MATH 654 Introduction to Fluid Dynamics Fall 2008

Homework #4 due: Tuesday, November 25

1. Recall the expression $W(z) = U(z + \frac{R^2}{z}) + \frac{\Gamma}{2\pi i}\log z$ for potential flow past a cylinder with circulation. Consider the case $\frac{\Gamma}{2\pi R} > 2U$, where $U > 0$, $R > 0$, $\Gamma > 0$. In class it was stated that there is a single stagnation point in the interior of the flow in this case. Find the location of the stagnation point in terms of U, R, Γ. Plot some streamlines for the case $U = R = 1, \Gamma = 6\pi$.

2. Recall the expression $W(z) = U\sqrt{z^2 + a^2}$ for potential flow past a flat plate. The flow is unrealistic due to the absence of a wake, so consider a simple model in which the complex potential is modified by adding a counter-rotating pair of point vortices, one at $z = z_0$ with circulation Γ and the other at $z = \overline{z_0}$ with circulation $-\Gamma$. Assume that $U > 0$, real $z_0 > 0$.

a) Find the complex potential of the modified flow in the form $W_1(z) = W_2(\zeta(z))$, where $\zeta = z + \sqrt{z^2 + a^2}$ is the conformal mapping from the exterior of a plate in the z-plane to the exterior of a cylinder in the ζ-plane and $W_2(\zeta)$ is the modified complex potential in the ζ-plane.

b) Find the circulation value Γ^* satisfying the Kutta condition at the edges of the plate.

c) Plot some streamlines in the z-plane and ζ-plane for the case $U = a = 1$, $z_0 = 1 + i$, $\Gamma = \Gamma^*$.

3. Let $(x_j(t), y_j(t))$, $j = 1, \ldots, N$ denote a set of point vortices. Show that the following quantities are invariant in time.

$$X = \sum_{j=1}^{N} \Gamma_j\, x_j \quad , \quad Y = \sum_{j=1}^{N} \Gamma_j\, y_j \quad , \quad R^2 = \sum_{j=1}^{N} \Gamma_j\, (x_j^2 + y_j^2)$$

4. Consider two point vortices z_1, z_2 whose strengths satisfy $\Gamma_1 > \Gamma_2 > 0$ and define the center of vorticity by $Z = (\Gamma_1 z_1 + \Gamma_2 z_2)/(\Gamma_1 + \Gamma_2)$. The previous exercise shows that Z is invariant in time. Show that the point vortices z_1, z_2 travel on circles centered at Z, with the same angular velocity, but different radii R_1, R_2. What happens to Z, R_1, R_2 in the limit $\Gamma_2 \to \Gamma_1$?

5. The stream function of a vortex-blob is $\psi_\delta(x, y) = -\frac{1}{2\pi}\log\sqrt{x^2 + y^2 + \delta^2}$, where δ is a smoothing parameter. Note that ψ_0 is the stream function of a point vortex.

a) Find the vorticity of a vortex-blob, $\omega_\delta = -\Delta\psi_\delta$, and show that $\int_{\mathbf{R}^2} \omega_\delta(x, y)\, dx\, dy = 1$ for all δ. Plot $w_\delta(x, 0)$ for $-1 \le x \le 1$ with $\delta = 0.2,\ 0.1,\ 0.05$ (all on the same plot).

b) The evolution equations for a set of vortex-blobs are the same as for a set of point vortices except that ψ_δ is used instead of ψ_0. Compute the motion of a set of vortex-blobs having initial locations $x_j(0) = \cos\theta_j$, $y_j(0) = 0$, $\theta_j = (-1 + \frac{j}{N+1})\pi$ and strengths $\Gamma_j = \frac{\pi}{N+1}\cos\theta_j$, for $j = 1, \ldots, N$. Take $N = 200$, $\delta = 2 \cdot 10^{-1}$ and use any convenient time integration scheme (code it yourself or use Matlab). Plot the blob locations at $t = 0, 1, 2, 4, 8, 16$ (use `axis square` in Matlab for proper scaling). The results represent a 2D model for the wake behind an airplane; see pages 50-51 in Van Dyke's photo album. Accounting for 3D effects is an important ongoing research problem.