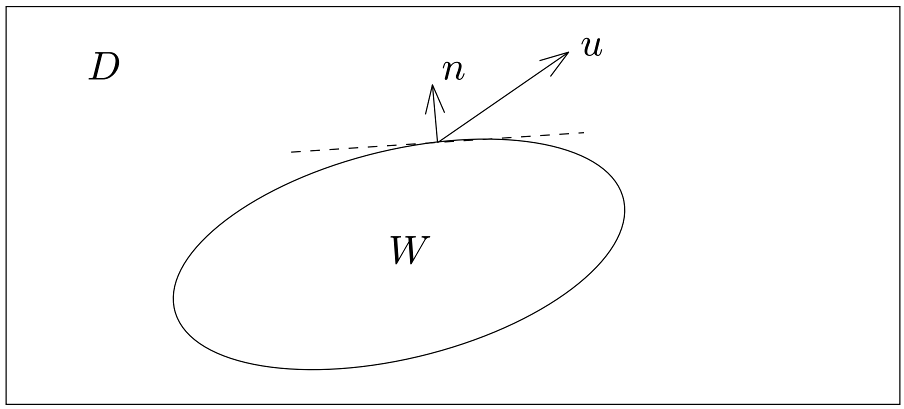
1.1 Euler equations

molecular model / continuum model

conservation laws : mass , momentum , energy

D : fluid domain (1D, 2D or 3D) , $W \subset D$



$(x, y, z) = x \in D$: Cartesian coordinates

$(u, v, w) = u$, $u = u(x, y, z, t)$: velocity , units : $[u] = \dfrac{L}{T}$

$\rho(x, t)$: mass density per unit volume , $[\rho] = \dfrac{M}{L^3}$

$\int_W \rho(x, t)\, dV$ = total mass of fluid in volume W at time t

n : unit outward normal vector to ∂W

$u \cdot n$: volume flow rate per unit area crossing ∂W in the outward direction

$$[u \cdot n] = \frac{L}{T} = \frac{L^3}{T} \cdot \frac{1}{L^2}$$

$\rho u \cdot n$: mass flow rate , mass flux ... , $[\rho u \cdot n] = \dfrac{M}{L^3} \cdot \dfrac{L}{T} = \dfrac{M}{T} \cdot \dfrac{1}{L^2}$

conservation of mass

$$\frac{d}{dt} \int_W \rho\, dV = -\int_{\partial W} \rho u \cdot n\, dA \quad : \quad \text{integral form}$$

rate of change of mass in W = rate at which mass crosses ∂W

<u>note</u>

$$\frac{d}{dt}\int_W \rho\, dV = \int_W \rho_t\, dV \quad , \quad \int_{\partial W} \rho u \cdot n\, dA = \int_W \nabla \cdot (\rho u)\, dV \quad : \text{ divergence thm}$$

$$\Rightarrow \quad \int_W (\rho_t + \nabla \cdot (\rho u))\, dV = 0$$

$$\Rightarrow \quad \rho_t + \nabla \cdot (\rho u) = 0 \quad : \quad \text{differential form}$$

<u>conservation of momentum</u> : $F = ma$

<u>acceleration</u>

$x(t) = (x(t), y(t), z(t))$: <u>path</u> of a fluid particle

$$\frac{dx}{dt}(t) = u(x(t), t) \quad : \quad \left.\begin{array}{l} \dot{x} = u(x,y,z,t) \\ \dot{y} = v(x,y,z,t) \\ \dot{z} = w(x,y,z,t) \end{array}\right\} \quad : \text{ 1st order system of ODEs}$$

Let $f(x, y, z, t)$ be given (scalar or vector).

$$\frac{d}{dt} f(x(t), y(t), z(t), t) = f_t + f_x \dot{x} + f_y \dot{y} + f_z \dot{z} = f_t + f_x u + f_y v + f_z w$$

$$= f_t + \nabla f \cdot u = f_t + (u \cdot \nabla) f = \frac{Df}{Dt} \quad : \text{ derivative of } f \text{ at a } \underline{\text{material point}}$$

$$\frac{D}{Dt} = \partial_t + (u \cdot \nabla) \quad : \quad \underline{\text{material derivative}}$$

<u>note</u> : $\rho_t + \nabla \cdot (\rho u) = 0 \;\Rightarrow\; \rho_t + \rho \nabla \cdot u + \nabla \rho \cdot u = 0 \;\Rightarrow\; \dfrac{D\rho}{Dt} = -\rho \nabla \cdot u$

<u>ex</u>

$f \to u$: velocity

$$\frac{Du}{Dt} = u_t + (u \cdot \nabla) u \quad : \quad \underline{\text{acceleration}} \text{ of a fluid particle}$$

<u>force</u>

A fluid volume W is subject to two types of forces. A <u>body force</u> or <u>external force</u> exerts a force per unit volume due to an external physical effect (e.g. gravity, electric force).

$b(x, t)$: body force per unit mass , $\rho b(x, t)$: body force per unit volume

$\int_W \rho b\, dV$: total body force on W

A <u>stress force</u> or <u>surface force</u> exerts a force per unit area on ∂W due to the fluid outside W (e.g. surface tension, friction, pressure).

<u>ideal fluid model</u>

The stress force has the form pn, where $p = p(x,t)$ is the fluid <u>pressure</u>, i.e. an ideal fluid has zero tangential stress.

total stress force on $W \;=\; -\int_{\partial W} pn\,dA \quad : \quad$ express as a volume integral

Let e be any constant vector.

$$\int_{\partial W} pn\,dA \cdot e \;=\; \int_{\partial W} pn \cdot e\,dA \;=\; \int_{\partial W} pe \cdot n\,dA \;=\; \int_W \nabla \cdot (pe)\,dV \;=\; \int_W \nabla p \cdot e\,dV$$

$$= \int_W \nabla p\; dV \cdot e \quad \Rightarrow \quad \int_{\partial W} pn\,dA \;=\; \int_W \nabla p\,dV$$

$\Rightarrow \quad -\nabla p \;:\;$ pressure force per unit volume due to surface stress

$F = ma \quad \Rightarrow \quad \rho\dfrac{Du}{Dt} = -\nabla p + \rho b \;:\;$ <u>momentum equation</u>

<u>integral form</u>

$\rho u \;:\;$ <u>momentum density</u> per unit volume $\quad , \quad \dfrac{d}{dt}\displaystyle\int_W \rho u\,dV = \,?$

$$(\rho u)_t = \rho u_t + \rho_t u$$

$$\rho\frac{Du}{Dt} = -\nabla p + \rho b \quad \Rightarrow \quad \rho(u_t + (u\cdot\nabla)u) = -\nabla p + \rho b$$

$$\rho_t + \nabla\cdot(\rho u) = 0 \quad \Rightarrow \quad \rho_t u + (\nabla\cdot(\rho u))u = 0$$

$$(\rho u)_t = -\nabla p + \rho b - \rho(u\cdot\nabla)u - (\nabla\cdot(\rho u))u$$

note : $\rho(u\cdot\nabla)u_i + (\nabla\cdot(\rho u))u_i = \nabla\cdot(\rho u u_i)$, <u>pf</u> : hw

$$(\rho u)_t = -\nabla p + \rho b - \nabla\cdot(\rho u u) \;,\; \text{ where } \nabla\cdot(\rho u u) = \begin{pmatrix} \nabla\cdot(\rho u u_1) \\ \nabla\cdot(\rho u u_2) \\ \nabla\cdot(\rho u u_3) \end{pmatrix}$$

$$\int_W (\rho u)_t\,dV = -\int_W \nabla p\,dV + \int_W \rho b\,dV - \int_W \nabla\cdot(\rho u u)\,dV$$

$$\frac{d}{dt}\int_W \rho u\, dV = -\int_{\partial W} pn\, dA + \int_W \rho b\, dV - \int_{\partial W} \rho u(u\cdot n)\, dA \quad : \quad \text{integral form}$$

$\rho u(u\cdot n)$: <u>momentum flow rate</u> or <u>momentum flux</u> per unit area crossing ∂W

rate of change of momentum in W

= stress force + body force + rate at which momentum crosses ∂W

<u>note</u>

$$\begin{array}{ccc} x & \rightarrow & \phi(x,t) \\ \uparrow & & \uparrow \\ \text{initial location} & & \text{location at time } t \end{array} \quad : \ \underline{\text{flow map}} \quad , \quad x \ : \ \underline{\text{Lagrangian parameter}}$$

$$\phi_t(x,t) = u(\phi(x,t),t) \quad , \quad \phi(x,0) = x$$

$$\frac{d}{dt} f(\phi(x,t),t) = (f_t + \nabla f\cdot\phi_t)(\phi(x,t),t) = \frac{Df}{Dt}(\phi(x,t),t)$$

<u>transport theorem</u>

$$\frac{d}{dt}\int_{W_t} \rho f\, dV = \int_{W_t} \rho\frac{Df}{Dt}\, dV \quad , \ \text{where } W_t = \phi(W_0,t) \qquad (\text{later} : f = 1\,,\, u\,,\, \tfrac{1}{2}|u|^2\,)$$

<u>pf</u>

$$\int_{W_t} \rho f\, dV = \int_{W_t} \rho f(y,t)\, dV_y \quad , \quad y = \phi(x,t)$$

$$= \int_{W_0} \rho f(\phi(x,t),t) J(x,t)\, dV_x \quad , \quad J = |\phi_x| \ : \ \text{Jacobian determinant}$$

$$\frac{d}{dt}\int_{W_t} \rho f\, dV = \int_{W_0} \frac{d}{dt}(\rho f(\phi(x,t),t) J(x,t))\, dV_x$$

$$= \int_{W_0} (\rho f(\phi(x,t),t) J_t(x,t) + \frac{D(\rho f)}{Dt}(\phi(x,t),t) J(x,t))\, dV_x$$

(<u>lemma</u> : $J_t(x,t) = (\nabla\cdot u)(\phi(x,t),t) J(x,t)$ <u>pf</u> : soon)

$$\frac{d}{dt}\int_{W_t} \rho f\, dV = \int_{W_0} \Big(\rho f(\nabla\cdot u) + \frac{D(\rho f)}{Dt}\Big)(\phi(x,t),t) J(x,t)\, dV_x$$

$$= \int_{W_t} \Big(\rho f(\nabla\cdot u) + \frac{D(\rho f)}{Dt}\Big)(y,t)\, dV_y$$

$$\frac{D(\rho f)}{Dt} = \rho\frac{Df}{Dt} + \frac{D\rho}{Dt} f = \rho\frac{Df}{Dt} + (-\rho\nabla\cdot u) f \qquad (\underline{\text{pf}} : \text{hw}) \qquad \underline{\text{ok}}$$

<u>pf</u> : lemma

$$\phi(x,t) = (\,\xi(x,y,z,t),\, \eta(x,y,z,t),\, \zeta(x,y,z,t)\,)$$

$$\phi_t(x,t) = u(\phi(x,t),t)$$

$$\Rightarrow \begin{cases} \xi_t(x,y,z,t) \;=\; u(\,\xi(x,y,z,t),\, \eta(x,y,z,t),\, \zeta(x,y,z,t),\, t\,) \\ \\ \eta_t(\ldots\ldots\ldots) = \; v(\ldots\ldots\ldots) \\ \\ \zeta_t(\ldots\ldots\ldots) = \; w(\ldots\ldots\ldots) \end{cases}$$

$$J = |\phi_x| = \begin{vmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{vmatrix} \quad , \quad J_t = \lim_{h\to 0} \frac{J(t+h)-J(t)}{h}$$

$$J(t+h) = \begin{vmatrix} \xi_x(t+h) & \xi_y \ldots & \xi_z \ldots \\ \eta_x \ldots & \eta_y \ldots & \eta_z \ldots \\ \zeta_x \ldots & \zeta_y \ldots & \zeta_z \ldots \end{vmatrix} \approx \begin{vmatrix} \xi_x + h\xi_{xt} & \xi_y + h\xi_{yt} & \xi_z + h\xi_{zt} \\ \eta_x + h\eta_{xt} & \eta_y + h\eta_{yt} & \eta_z + h\eta_{zt} \\ \zeta_x + h\zeta_{xt} & \zeta_y + h\zeta_{yt} & \zeta_z + h\zeta_{zt} \end{vmatrix}$$

