MATH 654 Introduction to Fluid Dynamics Fall 2008

Homework #2 Due: Thursday, October 23

1. Consider steady homogeneous ideal 2D flow with zero body force.

a) Consider the velocity field $(u, v) = \gamma(x, -y)$ with strain rate γ. Show that the pressure has the form $p(r) = p_0 - \frac{1}{2}\rho_0\gamma^2 r^2$, where $r^2 = x^2 + y^2$.

b) Consider the velocity field $(u, v) = \frac{\omega}{2}(-y, x)$ with vorticity ω. Show that the pressure has the form $p(r) = p_0 + \frac{1}{8}\rho_0\omega^2 r^2$.

(Note that the pressure is maximum at $r = 0$ in (a) and minimum at $r = 0$ in (b)).

2. Let $S = \frac{1}{2}(\nabla\mathbf{u} - \nabla\mathbf{u}^T)$ and $\omega = \nabla \times \mathbf{u}$. Show that $S\omega = 0$. Find the eigenvalues of S.

3. Prove the following identities.

a) $\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$

b) $\nabla \cdot \nabla u = \Delta u$, $\nabla \cdot \nabla u^T = \nabla(\nabla \cdot u)$

c) $\mathbf{u} \cdot \Delta\mathbf{u} = \nabla \cdot ((\nabla\mathbf{u})^T\mathbf{u}) - |\nabla\mathbf{u}|^2$, where $|\nabla\mathbf{u}|^2 = |\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2$

d) $\nabla \times \mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial\theta} - \frac{\partial u_\theta}{\partial z}\right)e_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)e_\theta + \left(\frac{1}{r}\frac{\partial(ru_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)e_z$

4. Show that the vorticity relation $\omega(\phi(\mathbf{x}, t), t) = \phi_{\mathbf{x}}(\mathbf{x}, t)\,\omega(\mathbf{x}, 0)$ in 3D ideal flow reduces to $\omega(\phi(\mathbf{x}, t), t) = \omega(\mathbf{x}, 0)$ in 2D ideal flow.

5. Consider 2D incompressible flow with stream function $\psi(x, y, t)$ in a flow domain D. Let (x_1, y_1), (x_2, y_2) be two points in D and assume $\rho = 1$. Show that $\psi(x_2, y_2, t) - \psi(x_1, y_1, t)$ is the total mass flux across any curve in D connecting the two points.

6. page 45 / 2 (Navier-Stokes equations in cylindrical coordinates; use results from hw#1)

7. Consider steady incompressible viscous flow between two cylinders, $R_1 \leq r \leq R_2$, in the following two cases. Find the velocity $u(r)$ and vorticity $\omega(r)$ in each case.

a) The cylinders are stationary and the flow is driven by a constant pressure gradient p_z in the axial direction (axisymmetric annular Poiseuille flow). Assume that $\mathbf{u}(r, \theta, z) = u(r)\mathbf{e}_z$.

b) The cylinders are rotating (axisymmetric Couette flow). Assume that $p = p(r)$, $\mathbf{u}(r, \theta, z) = u(r)\mathbf{e}_\theta, u(R_1) = U_1, u(R_2) = U_2$.

8. Plot the streamlines associated with the following streamfunctions. Use a contour plot command in Matlab. Use `subplot` to conserve paper and `axis square` to get correct scaling.

a) $\psi(x, y) = \gamma xy - \frac{\omega}{4}(x^2 + y^2)$

This is a strained vortex. Take $\gamma = 1,\ \omega = 0, 0.4, 1, 2, 3, 40$.

b) $\psi(x, y) = \frac{\Gamma_1}{2\pi}\log((x - x_1)^2 + (y - y_1)^2)^{1/2} + \frac{\Gamma_2}{2\pi}\log((x - x_2)^2 + (y - y_2)^2)^{1/2} + Vx$

This is a pair of point vortices. Take $x_1 = -x_2 = 1$, $y_1 = y_2 = 0$.

case 1: $\Gamma_1 = \Gamma_2 = 2\pi, V = 0$ (co-rotating)

case 2: $\Gamma_1 = -\Gamma_2 = 2\pi, V = 0$ (counter-rotating)

case 3: $\Gamma_1 = -\Gamma_2 = 2\pi, V = 0.5$ (counter-rotating in a uniform stream)