Math 572 Numerical Methods for Differential Equations Winter 2025

hw7 , due : Tuesday, April 22 at 4pm

1. Fritz John wrote in his PDE book, "Instability of a difference scheme under small perturbations does not exclude the possibility that in special cases the scheme converges towards the correct function, if no errors are permitted in the data or the computation." He gave the following example to illustrate this idea.

a) Consider the wave equation $v_t + cv_x = 0$, $v(x,0) = f(x)$ with $c > 0$. Show that the numerical solution given by the upwind scheme can be expressed as

$$u_j^n = ((1-c\lambda)I + c\lambda S_-)^n f_j = \sum_{\ell=0}^{n} \binom{n}{\ell} (1-c\lambda)^{n-\ell} (c\lambda)^\ell f_{j-\ell},$$

and the numerical solution given by the downwind scheme can be expressed as

$$u_j^n = ((1+c\lambda)I - c\lambda S_+)^n f_j = \sum_{\ell=0}^{n} \binom{n}{\ell} (1+c\lambda)^{n-\ell} (-c\lambda)^\ell f_{j+\ell},$$

where the shift operators are defined by $S_- f_j = f_{j-1}, S_+ f_j = f_{j+1}$.

b) Let $f(x) = e^{\alpha x}$ and take $t = t_n = nk$, $x = x_j = jh$ with $\lambda = k/h$ fixed. Using the formulas derived above for each scheme, show that the numerical solution u_j^n converges to the correct value $v(x,t) = e^{\alpha(x-ct)}$ as $n \to \infty$ for any value of λ.

Hence the numerical solution converges even when the scheme is unstable. Fritz John noted that this is consistent with the CFL condition since an analytic function (e.g. $f(x) = e^{\alpha x}$) is determined by its values in any interval. Note further that in computing the solution of a physically unstable problem (e.g. Kelvin-Helmholtz instability), it is necessary to use an unstable scheme to ensure consistency with the physics, but one must guard against the introduction of spurious perturbations whether from finite precision arithmetic or the scheme itself.

2. Consider the scalar wave equation $v_t + v_x = 0$ with two cases of initial data $v(x,0)$,

$$f_1(x) = \begin{cases} 1 , & x < 0 \\ 0 , & x = 0 \\ -1 , & x > 0 \end{cases} \quad , \quad f_2(x) = \begin{cases} -1 , & x < 0 \\ 1 - 2|x-1| , & 0 \le x \le 2 \\ -1 , & x > 2 \end{cases} .$$

Compute the solution for $-1 \le x \le 5$, $0 \le t \le 2$ using the upwind scheme and the downwind scheme with $h = 0.05, k = 0.04$. For each scheme, plot the exact solution and numerical solution (on the same plot) at $t = 0, 1, 2$. Discuss the results.

3. Consider the scalar wave equation, $v_t + cv_x = 0$, with $c > 0$, and the difference operator for the upwind scheme, $L_h u_j^n = \dfrac{u_j^{n+1} - u_j^n}{k} + cD_- u_j^n = 0$. In class we showed that the local truncation error satisfies $L_h v_j^n = O(h)$, and that the scheme converges with 1st-order accuracy, $u_j^n = v_j^n + O(h)$ as $h, k \to 0$ with fixed $t = t_n = nk$ and $\lambda = h/k$ for $c\lambda \le 1$. Now consider the model equation for the upwind scheme, $\phi_t + c\phi_x = \frac{1}{2}hc(1-c\lambda)\phi_{xx}$.

a) Show that $L_h \phi_j^n = O(h^2)$. b) Show that $u_j^n = \phi_j^n + O(h^2)$ for $c\lambda \le 1$.

Hence the numerical solution u_j^n is closer to the solution of the model equation ϕ_j^n than it is to the exact solution of the PDE v_j^n.

4. Consider the scalar wave equation, $v_t + cv_x = 0$, discretized by the Lax-Friedrichs scheme, $\dfrac{u_j^{n+1} - \frac{1}{2}(u_{j+1}^n + u_{j-1}^n)}{k} + cD_0 u_j^n = 0$, where the wave speed c may be positive or negative.

a) Show that the CFL condition is satisfied if $|c|\lambda \le 1$, where $\lambda = k/h$.

b) Find the amplification factor $\rho(\xi h)$.

c) Show that the Lax-Friedrichs scheme is stable in the 2-norm if $|c|\lambda \le 1$.

d) Show that the local truncation error is $\tau_j^n = \dfrac{h}{2\lambda}(c^2\lambda^2 - 1)(v_{xx})_j^n + O(k^2)$.

Hence it follows by the usual reasoning that the Lax-Friedrichs scheme is 1st order accurate, $u_j^n = v_j^n + O(h)$, and that the model equation is $\phi_t + c\phi_x = \dfrac{h}{2\lambda}(1 - c^2\lambda^2)\phi_{xx}$.

e) Consider the upwind scheme and the Lax-Friedrichs scheme with $c = 1, k = 0.01, h = 0.1$. For which scheme is the coefficient of artificial viscosity smaller?

Announcement. The final exam is on Thursday, April 24 at 4-6pm in the usual classroom; it will cover everything up to the last class on Tuesday April 22; no calculators; 2 pages of notes are allowed; no photocopies of lecture notes; if you need accommodation, please contact me.