$\det(a_1,a_2,a_3)$, $a_i = i$th row , det is a multilinear function of the rows

$$\det(a_1+hb_1, a_2+hb_2, a_3+hb_3)$$
$$= \det(a_1,a_2,a_3) + h(\,\det(b_1,a_2,a_3) + \det(a_1,b_2,a_3) + \det(a_1,a_2,b_3)) + O(h^2)$$

$$J(t+h) = J(t) + h\left(\begin{vmatrix} \xi_{xt} & \xi_{yt} & \xi_{zt} \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{vmatrix} + \begin{vmatrix} \xi_x & \xi_y & \xi_z \\ \eta_{xt} & \eta_{yt} & \eta_{zt} \\ \zeta_x & \zeta_y & \zeta_z \end{vmatrix} + \begin{vmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_{xt} & \zeta_{yt} & \zeta_{zt} \end{vmatrix} \right) + O(h^2)$$

$$\begin{vmatrix} \xi_{xt} & \xi_{yt} & \xi_{zt} \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{vmatrix} = \begin{vmatrix} u_x\xi_x + u_y\eta_x + u_z\zeta_x & u_x\xi_y + u_y\eta_y + u_z\zeta_y & u_x\xi_z + u_y\eta_z + u_z\zeta_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{vmatrix}$$

$$= \; u_x \begin{vmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{vmatrix} + u_y \begin{vmatrix} \eta_x & \eta_y & \eta_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{vmatrix} + u_z \begin{vmatrix} \zeta_x & \zeta_y & \zeta_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{vmatrix} = u_x J$$

similarly : $\begin{vmatrix} \xi_x & \xi_y & \xi_z \\ \eta_{xt} & \eta_{yt} & \eta_{zt} \\ \zeta_x & \zeta_y & \zeta_z \end{vmatrix} = v_y J$, $\begin{vmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_{xt} & \zeta_{yt} & \zeta_{zt} \end{vmatrix} = w_z J$: hw

$$\Rightarrow \quad J_t = (u_x + v_y + w_z) \cdot J = (\nabla \cdot u)\, J \qquad \underline{\text{ok}}$$

<u>application</u> of transport theorem

$$\frac{d}{dt}\int_{W_t}\rho f\,dV = \int_{W_t}\rho\,\frac{Df}{Dt}\,dV$$

1. $f=1 \;\Rightarrow\; \dfrac{d}{dt}\displaystyle\int_{W_t}\rho\,dV = 0$: conservation of mass

$$\Rightarrow\; \int_{W_t}\rho\,dV = \int_{W_0}\rho\,dV$$

$$\Rightarrow\; \int_{W_t}\rho(y,t)\,dV_y = \int_{W_0}\rho(\phi(x,t),t)J(x,t)\,dV_x = \int_{W_0}\rho(x,0)\,dV_x$$

$$\Rightarrow\; \rho(\phi(x,t),t)J(x,t) = \rho(x,0)$$

The density following a fluid particle is inversely proportional to local changes in the fluid volume.

recall : $\dfrac{D\rho}{Dt} = -(\nabla\cdot u)\,\rho$, $J_t = (\nabla\cdot u)\,J$

2. $f=u \;\Rightarrow\; \dfrac{d}{dt}\displaystyle\int_{W_t}\rho u\,dV = \int_{W_t}\rho\frac{Du}{Dt}\,dV = \int_{W_t}(-\nabla p + \rho b)\,dV$

$\Rightarrow\; \dfrac{d}{dt}\displaystyle\int_{W_t}\rho u\,dV = -\int_{\partial W_t} pn\,dA + \int_{W_t}\rho b\,dV$: conservation of momentum

rate of change of momentum in W_t = stress force + body force

3. $f=\frac{1}{2}|u|^2 \;\Rightarrow\; \dfrac{d}{dt}\displaystyle\int_{W_t}\tfrac{1}{2}\rho|u|^2\,dV = \int_{W_t}\rho\,\frac{D(\frac{1}{2}|u|^2)}{Dt}\,dV = \int_{W_t}\rho u\cdot\frac{Du}{Dt}\,dV$ (hw)

$\Rightarrow\; \dfrac{d}{dt}\displaystyle\int_{W_t}\tfrac{1}{2}\rho|u|^2\,dV = \int_{W_t} u\cdot(-\nabla p + \rho b)\,dV$: conservation of kinetic energy

rate of change of kinetic energy in W_t

= rate at which work is done due to surface stress and body force

<u>note</u>

So far we derived the equations $\rho_t + \nabla\cdot(\rho u) = 0$, $\rho u_t + \rho(u\cdot\nabla)u = -\nabla p + \rho b$. These are 4 equations, but there are 5 variables (ρ, u, p), so something is missing.

thermodynamics

ϵ : internal energy density per unit mass due to molecular vibration

w : enthalpy , $w = \epsilon + \dfrac{p}{\rho}$

T : temperature

s : entropy

equation of state : $A = f(B, C)$, e.g. $p = f(\rho, T)$

1st law of thermodynamics : $dw = Tds + \dfrac{dp}{\rho}$

2nd ...."........... : $ds \geq 0$

We will assume the flow is isentropic, i.e. $ds = 0$.

1st LOT $\Rightarrow$ $w_t = \dfrac{p_t}{\rho}$, $\nabla w = \dfrac{\nabla p}{\rho}$, special case : $p = f(\rho)$ (more later)

conservation of energy (isentropic, ideal flow)

$e = \frac{1}{2}\rho|u|^2 + \rho\epsilon$: total energy density per unit volume (kinetic + internal)

$e_t = (\frac{1}{2}\rho|u|^2)_t + (\rho\epsilon)_t$

$$(\tfrac{1}{2}\rho|u|^2)_t = \rho(\tfrac{1}{2}|u|^2)_t + \rho_t \cdot \tfrac{1}{2}|u|^2 = \rho u \cdot u_t + \rho_t \cdot \tfrac{1}{2}|u|^2$$
$$= u \cdot (-\rho(u \cdot \nabla)u - \nabla p + \rho b) - (\nabla \cdot (\rho u)) \cdot \tfrac{1}{2}|u|^2$$

note : $u \cdot (u \cdot \nabla)u = u \cdot \nabla(\frac{1}{2}|u|^2)$ (hw)

$$(\tfrac{1}{2}\rho|u|^2)_t = -\rho u \cdot \nabla(\tfrac{1}{2}|u|^2) - (\nabla \cdot (\rho u)) \cdot \tfrac{1}{2}|u|^2 - u \cdot \nabla p + \rho u \cdot b$$
$$= -\nabla \cdot (\rho u \cdot \tfrac{1}{2}|u|^2) - \rho u \cdot \nabla w + \rho u \cdot b$$

$(\rho\epsilon)_t = \rho_t\epsilon + \rho\epsilon_t$

$$\epsilon = w - \frac{p}{\rho} \;\Rightarrow\; \epsilon_t = w_t - \frac{p_t}{\rho} + \frac{p}{\rho^2}\rho_t = \frac{p}{\rho^2}\rho_t$$

$$(\rho\epsilon)_t = \rho_t\epsilon + \rho\frac{p}{\rho^2}\rho_t = \rho_t(\epsilon + \frac{p}{\rho}) = -(\nabla \cdot (\rho u))w$$

$$e_t = (\tfrac{1}{2}\rho|u|^2)_t + (\rho\epsilon)_t = -\nabla \cdot (\rho u(\tfrac{1}{2}|u|^2 + w)) + \rho u \cdot b$$

$$\rho u(\tfrac{1}{2}|u|^2 + w) = u(\tfrac{1}{2}\rho|u|^2 + \rho w) = u(e - \rho\epsilon + \rho\epsilon + p) = u(e + p)$$

$$e_t + \nabla \cdot ((e + p)u) = \rho u \cdot b$$

<u>isentropic Euler equations</u>

$$\left.\begin{array}{l} \rho_t + \nabla\cdot(\rho u) = 0 \\ \rho u_t + \rho(u\cdot\nabla)u = -\nabla p + \rho b \\ e_t + \nabla\cdot((e+p)u) = \rho u\cdot b \\ e = \frac{1}{2}\rho|u|^2 + \rho\epsilon \\ \epsilon = \epsilon(\rho, p) \end{array}\right\} \ : \ \text{7 equations, 7 variables : } \rho\,,\ u\,,\ p\,,\ e\,,\ \epsilon$$

<u>boundary condition</u> : in ideal flow on a domain D we require $u\cdot n|_{\partial D} = 0$

<u>integral form</u> of conservation of energy

$$\frac{d}{dt}\int_W e\,dV = \int_W e_t\,dV = \int_W (-\nabla\cdot((e+p)u) + \rho u\cdot b)\,dV$$

$$= -\int_{\partial W} eu\cdot n\,dA - \int_{\partial W} pu\cdot n\,dA + \int_W \rho u\cdot b\,dV$$

rate of change of total energy in W

= rate at which total energy crosses ∂W

+ rate at which work is done due to surface stress and body force

<u>incompressible flow</u>

<u>thm</u>

The following conditions are equivalent.

1. $\int_{W_t} dV = \int_{W_0} dV$ for all t, W_0 : conservation of volume , <u>incompressible flow</u>
2. $J = 1$
3. $\nabla\cdot u = 0$
4. $\dfrac{D\rho}{Dt} = 0$ ($\rho_t + u\cdot\nabla\rho = 0$: <u>advection equation</u>)
5. $\rho(\phi(x,t),t) = \rho(x,0)$

<u>pf</u>

$1 \Rightarrow 2$: $\int_{W_t} dV = \int_{W_0} J\,dV = \int_{W_0} dV$

$2 \Rightarrow 3$: $J_t = (\nabla\cdot u)J$

$3 \Rightarrow 4$: $\dfrac{D\rho}{Dt} = -\rho\nabla\cdot u$

$4 \Rightarrow 5$: ok , $5 \Rightarrow 4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$: ok

<u>incompressible Euler equations</u> (variable density, <u>stratified flow</u>)

$$\left.\begin{array}{l} \rho_t + (u\cdot\nabla)\rho = 0 \\ \rho u_t + \rho(u\cdot\nabla)u = -\nabla p + \rho b \\ \nabla\cdot u = 0 \end{array}\right\} \ : \ \text{5 equations, 5 variables} : \ \rho\,,\ u\,,\ p$$

<u>note</u>

$$\frac{d}{dt}\int_W \tfrac{1}{2}\rho|u|^2\,dV = \ \ldots \quad \text{(hw)}$$

<u>incompressible Euler equations</u> (constant density, <u>homogeneous flow</u>, $b = 0$)

$$\left.\begin{array}{l} \rho_0 u_t + \rho_0(u\cdot\nabla)u = -\nabla p \\ \nabla\cdot u = 0 \end{array}\right\} \ : \ \text{4 equations, 4 variables} : \ u\,,\ p$$

<u>Bernoulli's theorem</u>

In steady homogeneous ideal flow with zero body force, the quantity $\frac{1}{2}\rho_0|u|^2 + p$ is constant along streamlines.

<u>streamline</u> : $x(s)$, $\dfrac{dx}{ds}(s) = u(x(s),t)$

particle path : $x(t)$, $\dfrac{dx}{dt}(t) = u(x(t),t)$

Streamlines and particle paths are different in general, but they coincide for <u>steady flow</u> ($u(x,t) = u(x)$).

<u>pf</u>

steady flow $\Rightarrow$ $\rho_0(u\cdot\nabla)u = -\nabla p$

$(u\cdot\nabla)u = \nabla(\frac{1}{2}|u|^2) - u\times(\nabla\times u)$: hw

$$\begin{aligned}\nabla(\tfrac{1}{2}\rho_0|u|^2 + p) &= \rho_0\nabla(\tfrac{1}{2}|u|^2) + \nabla p = \rho_0(\,(u\cdot\nabla)u + u\times(\nabla\times u)\,) + \nabla p \\ &= \rho_0 u\times(\nabla\times u)\end{aligned}$$

$$\frac{d}{ds}(\tfrac{1}{2}\rho_0|u|^2 + p)(x(s)) = \nabla(\tfrac{1}{2}\rho_0|u|^2 + p)|_{x(s)}\cdot x'(s) = (\rho_0 u\times(\nabla\times u))\cdot u = 0$$

<u>ok</u>

1.2 vorticity

def

$\omega = \nabla \times u$: vorticity

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ u & v & w \end{vmatrix} = \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix}$$

note

consider steady flow : $u(x) = (u(x,y,z),\, v(x,y,z),\, w(x,y,z))$

$$u(x) = u(x_0) + \nabla u(x_0) \cdot (x - x_0) + \dots$$

$$\nabla u = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$$

$\nabla u = \frac{1}{2}(\nabla u + \nabla u^T) + \frac{1}{2}(\nabla u - \nabla u^T) = D + S \quad , \quad D$: deformation matrix

$D = \frac{1}{2}(\nabla u + \nabla u^T) \;\Rightarrow\; D^T = D$: symmetric

$\Rightarrow$ D has 3 real e-values (γ_i, strain rate) and orthogonal e-vectors (strain axes)

trace $D = \gamma_1 + \gamma_2 + \gamma_3 = \nabla \cdot u$

$S = \frac{1}{2}(\nabla u - \nabla u^T) \;\Rightarrow\; S^T = -S$: anti-symmetric

$$2S = \nabla u - \nabla u^T = \begin{pmatrix} 0 & u_y - v_x & u_z - w_x \\ v_x - u_y & 0 & v_z - w_y \\ w_x - u_z & w_y - v_z & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

$$2Se = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \omega_2 e_3 - \omega_3 e_2 \\ \omega_3 e_1 - \omega_1 e_3 \\ \omega_1 e_2 - \omega_2 e_1 \end{pmatrix} = \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ e_1 & e_2 & e_3 \end{vmatrix}$$

$\Rightarrow \quad Se = \frac{1}{2}\omega \times e \quad$, for hw you will find the e-values of S

summary

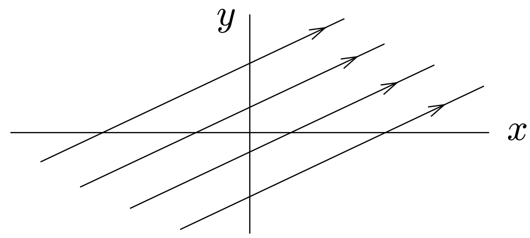
$$u(x) = u(x_0) + D(x_0) \cdot (x - x_0) + \tfrac{1}{2}\omega(x_0) \times (x - x_0) + \dots$$

We will examine the streamlines associated with each term for the case of incompressible flow ($\gamma_1 + \gamma_2 + \gamma_3 = 0$).

