Math 572 Numerical Methods for Differential Equations Winter 2024

hw3 , due : Tuesday, February 27 at 4pm

1. a) Find the coefficients $\gamma_i$, $\beta_i$ in the 4-step Adams-Bashforth method,

$$u_{n+1} = u_n + h(\gamma_0 f_n + \gamma_1 \nabla f_n + \gamma_2 \nabla^2 f_n + \gamma_3 \nabla^3 f_n),$$
$$u_{n+1} = u_n + h(\beta_0 f_n + \beta_1 f_{n-1} + \beta_2 f_{n-2} + \beta_3 f_{n-3}).$$

b) Find the coefficients $\gamma_i^*$, $\beta_i^*$ in the 3-step Adams-Moulton method,

$$u_{n+1} = u_n + h(\gamma_{-1}^* f_{n+1} + \gamma_0^* \nabla f_{n+1} + \gamma_1^* \nabla^2 f_{n+1} + \gamma_2^* \nabla^3 f_{n+1}),$$
$$u_{n+1} = u_n + h(\beta_{-1}^* f_{n+1} + \beta_0^* f_n + \beta_1^* f_{n-1} + \beta_2^* f_{n-2}).$$

c) Which method is more accurate? Give a reason to justify your answer.

2. Solve the difference equation for $u_n$.

a) $u_n - 2u_{n-1} - 3u_{n-2} = 0$ , $u_0 = 0$ , $u_1 = 1$

b) $u_n - 3u_{n-1} + 3u_{n-2} - u_{n-3} = 0$ , $u_0 = 1$ , $u_1 = 0$ , $u_2 = -3$

3. Consider the following 2-step methods for $y' = f(y)$,

scheme 1 : $u_n - u_{n-2} - \frac{h}{3}\Big(f(u_n) + 4f(u_{n-1}) + f(u_{n-2})\Big) = 0$ (Milne's method)

scheme 2 : $u_n + 4u_{n-1} - 5u_{n-2} - h\Big(4f(u_{n-1}) + 2f(u_{n-2})\Big) = 0$

Answer the following questions for each scheme.

a) Find $\rho(\zeta)$, $\sigma(\zeta)$. Is the scheme consistent? Is the root condition satisfied?

b) Find the local truncation error in the form $\tau_n = Cy^{(r+1)}(t)h^{r+1} + O(h^{r+2})$.

In parts (c) and (d), consider the test equation, $y' = \lambda y$.

c) Find the characteristic roots $\zeta_1(h)$, $\zeta_2(h)$ and show that they satisfy

scheme 1 : $\zeta_1(h) = e^{h\lambda} + O(h^5)$ , $\zeta_2(h) = -e^{-h\lambda/3} + O(h^3)$,

scheme 2 : $\zeta_1(h) = e^{h\lambda} + O(h^4)$ , $\zeta_2(h) = -5e^{-3h\lambda/5} + O(h^2)$.

d) Take $y(0) = 1$ for the exact solution and $u_0 = 1$, $u_1 = e^{h\lambda}$ for the numerical solution. Derive the following expressions for $u_n$. Compute $\lim_{n\to\infty} u_n$ for fixed $t = nh$. Are the schemes convergent? Explain.

scheme 1 : $u_n = (1 + O(h^5)) \cdot (e^{\lambda t} + O(h^4)) + O(h^5) \cdot ((-1)^n e^{-\lambda t/3} + O(h^2))$

scheme 2 : $u_n = (1 + O(h^4)) \cdot (e^{\lambda t} + O(h^3)) + O(h^4) \cdot ((-5)^n e^{-3\lambda t/5} + O(h))$

4. Consider the van der Pol equation, $y'' + (y^2 - 1)y' + y = 0$, an oscillator with nonlinear damping. The key feature of the equation is that it has a limit cycle, i.e. an attracting periodic solution. Express the equation as a first-order system, take $y(0) = 0.1$, $y'(0) = 0.1$ as initial data, and compute the numerical solution on the interval $0 \le t \le 40$ using the following schemes.

a) 2-step Adams-Bashforth

b) a predictor-corrector scheme using 2-step Adams-Bashforth as the predictor and one iteration of 1-step Adams-Moulton (trapezoid method) as the corrector; write out the equations

Take $h = 0.3$, $0.15$, $0.075$. Use the exact initial data and one step of Euler's method for the starting values of the numerical solution. Plot the numerical solution two ways: (1) in physical space ($y$ versus $t$), (2) in the phase plane ($y'$ versus $y$). Discuss the results. How do the two methods compare? What accounts for the difference between the two methods?

**Announcement**. The midterm exam is on Thursday, March 7 in class; it will cover everything up to the last lecture on ODEs; no calculators; 1 page of notes (1 side) is allowed; no photocopies of lecture notes.