Math 572 Numerical Methods for Differential Equations Winter 2024

hw5 , due : Thursday April 4 at 4pm

1. Consider the difference scheme $\dfrac{u_j^{n+1}-u_j^n}{k} = D_+D_-u_j^n - \frac{1}{12}h^2(D_+D_-)^2u_j^n$ for the heat equation $v_t = v_{xx}$ on $-\infty < x < \infty$.

a) Show that the local truncation error is $O(k^p)+O(h^q)$; find p, q.

b) Find the amplification factor $\rho(\xi h)$.

c) For which $\lambda = k/h^2$ is the scheme stable in the 2-norm?

2. Given u_j, $j = 0, \pm 1, \ldots$, define $\hat{u}(\xi h) = \sum_{j=-\infty}^{\infty} u_j e^{ij\xi h}$. Verify the following formulas.

a) $u_j = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \hat{u}(\xi h)e^{-ij\xi h}d(\xi h)$, b) $\displaystyle\sum_{j=-\infty}^{\infty} |u_j|^2 = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} |\hat{u}(\xi h)|^2 d(\xi h)$

3. Consider the difference scheme $u_j^{n+1} = u_j^n + kD_+D_-u_j^{n+1}$ for the heat equation $v_t = v_{xx}$, corresponding to backward Euler in time and 2nd order central differencing in space. Assume free-space boundary conditions on $-\infty < x < \infty$.

a) Find the amplification factor $\rho(\xi h)$; show that $0 \le \rho(\xi h) \le 1$ for all ξh; plot $\rho(\xi h)$ for $0 \le \xi h \le \pi$ and $\lambda = 1/2, 1, 10$. Show that the scheme is unconditionally stable in the 2-norm using b) Fourier analysis, c) energy method.

4. Consider the heat equation $v_t = v_{xx}$ on the interval $0 \le x \le 1$ with zero Dirichlet boundary conditions $v(0,t) = v(1,t) = 0$ and initial condition $v(x,0) = 1 - 2|x - \frac{1}{2}|$. On page 39 of the notes the problem was solved using forward Euler in time and 2nd order central differencing in space with grid spacing $h = 0.05$ and time step $k = 0.0013$. Now solve the problem by the Crank-Nicolson method with the same values of h, k as before. Solve the linear system at each time step by the tridiagonal LU factorization scheme discussed in class. Compare the Crank-Nicolson and forward Euler results and explain any differences.

5. Consider the heat equation $v_t = v_{xx}$ on $0 \le x \le 1$ with Neumann boundary conditions, $v_x(0,t) = v_x(1,t) = 0$. In this case the boundary values $v(0,t)$ and $v(1,t)$ vary in time, but there is zero flux through the boundaries. Consider the method of lines, $u_j' = D_+D_-u_j$ for $j = 0 : N$, $Nh = 1$; note that for $j = 0$ this calls for u_{-1}^n which falls outside the domain; in this case use a central difference for the boundary condition, $D_0u_0^n = (u_1^n - u_{-1}^n)/2h = 0$, which yields $u_{-1}^n = u_1^n$; use a similar treatment for $j = N$.

a) Write the system in the form $u' = Au$, where $u = (u_0, \ldots, u_N)^T$. (hint: A has a constant diagonal.) Find the eigenvalues and eigenvectors of A. Note that this is not the same matrix A that we examined in class for the case of Dirichlet boundary conditions.

b) Plot the eigenvectors for the case $N = 16$ and wavenumbers $m = 0, 1, 2, 4, 8, 16$ using a similar format as on page 42 of the notes.

c) Suppose forward Euler is used to solve $u' = Au$. Show that the scheme is absolutely stable if $\lambda \le 1/2$, where $\lambda = k/h^2$.

d) The heat equation with Neumann boundary conditions also describes the diffusion of gas in a closed container, where $v(x,t)$ is the gas density at location x and time t. Consider the initial condition $v(x,0) = \begin{cases} 0 \text{ if } 0 \le x < \frac{1}{2}, \\ 1 \text{ if } \frac{1}{2} < x \le 1, \end{cases}$ and set $v(\frac{1}{2}, 0) = \frac{1}{2}$. The initial condition defines a state in which the left half of the container is a vacuum and the right half contains gas of uniform density. Solve the problem numerically using the scheme described above with $h = 0.05$ and $k = 0.001275, 0.001225$. For each k plot the results at time $t = 0, 25k, 50k, 100k$. Explain the results. What is the long-time asymptotic state of the system?