2D flow $\Rightarrow$ $u(x) = (u(x,y), v(x,y), 0) \Rightarrow \gamma_3 = 0\ ,\ \gamma_1 + \gamma_2 = 0$

1. $u(x) = u(x_0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$: <u>translation</u>

streamlines : $\begin{cases} \dfrac{dx}{ds} = u_0 \\ \dfrac{dy}{ds} = v_0 \end{cases} \Rightarrow$



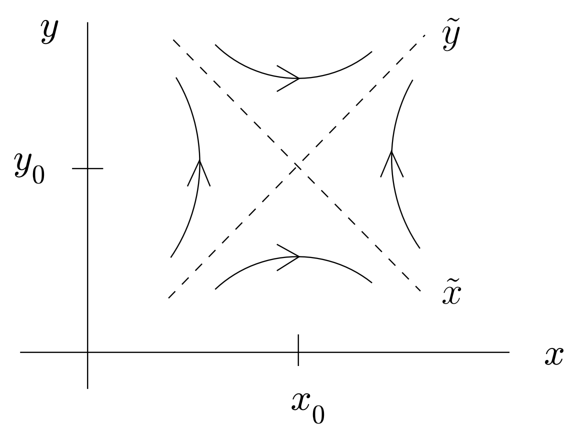
2. $u(x) = D(x_0)(x - x_0) = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$: <u>deformation</u> , <u>strain flow</u>

streamlines : $\begin{cases} \dfrac{dx}{ds} = a(x - x_0) + b(y - y_0) \\ \dfrac{dy}{ds} = b(x - x_0) - a(y - y_0) \end{cases}$

$QDQ^{-1} = \tilde{D} = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix}$, $\gamma > 0$, Q : orthogonal

$\tilde{x} = Q(x - x_0) \quad \Rightarrow \quad \dfrac{d\tilde{x}}{ds} = Q\dfrac{dx}{ds} = QD(x - x_0) = QDQ^{-1}\tilde{x} = \tilde{D}\tilde{x}$

$\left.\begin{aligned} \dfrac{d\tilde{x}}{ds} &= \ \ \gamma\tilde{x} \\ \dfrac{d\tilde{y}}{ds} &= -\gamma\tilde{y} \end{aligned}\right\} \Rightarrow$



$\Rightarrow$ an arbitrary patch turns into a line

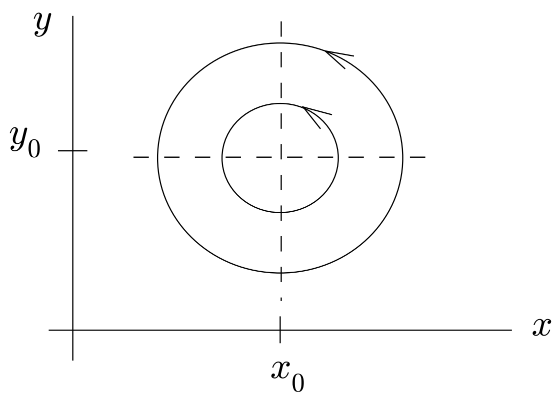
3D $\Rightarrow$ $\gamma_1 > 0\ ,\ \gamma_3 < 0\ ,\ \begin{cases} \gamma_2 > 0 \ \Rightarrow \text{ an arbitrary volume turns into a sheet} \\ \gamma_2 < 0 \ \Rightarrow \ \ldots\ldots\ldots\ldots\ldots\ \text{''}\ \ldots\ldots\ldots\ldots\ldots\ \text{tube} \end{cases}$

3. $u(x) = \frac{1}{2}\omega(x_0) \times (x - x_0)$: <u>rotation</u>

$$\text{2D flow} \quad \Rightarrow \quad \omega = \nabla \times u = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ u & v & w \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ \omega(x,y) \end{pmatrix}$$

$$x - x_0 = \tilde{x} \;\Rightarrow\; \omega \times (x - x_0) = \omega \times \tilde{x} = \begin{vmatrix} i & j & k \\ 0 & 0 & \omega \\ \tilde{x} & \tilde{y} & \tilde{z} \end{vmatrix} = \begin{pmatrix} -\omega\tilde{y} \\ \omega\tilde{x} \\ 0 \end{pmatrix}$$

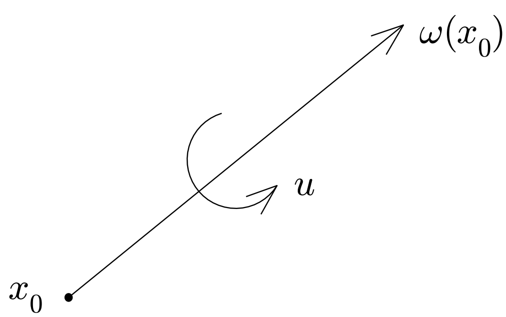
$$\text{streamlines} \; : \; \begin{cases} \dfrac{d\tilde{x}}{ds} = -\frac{1}{2}\omega\tilde{y} \\ \dfrac{d\tilde{y}}{ds} = \frac{1}{2}\omega\tilde{x} \end{cases} \Rightarrow \quad \frac{d^2\tilde{x}}{ds^2} + \tfrac{1}{4}\omega^2\tilde{x} = 0 \; : \; \text{simple harmonic motion}$$



$\omega > 0 \quad \Rightarrow \quad$ counterclockwise rotation

<u>note</u>

In 3D flow, $\omega(x_0)$ is an arbitrary vector and $u(x) = \frac{1}{2}\omega(x_0) \times (x - x_0)$ corresponds to rotation about the axis $\omega(x_0)$ with angular frequency $\frac{1}{2}|\omega(x_0)|$.



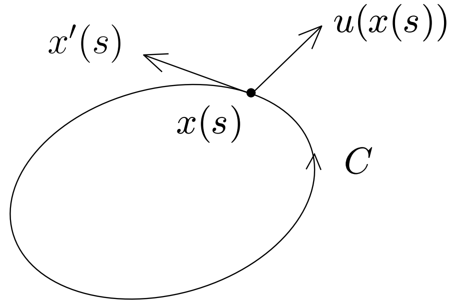
<u>summary</u>

$$u(x) = u(x_0) \;+\; D(x_0) \cdot (x - x_0) \;+\; \tfrac{1}{2}\omega(x_0) \times (x - x_0) \;+\; \cdots$$
$$= \text{translation} \;+\; \text{deformation} \;+\; \text{rotation} + \cdots$$

<u>def</u>

$C = \{x(s)\,,\, 0 \le s \le 1\,,\, x(1) = x(0)\}$: closed curve in flow domain (2D or 3D)

$\Gamma_C = \int_C u \cdot ds = \int_0^1 u(x(s)) \cdot x'(s)\, ds$: <u>circulation</u> of u around C



Let $C = \partial S$, where S is a surface.

$$\Gamma_C = \int_{\partial S} u \cdot ds \underset{\substack{\uparrow \\ \text{Stokes theorem}}}{=} \int_S (\nabla \times u) \cdot n\, dA = \int_S \omega \cdot n\, dA$$: <u>vorticity flux</u> through S

<u>thm</u> (Kelvin)

$\Gamma_{C_t} = \Gamma_{C_0}$, where $C_t = \phi(C_0, t)$ (homogeneous, ideal, zero body force)

<u>pf</u>

Let C_0 be parametrized by $x(s)$. Then C_t is parametrized by $\phi(x(s), t)$.

$$\frac{d}{dt} \int_{C_t} u \cdot ds = \frac{d}{dt} \int_0^1 u(\phi(x(s), t), t) \cdot \frac{d}{ds} \phi(x(s), t)\, ds$$

$$= \int_0^1 \left(u(\phi(x(s), t), t) \cdot \frac{d}{dt}\frac{d}{ds}\phi(x(s), t) + \frac{d}{dt} u(\phi(x(s), t), t) \cdot \frac{d}{ds}\phi(x(s), t)\right) ds$$

$$= \int_0^1 u(\phi(x(s), t), t) \cdot \frac{d}{ds} u(\phi(x(s), t), t)\, ds + \int_{C_t} \frac{Du}{Dt} \cdot ds$$

$$= \int_{C_t} \frac{d}{ds}\left(\tfrac{1}{2}|u|^2\right) ds + \int_{C_t} \frac{-\nabla p}{\rho_0} \cdot ds = \left(\tfrac{1}{2}|u|^2 + \frac{-p}{\rho_0}\right)\Bigg|_{\phi(x(0),t)}^{\phi(x(1),t)} = 0 \qquad \underline{\text{ok}}$$

<u>note</u>

Kelvin's theorem says that the circulation around a <u>material curve</u> is invariant in time, so if the length of C_t increases (or decreases) in time, then the average tangential fluid velocity around C_t decreases (or increases). Similarly, if we define $S_t = \phi(S_0, t)$ to be a <u>material surface</u>, then a similar relation holds between the area of S_t and the average vorticity flux through S_t.

<u>thm</u>

1. $\dfrac{D\omega}{Dt} = (\omega \cdot \nabla) u$

2. $\omega(\phi(x,t),t) = \phi_x(x,t)\,\omega(x,0)$

<u>pf</u>

1. $\dfrac{Du}{Dt} = u_t + (u\cdot\nabla)u = u_t + \nabla(\tfrac{1}{2}|u|^2) - u\times\omega = \dfrac{-\nabla p}{\rho_0}$

take curl : $\omega_t - \nabla\times(u\times\omega) = 0$, since $\nabla\times\nabla f = 0$

$\nabla\times(u\times\omega) = u(\nabla\cdot\omega) - \omega(\nabla\cdot u) + (\omega\cdot\nabla)u - (u\cdot\nabla)\omega$: hw2

$\omega_t - (\omega\cdot\nabla)u + (u\cdot\nabla)\omega = 0$, since $\nabla\cdot u = 0$, $\nabla\cdot\omega = \nabla\cdot(\nabla\times u) = 0$ <u>ok</u>

2. set $a(x,t) = \omega(\phi(x,t),t)$, $b(x,t) = \phi_x(x,t)\,\omega(x,0)$

$a_t(x,t) = \dfrac{D\omega}{Dt}(\phi(x,t),t) = (\omega\cdot\nabla)u(\phi(x,t),t) = (a(x,t)\cdot\nabla)u(\phi(x,t),t)$

$b_t(x,t) = \phi_{xt}(x,t)\,\omega(x,0) = \partial_x(u(\phi(x,t),t))\,\omega(x,0)$

$= \nabla u(\phi(x,t),t)\,\phi_x(x,t)\,\omega(x,0) = \nabla u(\phi(x,t),t)\,b(x,t) = (b(x,t)\cdot\nabla)u(\phi(x,t),t)$

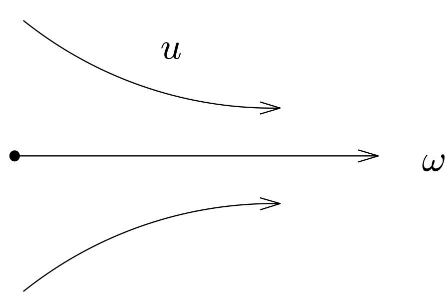
$\Rightarrow$ $a(x,t)$ and $b(x,t)$ satisfy the same ODE in time t

$$\left.\begin{array}{l} a(x,0) = \omega(\phi(x,0),0) = \omega(x,0) \\ b(x,0) = \phi_x(x,0)\,\omega(x,0) = \omega(x,0) \end{array}\right\} \Rightarrow a(x,t) = b(x,t)$$ <u>ok</u>

<u>note</u>

$\dfrac{D\omega}{Dt} = (\omega\cdot\nabla)u = (\nabla u)\,\omega = (D+S)\omega = D\omega$, since $S\omega = \tfrac{1}{2}\omega\times\omega = 0$

$\Rightarrow$ If ω is aligned with a positive strain axis, then $|\omega|$ increases in time.



: <u>vortex stretching</u>

<u>open question</u> : Can $\omega\to\infty$ in finite time?

<u>note</u>

1. $\omega(x,0)=0 \;\Rightarrow\; \omega(\phi(x,t),t)=0$

2. A <u>vortex line</u> is an integral curve of the vorticity field, i.e. a curve $x(s)$ st $x'(s)=\omega(x(s),t)$. If $x(s)$ is a vortex line at time $t=0$, then $y(s)=\phi(x(s),t)$ is a vortex line at time $t>0$, i.e. "vortex lines move with the fluid".

