Math 572 Numerical Methods for Differential Equations Winter 2024

hw6 , due : Tuesday, April 23 at 4pm

1. Consider the 2D heat equation, $v_t = v_{xx} + v_{yy}$, on the unit square, $0 \le x, y \le 1$, with Dirichlet boundary conditions, $v(0, y, t) = v(1, y, t) = v(x, 0, t) = 0$, $v(x, 1, t) = 1$, and initial condition $v(x, y, 0) = 0$. The solution $v(x, y, t)$ represents the temperature of a square plate that is heated on one side and cooled on the other three sides. Solve the problem numerically using the forward Euler/central difference scheme $u^{n+1} = u^n + k(D_+^x D_-^x + D_+^y D_-^y)u^n$ with $h = 0.1$ and $k = 0.0025$. Is the scheme stable for this choice of parameters? Make a contour plot and a surface plot of the numerical solution at time $t = 2$ (including the boundary values); the relevant commands in Matlab are `contour` and `mesh` (or `surf`) or use the equivalent in Python. Note that the boundary conditions are discontinuous at two corners of the plate, $(x, y) = (0, 1)$ and $(x, y) = (1, 1)$; the boundary conditions at these points are not used in the numerical scheme (do you see why?), but for plotting purposes set the values to $v(0, 1, t) = v(1, 1, t) = 1$.

2. Fritz John wrote in his textbook on partial differential equations, "Instability of a difference scheme under small perturbations does not exclude the possibility that in special cases the scheme converges towards the correct function, if no errors are permitted in the data or the computation." He gave the following example to illustrate this idea.

a) Consider the wave equation $v_t + cv_x = 0$, $v(x, 0) = f(x)$, $c > 0$. Show that the numerical solution given by the upwind scheme can be expressed as

$$u_j^n = ((1 - c\lambda)I + c\lambda S_-)^n f_j = \sum_{\ell=0}^{n} \binom{n}{\ell} (1 - c\lambda)^{n-\ell}(c\lambda)^\ell f_{j-\ell},$$

and the numerical solution given by the downwind scheme can be expressed as

$$u_j^n = ((1 + c\lambda)I - c\lambda S_+)^n f_j = \sum_{\ell=0}^{n} \binom{n}{\ell} (1 + c\lambda)^{n-\ell}(-c\lambda)^\ell f_{j+\ell}.$$

b) Let $f(x) = e^{\alpha x}$ and take $t = t_n = nk$, $x = x_j = jh$ with $\lambda = k/h$ fixed. Using the formulas derived above for each scheme show that the numerical solution u_j^n converges to the correct value $v(x, t) = e^{\alpha(x-ct)}$ as $n \to \infty$ for any value of λ.

Hence the numerical solution converges even when the scheme is unstable. Fritz John noted that despite appearances, this result is consistent with the CFL condition since an analytic function (like $f(x) = e^{\alpha x}$) is determined by its values in any interval. It can be further noted that in computing the solution of a physically unstable problem (e.g. Kelvin-Helmholtz instability of a shear layer), it is necessary to use an unstable scheme to ensure consistency with the physics, but one must guard against the introduction of spurious perturbations whether from finite precision arithmetic or the scheme itself.

3. Consider the wave equation $v_t + v_x = 0$ with two cases of initial data $v(x, 0)$,

$$f_1(x) = \begin{cases} 1 \ , & x < 0 \\ 0 \ , & x = 0 \\ -1 \ , & x > 0 \end{cases} \quad , \quad f_2(x) = \begin{cases} -1 \ , & x < 0 \\ 1 - 2|x - 1| \ , & 0 \le x \le 2 \\ -1 \ , & x > 2 \end{cases} .$$

Compute the solution for $-1 \le x \le 5$, $0 \le t \le 2$ using the upwind scheme and the downwind scheme with $h = 0.05, k = 0.04$. For each scheme, plot the exact solution and numerical solution (on the same plot) at $t = 0, 1, 2$. Discuss the results.

4. Consider the scalar wave equation, $v_t + cv_x = 0$, discretized by the Lax-Friedrichs scheme, $\dfrac{u_j^{n+1} - \frac{1}{2}(u_{j+1}^n + u_{j-1}^n)}{k} + cD_0 u_j^n = 0$, where c may be positive or negative.

a) Show that the CFL condition is satisfied if $|c|\lambda \leq 1$.

b) Find the amplification factor $\rho(\xi h)$.

c) Show that the scheme is stable in the 2-norm if $|c|\lambda \leq 1$.

d) Show that the scheme is stable in the ∞-norm if $|c|\lambda \leq 1$.

e) The truncation error τ_j^n is defined by $\dfrac{v_j^{n+1} - \frac{1}{2}(v_{j+1}^n + v_{j-1}^n)}{k} + cD_0 v_j^n = \tau_j^n$, where v_j^n is the exact solution. Show that $\tau_j^n = \dfrac{h}{2\lambda}(c^2\lambda^2 - 1)(v_{xx})_j^n + O(k^2)$.

f) Show that the scheme is 1st order accurate in the ∞-norm, i.e. $||v^n - u^n||_\infty = O(k)$.

g) Show that the model equation is $\phi_t + c\phi_x = \dfrac{h}{2\lambda}(1 - c^2\lambda^2)\phi_{xx}$, i.e. $||\phi^n - u^n||_\infty = O(k^2)$.

Announcement. The final exam is on Thursday, May 2 at 1:30-3:30pm in the usual classroom; it will cover everything up to the last class on Tuesday April 23; no calculators; 2 pages of notes are allowed; no photocopies of lecture notes.