Homework #5 due: Tuesday, December 9

1. Plot the streamlines for the potential flow past the ellipse $4x^2 + y^2 = 1$ which approaches the uniform stream $(u, v) = (1, 0)$ at infinity and has zero circulation around the ellipse. Use any method of your choice.

2. Consider the problem of unsteady incompressible viscous flow on the plane $\mathbf{R}^2$ with initial vorticity $\omega(x, y, 0) = \delta(x, y)$, a delta-function of unit strength at the origin. Since the domain and initial condition are radially symmetric, the flow is assumed to be radially symmetric for $t > 0$ as well, and by dimensional reasoning we may assume that $\omega(r, t) = (1/t)f(\eta)$, where η is an appropriate nondimensional variable. Find the vorticity $\omega(r, t)$ and azimuthal velocity $v(r, t)$, and sketch their profiles for viscosity values $\nu = 10^{-2}, 10^{-4}, 10^{-6}$ and times $t = 10^{-1}, 1, 10$.

3. Consider the two-point boundary value problem $\epsilon^2 u'' - u = x - 1$, $u(0) = u(1) = 0$. Find the exact solution and use the method of matched asymptotic expansions to find a uniform asymptotic approximation in the limit $\epsilon \to 0$. Plot the exact solution, inner solution, outer solution, and the uniform approximation for $\epsilon = 0.4, 0.3, 0.2, 0.1$ (four separate plots).

4. In class we derived the Rayleigh equation for inviscid linear stability of a parallel shear flow and applied it to analyze the stability of a layer of constant vorticity in free-space,

$$U(y) = \begin{cases} -\omega_0\, d & \text{if} \quad y > d, \\ -\omega_0\, y & \text{if} \ -d \le y \le d, \\ \omega_0\, d & \text{if} \quad y < -d, \end{cases}$$

where d is the width of the vorticity layer and ω_0 is the vorticity magnitude. Suppose now that the flow is confined to a channel, $-H \le y \le H$, where $H > d$. Find the dispersion relation in this case and show that it reduces to the free-space dispersion relation in the limit $H \to \infty$ with d, ω_0, k fixed. Plot the dispersion relation (kc/ω_0 vs. kd) for the cases $H = d, 2d, 4d, 8d$ (four separate plots), indicating unstable modes with a solid line and stable modes with a dashed line. Is the channel flow more stable or less stable than the free-space flow?

5. (a) Show that $\mathrm{pv}\displaystyle\int_{-1}^{1} \frac{\sqrt{1-s^2}}{x-s}\, ds = \pi x$ for $-1 < x < 1$. Can you find two derivations?

(b) Consider a flat vortex sheet of finite length, initially on the interval $-1 < x < 1$, with vortex sheet strength $\sigma(x) = \sqrt{1-x^2}$. Find the circulation $\Gamma(x)$ associated with the initial data and use the Birkhoff-Rott integral to evaluate the initial velocity at a point on the sheet. How do you think the sheet evolves for $t > 0$?