions.

b) Now consider the midpoint applied to the test equation $y' = \lambda y$ and assume λ is a negative real number. Show that $h\lambda$ is contained in the region of absolute stability if and only if $-2 \le h\lambda \le 0$.

c) Show that no purely imaginary value of $h\lambda$ is contained in the region of absolute stability.

2. This problem asks for the local truncation error of a numerical method; give the answer in the form $\tau_n = cy_n^{(r+1)}h^{r+1} + O(h^{r+2})$; i.e. you need to find the constants c and r.

a) Find the local truncation error of the backward Euler method, $u_{n+1} = u_n + hf(u_{n+1})$.

b) Find the local truncation error of the trapezoid method, $u_{n+1} = u_n + \frac{h}{2}\Big(f(u_n) + f(u_{n+1})\Big)$, and show that the scheme is A-stable.

3. Consider the initial value problem $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} -11 & 9 \\ 9 & -11 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_0 = \begin{pmatrix} 1.0 \\ 1.2 \end{pmatrix}$.

a) Find the first component of the exact solution $y_1(t)$ and the corresponding numerical solutions $u_{1,n}$ given by the forward Euler method and the backward Euler method. In each case show that $\lim_{h\to 0} u_{1,n} = y_{1,n}$ at a fixed time $t = t_n = nh$.

b) Compute the numerical solution for $0 \le t \le 2$ using forward Euler and backward Euler with time step $h = 0.11,\ 0.10,\ 0.09$. In each case plot the first component of the exact solution $y_1(t)$ and the numerical solution $u_{1,n}$ versus time. Display the results as on page 17 of the notes (in Matlab use the `subplot` command). Discuss the results. Which cases are absolutely stable and which are not absolutely stable?

4. Consider the initial value problem $y'' + y = 0, y(0) = 1, y'(0) = 0$ with exact solution $y(t) = \cos t$ and note that it is a circle of radius one in phase space. Express the equation as a first order linear system,

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Solve the system using (a) forward Euler, (b) backward Euler, (c) RK4, up to time $t = 4\pi$ with time step $h = 4\pi/100$. Plot the numerical solution in phase space, i.e. plot $u_{1,n}$ on the x-axis and $u_{2,n}$ on the y-axis. Present the results for all three methods in a single plot which is correctly scaled to ensure that a circle appears as a circle rather than an ellipse (this can be done in Matlab using the `axis equal` command). Which of the three methods gives the best results? Why? Does absolute stability play a role?