MATH 654 Introduction to Fluid Dynamics Fall 2008

Homework #3 Due: Thursday, November 6

Some problems require plotting streamlines. You may use Matlab. Include any streamline that connects two stagnation points or that connects a stagnation point to infinity (if necessary, draw these by hand). Draw arrows on the streamlines to indicate the flow direction.

1. Consider inviscid incompressible flow with constant density $\rho_0 = 1$. Show that the pressure satisfies the Poisson equation $\Delta p = |S|^2 - |D|^2$, where $\nabla u = D + S$, D is the deformation matrix, S is the rotation matrix, and $|S|^2$, $|D|^2$ denote the sum of squares of the matrix elements. (This is consistent with the results of problem 1 on hw2 showing that strain is a source of high pressure and vorticity is a source of low pressure.)

2. Find the deformation matrix D at $(x, y) = (0, 1)$ induced by a point vortex of unit strength located at $(x, y) = (0, 0)$. Plot some streamlines associated with D in a neighborhood of $(x, y) = (0, 1)$. (This shows that a concentrated vortex induces strain in the far-field.)

3. Show that $g(r, \theta, \phi) = 1/(4\pi r)$ is a fundamental solution of the Laplace equation on $\mathbf{R}^3$, i.e. $\Delta g = -\delta$, where (r, θ, ϕ) are spherical coordinates, and the Laplace operator is

$$\Delta g = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial g}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial g}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 g}{\partial^2 \phi}.$$

Follow the same steps as the 2D case in class.

4. Given the complex potential $W(z)$, plot some streamlines of the flow.

a) $W(z) = \dfrac{1}{2\pi}\log z$: point source

b) $W(z) = \dfrac{1}{2\pi i}\log\left(\dfrac{z-i}{z+i}\right) + z/4\pi$: vortex pair (in a moving reference frame)

c) $W(z) = \dfrac{d}{dz}\left(\dfrac{1}{2\pi i}\log z\right)$: vortex-dipole

d) $W(z) = \dfrac{d^2}{dz^2}\left(\dfrac{1}{2\pi i}\log z\right)$: vortex-quadrupole

5. In class we saw that the complex potential $W(z) = U(z + R^2/z)$ describes flow past a circular cylinder, where R is the cylinder radius and U is the velocity at infinity. Suppose that a point vortex of strength Γ is placed in the flow at $z = z_0$, where $|z_0| > R$. Show that the condition of zero normal velocity on the boundary of the cylinder can be satisfied by placing a second point vortex of strength $-\Gamma$ inside the cylinder at $z = R^2/\overline{z_0}$. Plot some streamlines for the case $U = 1$, $R = 1$, $\Gamma = 10$, $z_0 = 2 + i$. (This illustrates the method of images, a technique for satisfying the boundary condition in potential flow.)