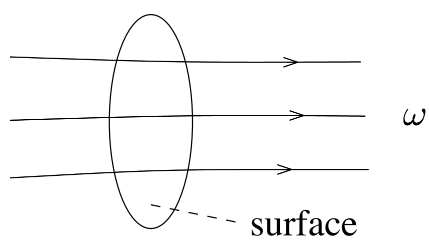
<u>pf</u> of 2

we have $x'(s)=\omega(x(s),0)$, need to show that $y'(s)=\omega(y(s),t)$

$y'(s)=\phi_x(x(s),t)x'(s)=\phi_x(x(s),t)\omega(x(s),0)=\omega(\phi(x(s),t),t)=\omega(y(s),t)$ <u>ok</u>

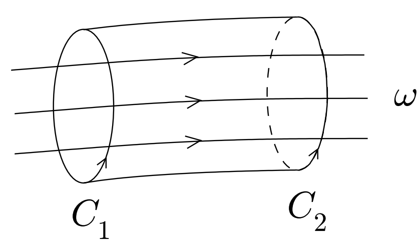
<u>def</u>

The set of vortex lines passing through a surface is called a <u>vortex tube</u>. The result above shows that "vortex tubes move with the fluid".



<u>thm</u> (Helmholtz)

Let C_1 and C_2 be closed curves surrounding a vortex tube oriented in the same direction around the tube. Then $\Gamma_{C_1}=\Gamma_{C_2}$.



<u>pf</u>

tube volume : W

tube surface : $\partial W = S_1\cup S_2\cup S$, $C_1=\partial S_1$, $C_2=\partial S_2$

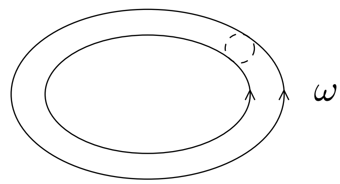
$$0=\int_W \nabla\cdot\omega\,dV=\int_{\partial W}\omega\cdot n\,dA=\int_{S_1}\omega\cdot n\,dA+\int_{S_2}\omega\cdot n\,dA+\int_S\omega\cdot n\,dA$$

$$=\int_{S_1}(\nabla\times u)\cdot n\,dA+\int_{S_2}(\nabla\times u)\cdot n\,dA=\int_{C_1}u\cdot ds-\int_{C_2}u\cdot ds \qquad \underline{\text{ok}}$$

<u>note</u>

1. The circulation around a vortex tube is well-defined (Helmholtz) and is called the <u>strength</u> of the tube. The tube strength is constant in time (Kelvin).

2. A closed vortex tube in the shape of a torus is called a <u>vortex ring</u>.



<u>2D flow</u>

$u = (u(x,y,t),\, v(x,y,t),\, 0)\ ,\ \omega = (0,\, 0,\, \omega(x,y,t))\ ,\ \omega = v_x - u_y$

$(\omega \cdot \nabla)u = (\omega_1\partial_x + \omega_2\partial_y + \omega_3\partial_z)u = 0$: no vortex stretching in 2D flow

$\frac{D\omega}{Dt} = 0 \ \Rightarrow\ \omega(\phi(x,t),t) = \omega(x,0)$: vorticity is advected

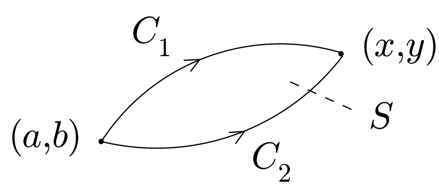
<u>claim</u>

$\nabla \cdot u = 0 \ \Rightarrow$ there exists $\psi(x,y,t)$ st $u = \psi_y\ ,\ v = -\psi_x$: <u>stream function</u>

<u>pf</u>

$\psi(x,y,t) = \int_{(a,b)}^{(x,y)} (-v dx + u dy)$

$\Rightarrow\ \psi_x = -v\ ,\ \psi_y = u$



must show that ψ is well-defined

$$\int_{C_2 - C_1} (-v dx + u dy) = \int_{\partial S} (P dx + Q dy) \underset{\substack{\uparrow \\ \text{Green's thm}}}{=} \int_S (Q_x - P_y)\, dA = \int_S (u_x + v_y)\, dA = 0$$

<u>ok</u>

<u>properties</u>

1. $\Delta\psi = \psi_{xx} + \psi_{yy} = -v_x + u_y = -\omega$: Poisson equation

2. C is a streamline of the flow $\left(\frac{dx}{ds} = u(x,y,t)\, ,\, \frac{dy}{ds} = v(x,y,t)\right)$

$\Leftrightarrow$ C is a level curve of the stream function ($\psi(x,y,t) =$ constant on C)

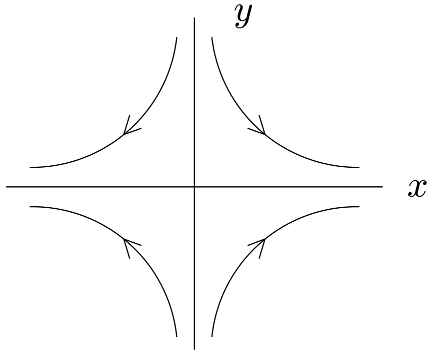
<u>pf</u>

$\Rightarrow)\ \frac{d}{ds}\psi(x(s),y(s),t) = \psi_x \frac{dx}{ds} + \psi_y \frac{dy}{ds} = -vu + uv = 0$ <u>ok</u>

$\Leftarrow)\ (\psi_x, \psi_y) \perp (\frac{dx}{ds}, \frac{dy}{ds}) \ \Rightarrow\ (-v, u) \perp (\frac{dx}{ds}, \frac{dy}{ds}) \ \Rightarrow\ (u,v) \,||\, (\frac{dx}{ds}, \frac{dy}{ds})$ <u>ok</u>

ex

1. $\psi(x,y) = \gamma xy \quad \Rightarrow \quad$ level curves : $xy = c$: hyperbolas

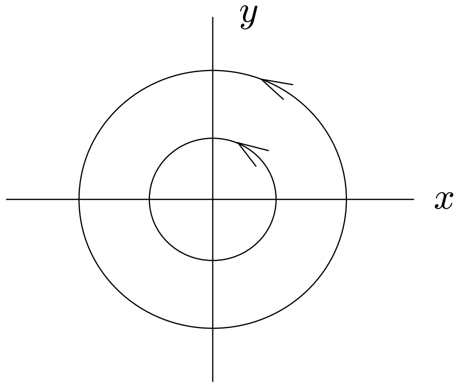


$$u = \gamma x \ , \ v = -\gamma y$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \gamma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \ : \ \text{deformation}$$

$$\omega = v_x - u_y = 0$$

2. $\psi(x,y) = -\dfrac{\omega}{4}(x^2+y^2) \quad \Rightarrow \quad$ level curves : $x^2+y^2 = c$: circles



$$u = -\frac{\omega}{2} y \ , \ v = \frac{\omega}{2} x$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{\omega}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \ : \ \text{rotation}$$

$$\omega = v_x - u_y = \omega$$

3. $\psi(x,y) = \gamma xy - \dfrac{\omega}{4}(x^2+y^2)$: hw2

<u>vorticity-stream form</u> (2D, ideal, homogeneous, zero body force)

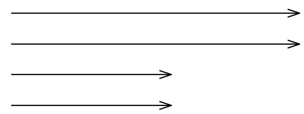
$\omega_t + u\omega_x + v\omega_y = 0$, $\omega(x,y,0)$: given , $(x,y) \in D$

$u = \psi_y$, $v = -\psi_x$

$\Delta\psi = -\omega$, $\psi|_{\partial D} = 0$

1.3 viscous flow

Consider a parallel shear flow with two layers of fluid moving at different speeds.



$$\left.\begin{array}{rcl} u(x,y,t) & = & u_0(y) \\ v(x,y,t) & = & 0 \\ p(x,y,t) & = & p_0 \end{array}\right\} \quad \Rightarrow \quad \left.\begin{array}{rcl} \nabla \cdot u & = & u_x + v_y \; = \; 0 \\ u_t + uu_x + vu_y & = & -p_x/\rho_0 \\ v_t + uv_x + vv_y & = & -p_y/\rho_0 \end{array}\right\} \qquad \underline{\text{ok}}$$

In ideal flow, each fluid layer slips past the adjacent layer and maintains its speed; there is no transfer of x-momentum in the y-direction. However in a real flow, the fast layer slows down and the slow layer speeds up, due to the effect of viscosity (fluid friction); x-momentum is transferred in the y-direction from the fast layer to the slow layer. In this case the fluid stress force has the form $-pn + \sigma n$, where σ is the viscous stress tensor.

conservation of momentum

integral form : $\displaystyle \frac{d}{dt}\int_W \rho u \, dV = -\int_{\partial W} (pn + \rho u(u \cdot n) - \sigma n) \, dA$

$\sigma_{12} = e_x \sigma e_y$: x-component of stress force acting on a surface with $n = e_y$

momentum equation : $\displaystyle \rho \frac{Du}{Dt} = -\nabla p + \nabla \cdot \sigma$ (div acts on each row of σ)

assumption

$\sigma = \mu(\nabla u + \nabla u^T)$: for incompressible flow

1. σ depends linearly on ∇u : Newtonian fluid

2. σ is symmetric

3. $\sigma_{12} = \mu(u_y + v_x)$

4. $\nabla \cdot \nabla u = \Delta u$, $\nabla \cdot \nabla u^T = \nabla(\nabla \cdot u)$: hw2

$\Rightarrow$ $\nabla \cdot \sigma = \mu \Delta u$

incompressible Navier-Stokes equations

$\nabla \cdot u = 0$

$$\frac{Du}{Dt} = -\frac{\nabla p}{\rho_0} + \nu \Delta u \quad , \quad \nu = \frac{\mu}{\rho_0} \quad : \quad \underline{\text{kinematic viscosity}}$$

$u|_{\partial D} = 0$: no-slip BC on solid boundaries

ex : unbounded parallel shear flow

$$\left.\begin{aligned} u(x,y,t) &= u_0(y,t) \\ v(x,y,t) &= 0 \\ p(x,y,t) &= p_0 \end{aligned}\right\} \Rightarrow \begin{aligned} \nabla \cdot u &= u_x + v_y = 0 \\ u_t + uu_x + vu_y &= -p_x/\rho_0 + \nu(u_{xx} + u_{yy}) \\ v_t + uv_x + vv_y &= -p_y/\rho_0 + \nu(v_{xx} + v_{yy}) \end{aligned}$$

ok if $u_0(y,t)$ satisfies $u_t = \nu u_{yy}$: diffusion equation , more later

kinetic energy , $\rho_0 = 1$

ideal flow

$$\frac{d}{dt}\int_W \tfrac{1}{2}|u|^2\, dV = -\int_{\partial W}(\tfrac{1}{2}|u|^2 + p)u \cdot n\, dA \quad , \quad \text{where } W \subset D \ : \ \text{hw1}$$

$$u \cdot n|_{\partial D} = 0 \ \Rightarrow \ \frac{d}{dt}\int_D \tfrac{1}{2}|u|^2\, dV = 0 \ : \ \text{energy conservation}$$

viscous flow

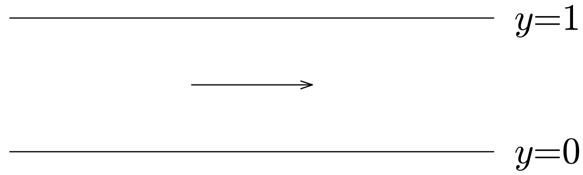
$$\begin{aligned} (\tfrac{1}{2}|u|^2)_t &= u \cdot u_t = u \cdot (-(u\cdot\nabla)u - \nabla p + \nu \Delta u) \\ &= -u \cdot \nabla(\tfrac{1}{2}|u|^2) - u \cdot \nabla p + \nu u \cdot \Delta u \ , \ \text{using } u \cdot (u \cdot \nabla)u = u \cdot \nabla(\tfrac{1}{2}|u|^2) \ : \ \text{hw1} \\ &= -\nabla \cdot ((\tfrac{1}{2}|u|^2 + p)u) + \nu u \cdot \Delta u \ , \ \text{using } \nabla \cdot u = 0 \end{aligned}$$

$$u \cdot \Delta u = \nabla \cdot ((\nabla u)^T u) - |\nabla u|^2 \ , \ \text{where } |\nabla u|^2 = |\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2 \ : \ \text{hw2}$$

$$\begin{aligned} \frac{d}{dt}\int_W \tfrac{1}{2}|u|^2\, dV &= \int_W (\tfrac{1}{2}|u|^2)_t\, dV \\ &= -\int_W \nabla \cdot ((\tfrac{1}{2}|u|^2 + p)u)\, dV + \nu \int_W (\nabla \cdot ((\nabla u)^T u) - |\nabla u|^2)\, dV \\ &= -\int_{\partial W} (\tfrac{1}{2}|u|^2 + p)u \cdot n\, dA + \nu \int_{\partial W} ((\nabla u)^T u) \cdot n\, dA - \nu \int_W |\nabla u|^2\, dV \end{aligned}$$

$$u|_{\partial D} = 0 \ \Rightarrow \ \frac{d}{dt}\int_D \tfrac{1}{2}|u|^2\, dV = -\nu \int_D |\nabla u|^2\, dV \ : \ \underline{\text{energy dissipation}}$$

