Math 572 Numerical Methods for Differential Equations Winter 2025

hw6 , due : Thursday, April 10 at 4pm

1. Consider a solvated mixture of two polymers labeled A and B. The state of the system can be described by the phase function, $v(x,t)$, where $v(x,t) = -1$ denotes pure polymer A, $v(x,t) = 1$ denotes pure polymer B, and intermediate values $-1 < v(x,t) < 1$ denote mixtures of A and B. The system is governed by the Allen-Cahn equation, $v_t = v_{xx} + \gamma(v - v^3)$, where the first term on the right is due to molecular diffusion (Fick's law) and the second term is due to intermolecular forces with strength $\gamma > 0$.

a) Consider first the case where there is no diffusion and the system is uniform in space. Then the phase function is a function of time, $y(t) = v(x,t)$, which is governed by the ordinary differential equation $y' = \gamma(y - y^3)$. Find the constant states of the system, $y(t) = c$, and determine whether they are stable or unstable. Find $\lim_{t\to\infty} y(t)$ in two cases, (i) $y_0 = 1/2$, (ii) $y_0 = -1/2$.

b) Solve the Allen-Cahn equation on the interval $0 \le x \le 1$ with Dirichlet boundary conditions $v(0,t) = v(1,t) = 0$ and initial condition $v(x,0) = 0.5 \sin 2\pi x$. Use the forward Euler method in time and central differencing in space, $u_j^{n+1} = u_j^n + k(D_+D_-u_j^n + \gamma(u_j^n - (u_j^n)^3))$. Take $h = 1/60, \gamma = 800$ and consider two cases for the time step, (i) $k = 0.000165$, (ii) $k = 0.000155$. In each case plot the solution at time step $n = 0, 10, 20, 40, 80$. At the final time you should see regions of almost pure polymers separated by an internal layer at $x = 0$ with boundary layers at $x = 0, 1$. This example illustrates the process of phase separation which can be seen if you mix oil and water in a container and let it stand for a while.

2. Consider the 2D heat equation, $v_t = v_{xx} + v_{yy}$, on the unit square, $0 \le x, y \le 1$, with Dirichlet boundary conditions, $v(0,y,t) = v(1,y,t) = v(x,0,t) = 0$, $v(x,1,t) = 1$, and initial condition $v(x,y,0) = 0$. The solution $v(x,y,t)$ represents the temperature of a square plate that is heated on one side and cooled on the other three sides. Solve the problem using the forward Euler/central difference scheme $u^{n+1} = u^n + k(D_+^x D_-^x + D_+^y D_-^y)u^n$ with $h = 0.1$ and $k = 0.0025$. Is the scheme stable for this choice of parameters? Make a contour plot and a surface plot of the numerical solution at time $t = 2$; at this time you are seeing essentially the steady state temperature on the plate; the relevant plotting commands in Matlab are `contour` and `mesh` (or `surf`) or use the equivalent commands in Python. The boundary values of the temperature should be included in the plots; note that the boundary values are discontinuous at the two upper corners of the plate, $(x,y) = (0,1)$ and $(x,y) = (1,1)$; however the temperature at the corner points is not used in the numerical scheme (do you see why?) and for plotting purposes you can set the values to $v(0,1,t) = v(1,1,t) = 1$.