Math 572 Numerical Methods for Differential Equations Winter 2024

hw4 , due : Thursday, March 21 at 4pm

1. Consider the 2-step BDF method, $\nabla u_n + \frac{1}{2}\nabla^2 u_n = hf(u_n)$.

a) Find the characteristic roots $\zeta_1(h), \zeta_2(h)$ for the test equation $y' = \lambda y$ and plot their magnitudes over the interval $-10 \leq h\lambda \leq 0$ using software (e.g. Matlab, Python, your choice).

b) Show analytically that the negative real axis is contained in the region of absolute stability.

note 1 : part (a) is meant to help with part (b)

note 2 : the scheme is actually A-stable, but showing that is not required here.

2. The Lorenz system is $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}' = \begin{pmatrix} \sigma(y_2 - y_1) \\ ry_1 - y_2 - y_1 y_3 \\ y_1 y_2 - by_3 \end{pmatrix}$.

The system was originally derived as a low-dimensional model of thermal convection, where the variables represent the temperature, density, and velocity in a fluid flow. It was discovered by numerical computations that the parameters $\sigma = 10, b = 8/3, r = 28$ yield chaotic dynamics. Compute the solution with these parameter values and initial conditions $y_1(0) = 0, y_2(0) = 1, y_3(0) = 0$ up to time $t = 100$. If you know Matlab, use the `ode45` solver; part of the problem is to read the online documentation and figure out how this command works. If you don't know Matlab, then use an alternative ODE solver of your choice, but describe it. Plot the projection of the orbit in the $y_1 y_3$-plane; the object displayed there is a strange attractor; it should look like the wings of a butterfly.

(Optional, not to be submitted) The full orbit in $y_1 y_2 y_3$-space can be seen in Matlab using `plot3`; use the rotate tool to get a better sense of what the orbit really looks like.

For a discussion of the background to all this, click here.

3. Consider the heat equation $v_t = v_{xx}$ on $-\infty < x < \infty$ with initial condition $v(x, 0) = f(x)$. No explicit boundary conditions are imposed as $x \to \pm\infty$, but we assume there is a unique solution $v(x, t)$. Suppose the equation is discretized by the finite-difference scheme, $u_j^{n+1} = u_j^n + kD_+D_-u_j^n$ with $u_j^0 = f(x_j)$. Show that the numerical solution has an asymptotic expansion, $u_j^n = v_j^n + kE_j^n + O(k^2)$ for $k \to 0$, where $v_j^n = v(x_j, t_n)$ is the exact solution, $E_j^n = E(x_j, t_n)$ is the principal error function, $x_j = jh, t_n = nk$, and $\lambda = k/h^2 \leq 1/2$. There are two steps: (a) derive the equation satisfied by $E(x, t)$, (b) prove the validity of the expansion. (hint: recall the proof for Euler's method applied to $y' = f(y)$ on page 7 of the notes). Something special happens to the error $u_j^n - v_j^n$ when $\lambda = 1/6$; what is it?

4. Compute the solution of the heat equation $v_t = v_{xx}$ on $0 \leq x \leq 1$ with boundary conditions $v(0, t) = v(1, t) = 0$ and initial condition (a) $v(x, 0) = 1 - 2|x - \frac{1}{2}|$, (b) $v(x, 0) = \sin \pi x$. Use forward Euler in time and 2nd order central differencing in space with $h = 0.05$ and two different values of the time step $k = 0.0013, 0.0012$. Plot the solution at time $t = 0, k, 25k, 50k$. (Note that case (a) reproduces a result in the notes). Explain the results.