<u>ex</u> : incompressible viscous flow in a channel



1. $u = u(y)$, $v = 0$, $p = p(x)$: steady pressure-driven parallel shear flow

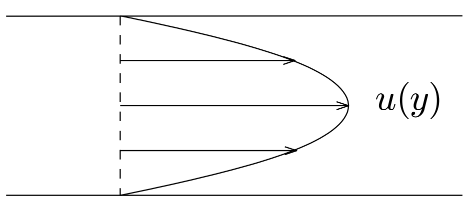
$u_x + v_y = 0$: ok

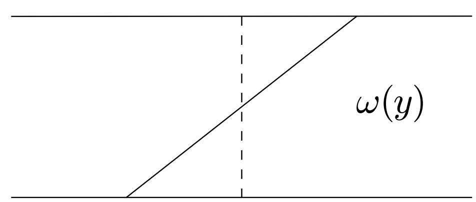
$u_t + uu_x + vu_y = -p_x/\rho_0 + \nu(u_{xx} + u_{yy})$

$v_t + uv_x + vv_y = -p_y/\rho_0 + \nu(v_{xx} + v_{yy})$: ok

$u_{yy} = \dfrac{p_x}{\nu\rho_0} \quad \Rightarrow \quad p_x = \text{constant} \quad (\text{assume } p_x < 0)$

$u(0) = u(1) = 0 \quad \Rightarrow \quad u(y) = \dfrac{-p_x}{2\nu\rho_0} y(1-y)$: <u>plane Poiseuille flow</u>

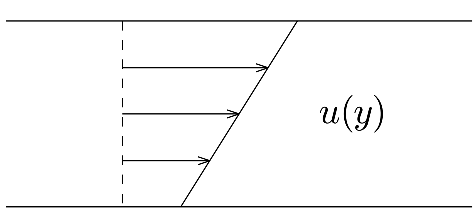


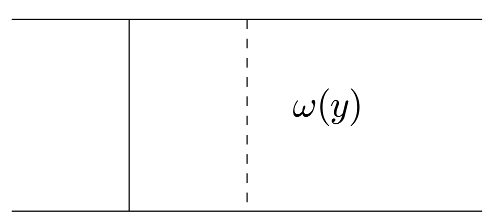


vorticity : $\omega = v_x - u_y = \dfrac{p_x}{2\nu\rho_0}(1-2y)$

2. $u = u(y)$, $v = 0$, $p_x = 0$, walls move in the x-direction

$u_{yy} = 0$, $u(0) = U_0$, $u(1) = U_1 \Rightarrow u(y) = U_0 + (U_1 - U_0)y$: <u>plane Couette flow</u>

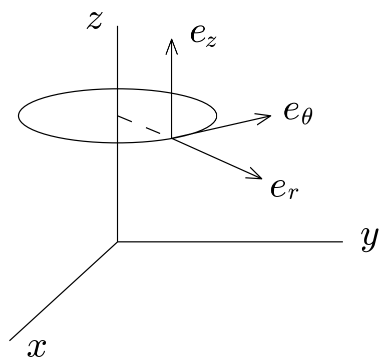




vorticity : $\omega = v_x - u_y = U_0 - U_1$

Navier-Stokes equations in cylindrical coordinates : hw2 , 45/2

r : radial , θ : azimuthal , z : axial



$$u = u_r e_r + u_\theta e_\theta + u_z e_z$$

$$\nabla = e_r \frac{\partial}{\partial r} + \frac{e_\theta}{r}\frac{\partial}{\partial \theta} + e_z \frac{\partial}{\partial z}$$

$$\nabla \cdot u = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (u \cdot \nabla) = \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}$$

$$\Delta = \nabla \cdot \nabla = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

$$\frac{Du}{Dt} = -\nabla p + \nu \Delta u$$

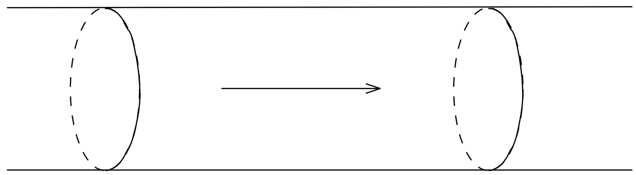
$$\frac{Du_r}{Dt} - \frac{u_\theta^2}{r} = -\frac{\partial p}{\partial r} + \nu\Big(\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\Big)$$

$$\frac{Du_\theta}{Dt} + \frac{u_r u_\theta}{r} = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \nu\Big(\Delta u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\Big)$$

$$\frac{Du_z}{Dt} = -\frac{\partial p}{\partial z} + \nu \Delta u_z$$

$$\nabla \times u = \Big(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\Big)e_r + \Big(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\Big)e_\theta + \Big(\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\Big)e_z$$

<u>ex</u> : incompressible viscous flow in a circular pipe : 45/1



$0 \le r \le R\ ,\ 0 \le \theta \le 2\pi\ ,\ \ -\infty < z < \infty$

$u = u(r)e_z\ ,\ p = p(z)$: steady pressure-driven axisymmetric axial flow

$\Rightarrow\ \nabla \cdot u = 0\ ,\ u \cdot n = 0$ on pipe wall

r-component of momentum equation : ok

θ-component .... : ok

z-component .... : $0 = -\dfrac{\partial p}{\partial z} + \nu\,\dfrac{1}{r}\dfrac{\partial}{\partial r}\Big(r\dfrac{\partial u}{\partial r}\Big)$

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big) = \frac{1}{\nu}\frac{\partial p}{\partial z} = \text{constant} = \frac{p_z}{\nu}$$

$$u(r) = \frac{p_z}{\nu}\frac{r^2}{4} + a\log r + b \quad , \quad \begin{cases} u(0) : \text{finite} & \Rightarrow\ a = 0 \\ u(R) = 0 & \Rightarrow\ b = \cdots \end{cases}$$

$u(r) = \dfrac{p_z}{4\nu}(r^2 - R^2)$: <u>axisymmetric Poiseuille flow</u>

vorticity : $\omega = -u'(r)e_\theta$: hw2

<u>mass flow rate</u> through the pipe

$$\int_S u\,dA \ = \ \int_0^{2\pi}\int_0^R \frac{p_z}{4\nu}(r^2 - R^2)\,r\,dr\,d\theta \ = \ \cdots \ = \ -\frac{\pi p_z}{8\nu}R^4$$

<u>hw2</u>

1. $R_1 \le r \le R_2\,,\ u = u(r)e_z\,,\ p = p(z)$: axisymmetric annular Poiseuille flow

2. $R_1 \le r \le R_2\,,\ u = u(r)e_\theta\,,\ p = p(r)\,,\ u(R_1) = U_1\,,\ u(R_2) = U_2$: Couette flow

vorticity equation

$$\frac{Du}{Dt} = -\frac{\nabla p}{\rho_0} + \nu \Delta u$$

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla) u + \nu \Delta \omega \quad : \quad \text{convection} \ , \ \text{stretching} \ , \ \text{diffusion}$$

$[\nu] = \dfrac{L^2}{T}$: (15°C , 1 atm)

	air	water	olive oil
ν (cm^2/s)	0.145	0.0114	1.08

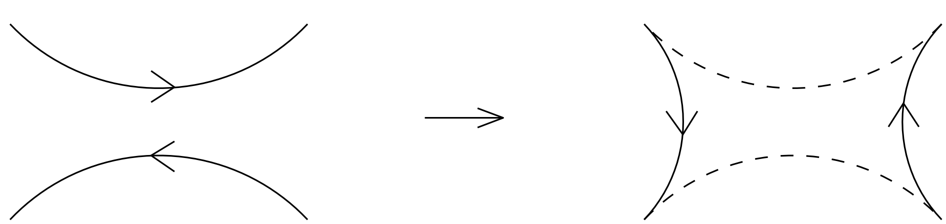
non-dimensionalization

$$x = x^* L \ , \ u = u^* U \ , \ t = t^* \frac{L}{U} \ , \ \omega = \omega^* \frac{U}{L}$$

$$\frac{D\omega^*}{Dt^*} \cdot \frac{U}{L} \cdot \frac{U}{L} = (\omega^* \cdot \nabla^*) u^* \cdot \frac{U}{L} \cdot \frac{1}{L} \cdot U \ + \ \nu \Delta^* \omega^* \cdot \frac{1}{L^2} \cdot \frac{U}{L}$$

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla) u \ + \frac{1}{Re} \Delta u \ , \ Re = \frac{UL}{\nu}$$: Reynold's number , flow similarity

inviscid flow	viscous flow
$\dfrac{D\omega}{Dt} = (\omega \cdot \nabla) u$	$\dfrac{D\omega}{Dt} = (\omega \cdot \nabla) u + \nu \Delta u$
$u \cdot n\vert_{\partial D} = 0$	$u\vert_{\partial D} = 0$
Γ_{C_t} invariant in time	fails
total kinetic energy conserved	dissipates
$\omega(\phi(x,t),t) = \phi_x(x,t)\omega(x,0)$	fails
vortex lines preserve their topology	vortex lines may reconnect due to viscous cancellation of vorticity



<u>thm</u> (Helmholtz decomposition)

Any vector field G on a domain D has a unique decomposition $G = F + \nabla f$, where $\nabla \cdot F = 0\,,\ F \cdot n|_{\partial D} = 0\,,$ and $F \perp \nabla f$ in $L^2(D)$.

<u>heuristic</u>

$$G = F + \nabla f \ \Rightarrow \begin{cases} \nabla \cdot G = \nabla \cdot F + \nabla \cdot \nabla f & \Rightarrow \ \Delta f = \nabla \cdot G \\ G \cdot n = F \cdot n + \nabla f \cdot n & \Rightarrow \ \partial_n f = G \cdot n \text{ on } \partial D \end{cases}$$

<u>pf</u>

given G, let f satisfy $\Delta f = \nabla \cdot G\ ,\ \partial_n f = G \cdot n$ on ∂D : <u>Neumann problem</u>

solvability condition : $\int_D \nabla \cdot G \, dV = \int_{\partial D} G \cdot n \, dA$: ok

set $F = G - \nabla f$

then $G = F + \nabla f\ ,\ \nabla \cdot F = \nabla \cdot G - \nabla \cdot \nabla f = \nabla \cdot G - \Delta f = 0$

$$F \cdot n|_{\partial D} = (G - \nabla f) \cdot n|_{\partial D} = 0$$

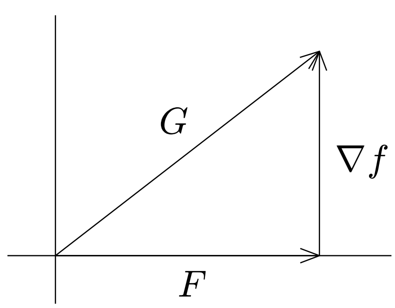
$$\int_D F \cdot \nabla f \, dV = \int_D (\nabla \cdot (fF) - f(\nabla \cdot F)) \, dV = \int_{\partial D} fF \cdot n \, dA = 0$$

$$G = F_1 + \nabla f_1 = F_2 + \nabla f_2 \ \Rightarrow \ F_1 - F_2 + \nabla(f_1 - f_2) = 0$$

$$\Rightarrow \int_D (|F_1 - F_2|^2 + (F_1 - F_2) \cdot \nabla(f_1 - f_2)) \, dV = 0 \ \Rightarrow \ \int_D |F_1 - F_2|^2 \, dV = 0$$

$\Rightarrow \ F_1 = F_2\ ,\ \nabla f_1 = \nabla f_2$ <u>ok</u>

<u>picture</u> gradient fields



divergence-free fields, parallel to boundary

$F = PG$, P : <u>projection</u> , P : linear , $PF = F$, $P \nabla f = 0$

<u>application</u> : incompressible Navier-Stokes

$$\nabla \cdot u = 0$$

$$u_t + (u \cdot \nabla)u = -\nabla p + \nu \Delta u$$

$$u_t + \nabla p = -(u \cdot \nabla)u + \nu \Delta u$$

$$P(u_t + \nabla p) = P(-(u \cdot \nabla)u + \nu \Delta u)$$

$u_t = P(-(u \cdot \nabla)u + \nu \Delta u)$: <u>projection method</u> (Chorin, Temam)

2.1 potential flow (2D or 3D)

def

potential flow : $u = \nabla\phi$, ϕ : potential function

irrotational flow : $\omega = \nabla \times u = 0$

note

1. If u is a potential flow, then u is irrotational.

2. If u is irrotational on a simply connected domain, then u is a potential flow.
(note : we'll discuss the case of a non-simply connected domain soon)

3. If u is irrotational on a simply connected domain and C is any closed curve in the domain, then $\Gamma_C = 0$.

4. If u is irrotational on a domain and C_1 is homologous to C_2, then $\Gamma_{C_1} = \Gamma_{C_2}$.

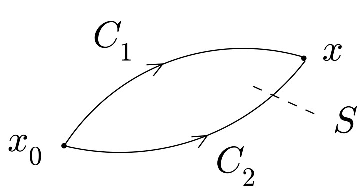
pf

1. $u = \nabla\phi \;\Rightarrow\; \nabla \times u = \nabla \times \nabla\phi = 0$ ok

2. define $\phi(x) = \int_{x_0}^{x} u \cdot ds$: line integral along any curve connecting x_0 and x

$$\phi(x) = \int_{x_0}^{x} u dx + v dy + w dz \;\Rightarrow\; \nabla\phi = u$$

need to show ϕ is well-defined , suppose C_1, C_2 are two such curves

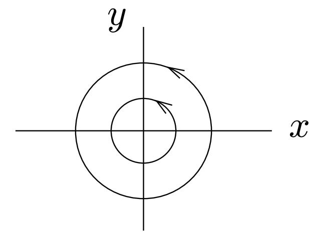


$$\int_{C_2 - C_1} u \cdot ds = \int_{\partial S} u \cdot ds = \int_S (\nabla \times u) \cdot n \, dA = 0 \quad \text{ok}$$

$$3.\ \Gamma_C = \int_C u \cdot ds = \begin{cases} \int_C \nabla\phi \cdot ds = 0 \\ \int_S (\nabla \times u) \cdot n \, dA = 0 \quad \text{ok} \end{cases}$$

4. ok

<u>ex</u> : <u>point vortex</u>



$$\psi(x,y) = \frac{-1}{4\pi}\log(x^2+y^2) = \frac{-1}{2\pi}\log r$$

streamlines : r =constant

$$u = \psi_y = \frac{-y}{2\pi(x^2+y^2)} \quad , \quad v = -\psi_x = \frac{x}{2\pi(x^2+y^2)}$$

$$(u,v) = \frac{1}{2\pi r}(-\sin\theta, \cos\theta) = \frac{1}{2\pi r}e_\theta \quad , \quad |u| \ \to \ \begin{cases} 0 & \text{as } r\to\infty \\ \infty & \text{as } r\to 0 \end{cases}$$

$$\omega = v_x - u_y = -\Delta\psi = -\frac{1}{r}(r\psi_r)_r = 0 \quad \Rightarrow \ u \text{ is irrotational for } (x,y)\neq(0,0)$$

$\Rightarrow$ u is a potential flow on any simply connected domain not containing $(0,0)$

$$u = \phi_x \ , \ v = \phi_y \ \Rightarrow \ \phi(x,y) = \frac{1}{2\pi}\tan^{-1}\left(\frac{y}{x}\right) = \frac{\theta}{2\pi} \quad : \ \text{multi-valued}$$

If C is any closed curve not surrounding $(0,0)$, then $\Gamma_C = \int_C u\cdot ds = 0$.

Alternatively, let $C = \{(x,y) : x^2+y^2 = R^2\}$, oriented counterclockwise.

$$\Rightarrow \ \Gamma_C = \int_C u\cdot ds = \begin{cases} \displaystyle\int_0^{2\pi}\frac{1}{2\pi R}e_\theta\cdot e_\theta\, R\,d\theta = 1 \\ \displaystyle\int_{r\leq R}\omega\,dA = 1 \ \Rightarrow \ \omega(x,y) = \delta(x,y) \ : \ \underline{\text{delta function}} \end{cases}$$

<u>def</u>

Consider the inner product $<f,g> = \int_{\mathbb{R}^2} f(x)g(x)dx$. The delta function $\delta(x)$ is the <u>distribution</u> satisfying the relation $<\delta,f> = \int_{\mathbb{R}^2}\delta(x)f(x)dx = f(0)$ for all <u>test functions</u> f in $C_0^\infty(\mathbb{R}^2)$. (Heuristically, $\delta(x) = 0$ for $x\neq 0$, but $\delta(0) = \infty$.)

<u>thm</u> : The vorticity of a point vortex is a delta function.

<u>pf</u>

$$\omega = -\Delta\psi \quad , \quad \psi = \frac{-1}{2\pi}\log r \quad , \quad \text{recall} : \ -\Delta\psi = 0 \ \text{ for } \ r\neq 0$$

We must show that $-\Delta\psi = \delta$ in the sense of distributions (in other words, $-\psi$ is a <u>fundamental solution</u> of the Laplace equation on $\mathbb{R}^2$). The <u>weak form</u> of the equation $-\Delta\psi = \delta$ is the statement that $<-\Delta\psi, f> \equiv <-\psi, \Delta f> = <\delta, f>$ for all test functions f.

$$<-\psi, \Delta f> = \int_0^{2\pi}\!\!\int_0^{\infty} -\psi(r)\left(\frac{1}{r}\,\partial_r(\,r\,\partial_r) + \frac{1}{r^2}\,\partial_\theta^2\right) f(r,\theta)\, r dr d\theta$$

$$= \int_0^{2\pi}\!\!\int_0^{\infty} \frac{1}{2\pi}\log r\;\, \partial_r(\,r\,\partial_r) f(r,\theta)\, dr d\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{2\pi}\log r \cdot r\,\partial_r f(r,\theta)\Big|_0^{\infty} - \int_0^{\infty} \frac{1}{2\pi r}\; r\,\partial_r f(r,\theta)\, dr\right) d\theta$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)\Big|_0^{\infty} d\theta \;=\; -\frac{1}{2\pi}\int_0^{2\pi} -f(0)\, d\theta \;=\; f(0) \;=\; <\delta, f> \qquad \underline{\text{ok}}$$

<u>note</u>

$\psi_\epsilon = \frac{-1}{2\pi}\log\sqrt{r^2+\epsilon^2} \;\Rightarrow\; -\Delta\psi_\epsilon = \delta_\epsilon$: approximate δ-function (<u>vortex-blob</u>, hw)

<u>2D incompressible irrotational flow</u>

$$\left.\begin{array}{lllll} \nabla\cdot u = 0 & \Rightarrow & u_x + v_y = 0 & \Rightarrow & u_x = -v_y \\ \nabla\times u = 0 & \Rightarrow & v_x - u_y = 0 & \Rightarrow & u_y = \;\;\, v_x \end{array}\right\} : \text{ Cauchy-Riemann eqs for } u\,, -v$$

$\Rightarrow\;\; F = u - iv$: <u>complex velocity</u> , analytic function of $z = x + iy$

$$\left.\begin{array}{lll} \nabla\cdot u = 0 & \Rightarrow & u = \psi_y \;,\; v = -\psi_x \\ \nabla\times u = 0 & \Rightarrow & u = \phi_x \;,\; v = \;\;\;\phi_y \end{array}\right\} \;\Rightarrow\; u - iv = \begin{cases} \phi_x + i\psi_x = (\phi + i\psi)_x \\ \psi_y - i\phi_y = (\phi + i\psi)_{iy} \end{cases}$$

$\Rightarrow\;\; W = \phi + i\psi$: <u>complex potential</u> , $F = W_x = W_{iy} = W_z$

<u>ex</u>

$$W = \frac{1}{2\pi i}\log z \;=\; \frac{1}{2\pi i}\,(\log|z| + i\arg z) \;=\; \frac{1}{2\pi}(\theta - i\log r)$$

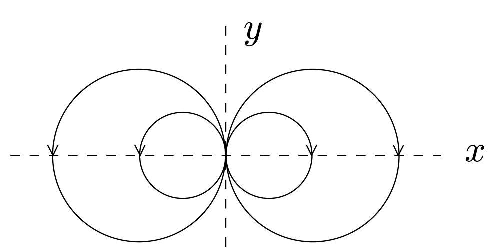
$$\Rightarrow\;\; \phi \;=\; \text{real}\, W = \frac{\theta}{2\pi}\;,\;\; \psi \;=\; \text{imag}\, W = \frac{-1}{2\pi}\log r \;\;:\;\; \text{point vortex}$$

$$W = \frac{1}{2\pi i z} \;=\; \frac{d}{dz}\left(\frac{1}{2\pi i}\log z\right) = \lim_{\epsilon\to 0}\frac{1}{2\pi i}\,\frac{\log(z+\epsilon) - \log(z-\epsilon)}{2\epsilon} \quad : \quad \underline{\text{vortex-dipole}}$$

$$= \frac{1}{2\pi i}\cdot\frac{x - iy}{x^2+y^2} \;=\; \frac{-y - ix}{2\pi\,(x^2+y^2)} \;=\; \phi + i\psi$$

$$\Rightarrow\;\; \psi = \frac{-x}{2\pi(x^2+y^2)} = c \;\Rightarrow\; x^2 + y^2 + \frac{x}{2\pi c} = 0 \;\Rightarrow\; \left(x + \frac{1}{4\pi c}\right)^2 + y^2 = \frac{1}{16\pi^2 c^2}$$

$\Rightarrow$ the streamlines are circles centered at $(x,y) = \left(-\dfrac{1}{4\pi c}\,,\,0\right)$ with radius $\dfrac{1}{4\pi|c|}$



$$y = 0 \;\;\Rightarrow\;\; \begin{cases} \psi = \dfrac{-1}{2\pi x} \\ v = -\psi_x = \dfrac{-1}{2\pi x^2} < 0 \end{cases}$$

question : what is the vorticity of a vortex-dipole?

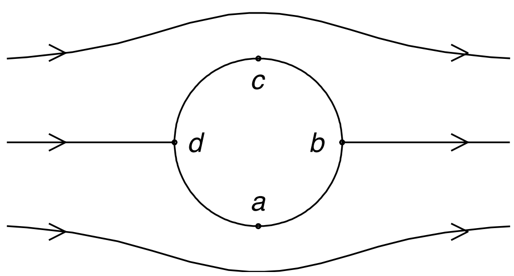
<u>ex</u> : potential flow past a cylinder

$W = U\Big(z + \dfrac{R^2}{z}\Big)$: uniform stream + vortex-dipole , $U > 0$

$F = U\Big(1 - \dfrac{R^2}{z^2}\Big) \to U$ as $|z| \to \infty$

$z = x \;\Rightarrow\; W = U\Big(x + \dfrac{R^2}{x}\Big) \;\Rightarrow\; \psi = \text{imag}\, W = 0$: streamline

$|z| = R \;\Rightarrow\; W = U\Big(z + \dfrac{R^2\,\overline{z}}{|z|^2}\Big) = U(z + \overline{z}) \;\Rightarrow\; \psi = \text{imag}\, W = 0$: streamline



$b\,,d$: stagnation points

<u>note</u> : $|F|$ attains it's max and min on the boundary of the flow domain

recall Bernoulli's thm : $\frac{1}{2}\rho_0|u|^2 + p \;=\;$ constant on streamlines

<u>slip velocity on cylinder surface</u>

$z = Re^{i\theta} \;\Rightarrow\; F = U(1 - e^{-2i\theta}) = U(1 - \cos 2\theta + i \sin 2\theta)$

$|u|^2 = |F|^2 = U^2((1 - \cos 2\theta)^2 + \sin^2 2\theta) = 2U^2(1 - \cos 2\theta) = 4U^2 \sin^2\theta$

$\Rightarrow\; 0 \le |u| \le 2U$

$|u| = 0 \;\Rightarrow\; \theta = 0\,,\pi$: $z = b\,,d$: $|u|$ is min , p is max

$|u| = 2U \;\Rightarrow\; \theta = \pm\frac{\pi}{2}$: $z = a\,,c$: $|u|$ is max , p is min

$\Gamma_{|z|=R} \;=\; \displaystyle\int_{|z|=R} u \cdot ds \;=\; 0$

<u>thm</u> (strong form of Bernoulli's theorem)
In steady homogeneous ideal <u>potential</u> flow, the quantity $\frac{1}{2}\rho_o|u|^2 + p$ is constant on the entire flow domain.

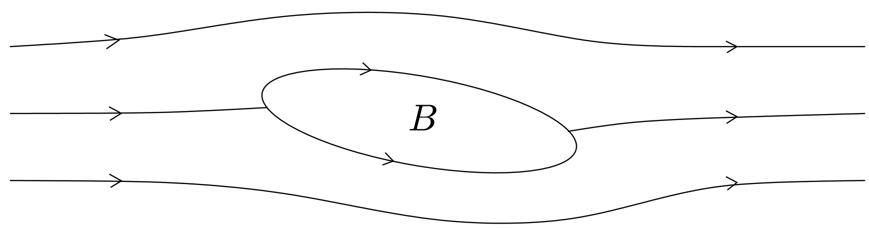
<u>pf</u>

$(u \cdot \nabla)u = -\dfrac{\nabla p}{\rho_0}\;,\; u = \nabla\phi \;\;\Rightarrow\;\; (u \cdot \nabla)u = \nabla(\frac{1}{2}|u|^2) - u \times (\nabla \times u) = \nabla(\frac{1}{2}|u|^2)$

$\Rightarrow\; \nabla\Big(\frac{1}{2}|u|^2 + \dfrac{p}{\rho_0}\Big) = 0$ <u>ok</u>

thm

Consider steady homogeneous ideal potential flow on the exterior of a compact body B and let $\mathcal{F}$ denote the force exerted by the fluid on B.



1. (Blasius) $\mathcal{F} = -\frac{1}{2} i\rho_0 \overline{\int_{\partial B} F^2\, dz}$, where $F = u - iv$

2. (Kutta-Joukowski) If $F \to U_\infty$ as $|z| \to \infty$, then $\mathcal{F} = -\rho_0 \Gamma |U_\infty| n$, where Γ is the circulation around ∂B and n is a unit vector normal to U_∞.

pf

$$1.\ \mathcal{F} = -\int_{\partial B} p\, n\, ds = i \int_{\partial B} p\, dz \quad , \quad p = p_0 - \tfrac{1}{2}\rho_0 |u|^2$$

$$= i \int_{\partial B} (p_0 - \tfrac{1}{2}\rho_0 |u|^2)\, dz = -\tfrac{1}{2} i\rho_0 \int_{\partial B} (u^2 + v^2)\, dz$$

$$\overline{F^2\, dz} = \overline{(u - iv)^2\, dz} = (u^2 - v^2 + 2iuv)(dx - idy)$$

$$= (u^2 - v^2)dx + 2uv\, dy + i(\, 2uv\, dx - (u^2 - v^2)dy\,)$$

note : $u \cdot n|_{\partial B} = 0 \;\Rightarrow\; (u, v) \cdot (dy, -dx) = 0 \;\Rightarrow\; u\, dy = v\, dx$

$$\overline{F^2\, dz} = (u^2 - v^2)dx + 2v^2 dx + i(2u^2 dy - (u^2 - v^2)dy)$$

$$= (u^2 + v^2)(dx + idy) = (u^2 + v^2)dz \qquad \underline{\text{ok}}$$

2. $F(z)$: analytic outside B , bounded as $|z| \to \infty$

$\Rightarrow F(z) = a_0 + \dfrac{a_1}{z} + \dfrac{a_2}{z^2} + \cdots$: <u>Laurent expansion</u>

ex : recall potential flow past a cylinder , $F(z) = U\left(1 - \dfrac{R^2}{z^2}\right)$

$$F^2 = a_0^2 + \frac{2a_0 a_1}{z} + \frac{2a_0 a_2 + a_1^2}{z^2} + \cdots$$

$a_0 = U - iV$, where $U_\infty = (U, V)$

$$a_1 = \frac{1}{2\pi i} \int_{\partial B} F\, dz = \frac{1}{2\pi i} \int_{\partial B} (u - iv)(dx + idy)$$

$$= \frac{1}{2\pi i} \int_{\partial B} udx + vdy + i(udy - vdx) = \frac{1}{2\pi i} \int_{\partial B} u \cdot ds = \frac{\Gamma}{2\pi i}$$

$$\mathcal{F} \;=\; -\tfrac{1}{2} i\rho_0 \overline{\int_{\partial B} F^2\,dz} \;=\! -\tfrac{1}{2}\, i\rho_0 \cdot \overline{2\pi i \cdot 2a_0 a_1} \;=\; -2\pi\rho_0 (U+iV)\frac{\Gamma}{-2\pi i}$$

$$\;=\; -i\rho_0\Gamma(U+iV) \;=\; -\rho_0\Gamma|U_\infty| n \qquad \underline{\text{ok}}$$

<u>def</u>

<u>drag</u> : component of $\mathcal{F}$ in direction of U_∞ , $\mathcal{F}_D = 0$

<u>lift</u> : component of $\mathcal{F}$ normal to U_∞ , $\mathcal{F}_L = -\rho_0\Gamma|U_\infty|$

$\Rightarrow$ if $\Gamma = 0$, then $\mathcal{F} = 0$: <u>d'Alembert's paradox</u>

<u>ex</u> : potential flow past a cylinder with circulation

$$W \;=\; U\Big(z + \frac{R^2}{z}\Big) + \frac{\Gamma}{2\pi i}\log z \;=\; \phi + i\psi \quad , \quad \Gamma\,,\, U > 0$$

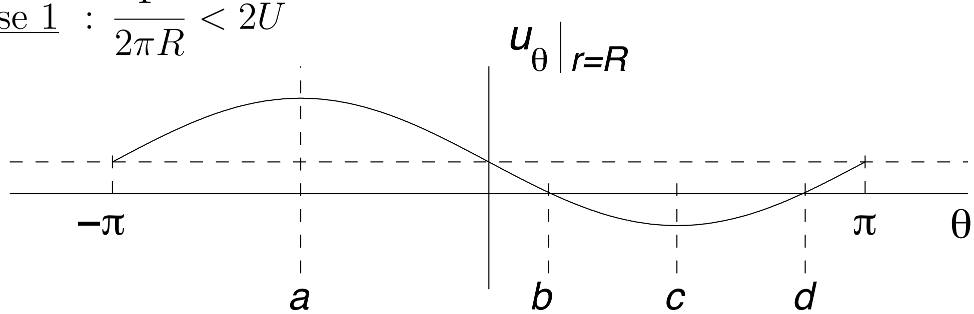
$$\phi(r,\theta) \;=\; U\cos\theta\Big(r + \frac{R^2}{r}\Big) + \frac{\Gamma\theta}{2\pi}$$

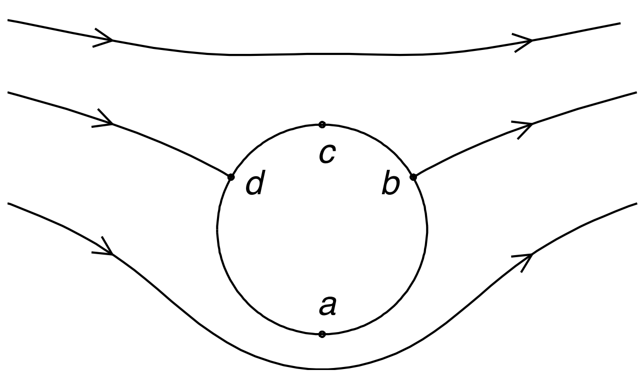
$u \;=\; \nabla\phi \;=\; u_r e_r + u_\theta e_\theta$, stagnation points : $u_r \;=\; u_\theta \;=\; 0$

$$u_r \;=\; \partial_r\phi \;=\; U\cos\theta\Big(1 - \frac{R^2}{r^2}\Big) \;=\; 0 \;\Rightarrow\; \theta = \pm\frac{\pi}{2} \;\text{ or }\; r = R$$

$$u_\theta \;=\; \frac{1}{r}\,\partial_\theta\phi \;=\; \frac{1}{r}\Big(-U\sin\theta\Big(r + \frac{R^2}{r}\Big) + \frac{\Gamma}{2\pi}\Big) \;\Rightarrow\; u_\theta|_{r=R} = -2U\sin\theta + \frac{\Gamma}{2\pi R}$$

<u>case 1</u> : $\dfrac{\Gamma}{2\pi R} < 2U$

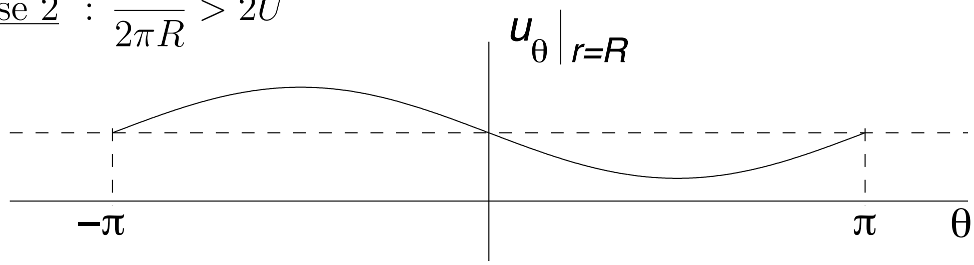


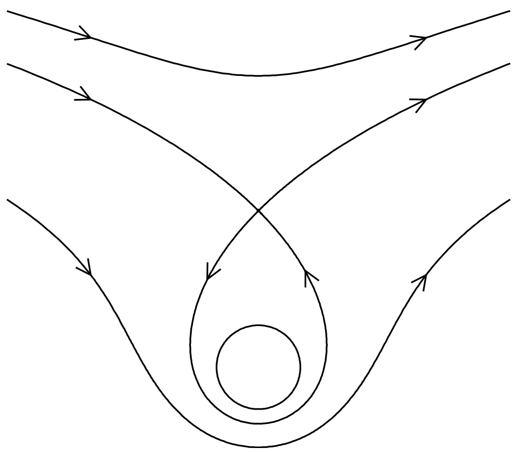


$$u_\theta(a) \;=\; 2U + \frac{\Gamma}{2\pi R} > 0\;,\;\; u_\theta(c) \;=\; -2U + \frac{\Gamma}{2\pi R} < 0$$

$$|u_\theta(a)| > |u_\theta(c)| \;\Rightarrow\; p(a) < p(c) < p(b) = p(d) \quad , \quad \mathcal{F} = -\rho_0\Gamma U e_y$$

case 2 : $\dfrac{\Gamma}{2\pi R} > 2U$





$u_\theta\big|_{r=R} > 0 \;\Rightarrow\;$ there are no stagnation points on ∂B , but there is a stagnation point in the interior of the flow (hw)

conformal mapping

$D_1 \longrightarrow D_2 \;\;,\;\; z \longmapsto \zeta$: analytic , $\zeta'(z) \neq 0$

claim

1. If $W_2(\zeta)$ is the complex potential of a flow on D_2, then $W_1(z) = W_2(\zeta(z))$ is the complex potential of a flow on D_1.
2. If z_0 is a stagnation point (or point vortex) of the flow on D_1, then $\zeta_0 = \zeta(z_0)$ is a stagnation point (or point vortex) of the flow on D_2.
3. If C_1 is a simple closed curve in D_1 and $C_2 = \zeta(C_1)$ is the corresponding curve in D_2, then $\Gamma_{C_1} = \Gamma_{C_2}$.

pf

1. $W_1(z)$ is analytic , just need to check bc

$\operatorname{imag} W_1(z) = \operatorname{imag} W_2(\zeta(z)) \Rightarrow$ streamlines correspond under the mapping ok

2. $\dfrac{dW_1}{dz} = \dfrac{dW_2}{d\zeta} \cdot \dfrac{d\zeta}{dz} \quad \ldots$ ok

3. $\Gamma_{C_1} = \int_{C_1} u \cdot ds = \int_{C_1} (u\,dx + v\,dy) = \text{real} \int_{C_1} (u - iv)(dx + idy)$

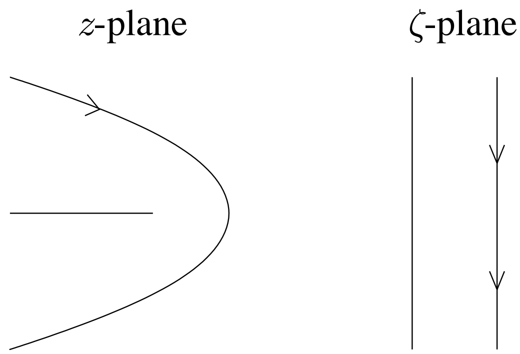
$$= \text{real} \int_{C_1} \frac{dW_1}{dz}\,dz = \text{real} \int_{C_2} \frac{dW_2}{d\zeta}\,d\zeta = \cdots = \Gamma_{C_2} \qquad \text{ok}$$

<u>ex 1</u> : potential flow around a sharp edge

$\zeta(z) = \sqrt{z} = \sqrt{r}e^{i\theta/2}$

D_1 = exterior of semi-infinite plate = $\{z = re^{i\theta} \; : \; r > 0 \; , \; -\pi < \theta < \pi\}$

D_2 = right half-plane = $\{\zeta \; : \; \text{real}\, \zeta > 0\}$



$W_2(\zeta) = i\zeta$: uniform stream , $u - iv = i \;\Rightarrow\; u = 0 \; , \; v = -1$

$W_1(z) = W_2(\zeta(z)) = i\sqrt{z}$: branch cut on plate

<u>streamlines</u>

$\psi_2 = \text{imag}\, W_2(\zeta) = \text{imag}(i\zeta) = \text{real}\, \zeta = c$: line

$\psi_1 = \text{imag}\, W_1(z) = \text{imag}(i\sqrt{r}\, e^{i\theta/2}) = \sqrt{r}\, \cos\frac{\theta}{2} = c$

$\Rightarrow \; r\cos^2\frac{\theta}{2} = c^2 \;\Rightarrow\; r(\frac{1}{2} + \frac{1}{2}\cos\theta) = c^2$

$\Rightarrow \; \sqrt{x^2+y^2} + x = 2c^2 \;\Rightarrow\; \sqrt{x^2+y^2} = 2c^2 - x$

$\Rightarrow \; x^2 + y^2 = 4c^4 - 4c^2x + x^2 \;\Rightarrow\; x = c^2 - \dfrac{y^2}{4c^2}$: parabola

<u>velocity on plate</u>

$$u - iv = \frac{dW_1}{dz} = \frac{i}{2\sqrt{z}} = \frac{i}{2\sqrt{r}}\,(\cos\tfrac{\theta}{2} - i\sin\tfrac{\theta}{2})$$

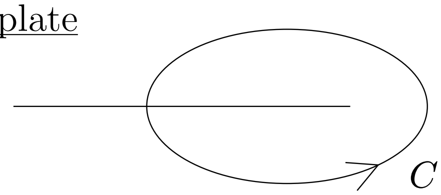
$$u = \frac{1}{2\sqrt{r}}\sin\tfrac{\theta}{2} \;\; , \;\; v = -\frac{1}{2\sqrt{r}}\cos\tfrac{\theta}{2} \quad\Rightarrow\quad \sqrt{u^2+v^2} \to \infty \;\text{ as }\; r \to 0$$

$$x < 0 \; , \; y = 0^+ \quad\Rightarrow\quad \theta = \pi \quad\Rightarrow\quad u = \frac{1}{2\sqrt{r}} \;\; , \;\; v = 0$$

$$x < 0 \; , \; y = 0^- \quad\Rightarrow\quad \theta = -\pi \quad\Rightarrow\quad u = -\frac{1}{2\sqrt{r}} \;\; , \;\; v = 0$$

For $x < 0$, u is discontinuous and v is continuous across $y = 0$, i.e. the velocity has a <u>tangential discontinuity</u> across the plate, i.e. there is a <u>bound vortex sheet</u> on the plate coinciding with the branch cut in the complex potential.

circulation on plate



$$\Gamma = \int_C u \cdot ds = \int_C \nabla\phi \cdot ds = \phi\Big|_{\theta=-\pi}^{\theta=\pi} = -2\sqrt{r} \;:\; \text{jump in potential across plate}$$

$$\phi \;=\; \text{real}\, W_1(z) \;=\; \text{real}(i\sqrt{r}\, e^{i\theta/2}) \;=\; -\sqrt{r}\,\sin\tfrac{\theta}{2}$$

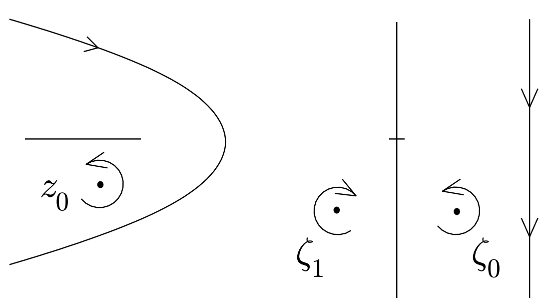
$$\omega \;=\; v_x - u_y \;=\; \delta(y)H(-x)\,\sigma(x) \quad , \quad H(x) = \begin{cases} 0\ , \ x<0 \\ 1\ , \ x>0 \end{cases} \;:\; \underline{\text{Heaviside function}}$$

$$\Gamma = \int_S \omega\, dA \;=\int_S \delta(y)H(-x)\,\sigma(x)dx$$

$$\sigma \;=\; \frac{d\Gamma}{dr} \;=\; -\frac{1}{\sqrt{r}} \;=\; \text{jump in velocity across plate} \;:\; \underline{\text{vortex sheet strength}}$$

<u>ex 2</u> : potential flow around a sharp edge + point vortex

z_0 : point vortex in z-plane , $\zeta_0 = \zeta(z_0) = \sqrt{z_0}$: point vortex in ζ-plane



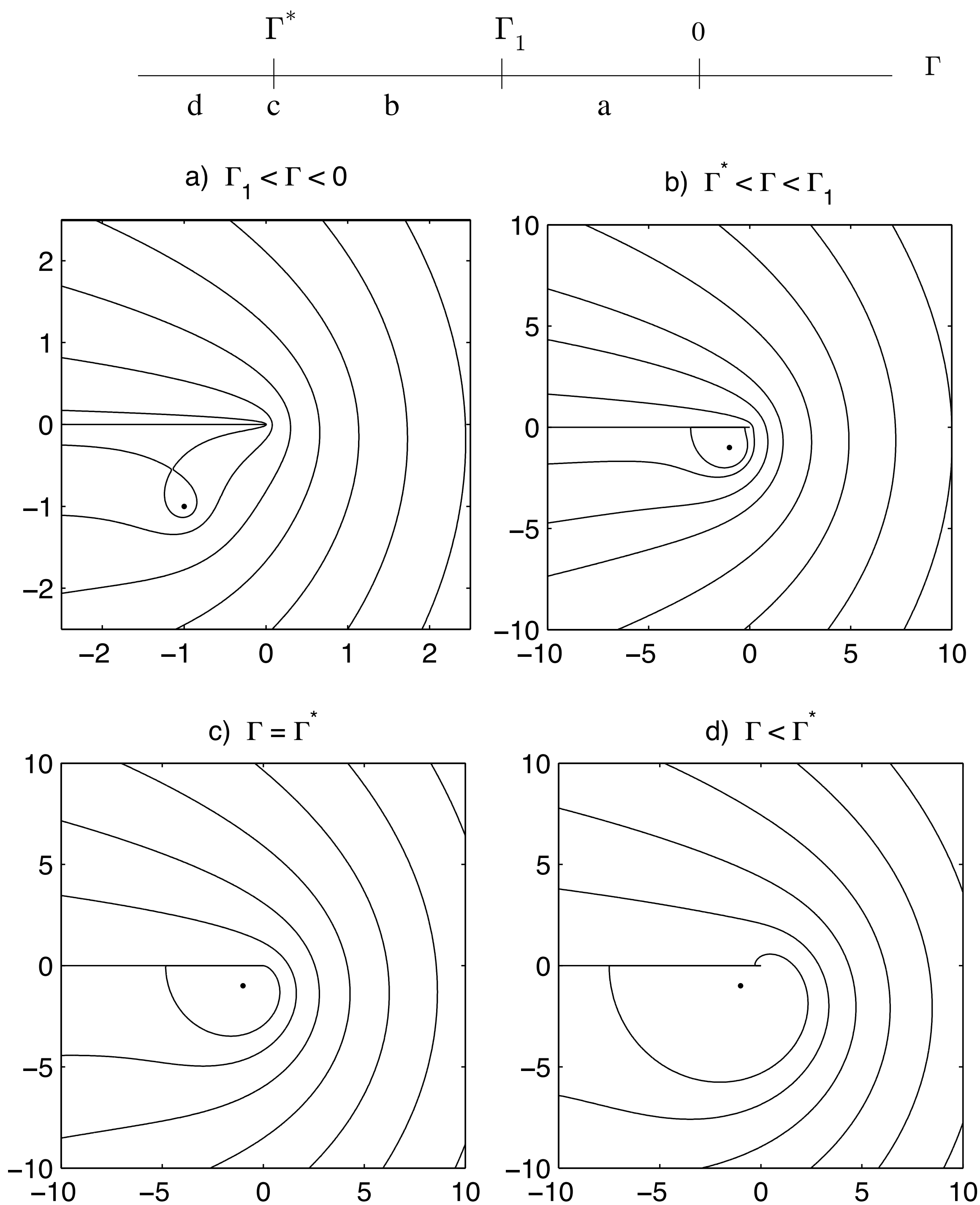
$$W_2(\zeta) \;=\; i\zeta + \frac{\Gamma}{2\pi i}\log(\zeta-\zeta_0) - \frac{\Gamma}{2\pi i}\log(\zeta-\zeta_1) \quad , \quad \zeta_1 = -\overline{\zeta_0} \;:\; \text{image vortex}$$

$$W_1(z) \;=\; W_2(\zeta(z)) \;=\; i\sqrt{z} + \frac{\Gamma}{2\pi i}\log(\sqrt{z}-\zeta_0) - \frac{\Gamma}{2\pi i}\log(\sqrt{z}-\zeta_1)$$

$$= i\sqrt{z} + \underbrace{\frac{\Gamma}{2\pi i}\log(z-z_0)}_{\substack{\uparrow \\ \text{point vortex}}} - \underbrace{\frac{\Gamma}{2\pi i}\log(\sqrt{z}+\zeta_0) - \frac{\Gamma}{2\pi i}\log(\sqrt{z}-\zeta_1)}_{\substack{\uparrow \\ \text{bound vortex sheet}}}$$

$$\frac{dW_1}{dz} = \frac{dW_2}{d\zeta}\cdot\frac{d\zeta}{dz} \quad , \quad \frac{d\zeta}{dz}\Big|_0 = \infty \;\Rightarrow\; \frac{dW_1}{dz}\Big|_0 = \infty \;\text{ unless }\; \frac{dW_2}{d\zeta}\Big|_0 = 0$$

$$\frac{dW_2}{d\zeta} = \ldots \;\Rightarrow\; \Gamma \;=\; \frac{-\pi|\zeta_0|^2}{\text{real}\,\zeta_0} = \Gamma^* \;:\; \underline{\text{Kutta condition}}$$



<u>note</u>

$z \to z_0$: circular streamlines , $z \to \infty$: parabolic streamlines

a) 1 stagnation point in flow, $|u| \to \infty$ as $z \to 0$

b) 2 stagnation points on lower side of plate, $|u| \to \infty$ as $z \to 0$

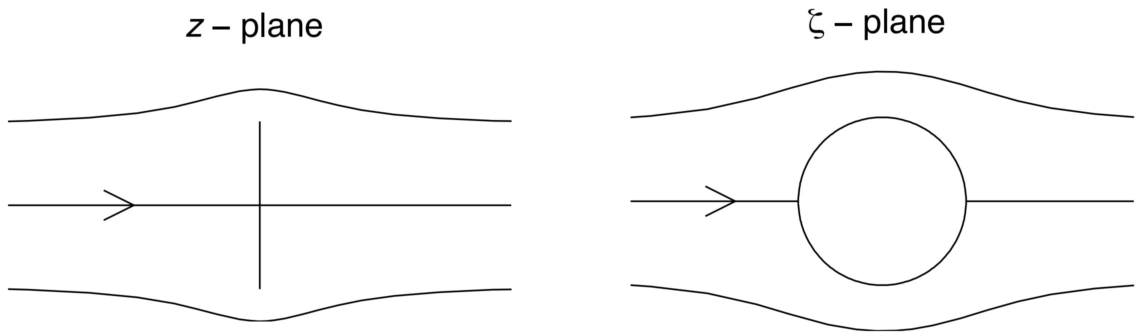
c) 1 stagnation point on lower side of plate, $|u|$ finite as $z \to 0$

d) 2 stagnation points, one on each side of plate, $|u| \to \infty$ as $z \to 0$

<u>ex 3</u> : potential flow past a finite plate

$D_1 \;=\;$ exterior of plate $\;=\; \{z \,:\, z \neq iy \;,\;\; -a \le y \le a\}$

$D_2 \;=\;$ exterior of circle $\;=\; \{\zeta \,:\, |\zeta| > a\}$



$\zeta(z) \;=\; z + \sqrt{z^2 + a^2}$: conformal map from D_1 onto D_2

<u>check</u>

choose $\sqrt{z^2+a^2} \;=\; \sqrt{(z-ia)(z+ia)} \;=\; \sqrt{|z^2+a^2|}e^{i(\theta_1+\theta_2)/2}$

where $\theta_1 \;=\; \arg(z-ia) \;,\; \theta_2 \;=\; \arg(z+ia) \;,\;\; -\frac{\pi}{2} < \theta_1 \;,\; \theta_2 < \frac{3\pi}{2}$

$\Rightarrow \quad \zeta(z)$ has a branch cut on the plate

$z \to \infty \quad \Rightarrow \quad \zeta \to \infty$

$z = iy + 0^+, \; |y| \le a \quad \Rightarrow \quad \theta_1 = -\frac{\pi}{2} \;,\; \theta_2 = \frac{\pi}{2} \quad \Rightarrow \quad \zeta = iy + \sqrt{a^2 - y^2}$

$z = iy + 0^-, \; |y| \le a \quad \Rightarrow \quad \theta_1 = \frac{3\pi}{2} \;,\; \theta_2 = \frac{\pi}{2} \quad \Rightarrow \quad \zeta = iy - \sqrt{a^2 - y^2}$ <u>ok</u>

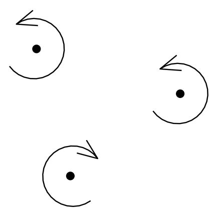
$$W_2(\zeta) \;=\; \frac{U}{2}\Big(\zeta + \frac{a^2}{\zeta}\Big) \quad \Rightarrow \quad W_1(z) \;=\; W_2(\zeta(z)) \;=\; \ldots \;=\; U\sqrt{z^2+a^2}$$

$$(\,\zeta - z = \sqrt{z^2+a^2} \quad \Rightarrow \quad \zeta^2 - 2z\zeta + z^2 = z^2 + a^2 \quad \Rightarrow \quad \zeta(\zeta - 2z) = a^2 \;\ldots)$$

As before, there are square-root singularities in velocity at the edges of the plate and a bound vortex sheet on the plate. The ideal flow is <u>attached</u>, but in reality the flow <u>separates</u> at the edges of the plate.

hw : add a pair of point vortices to model a wake point vortex model

Consider ideal flow on $\mathbb{R}^2$ generated by a set of point vortices located at z_j with circulation Γ_j, for $j = 1, \ldots, N$.



$$W(z) = \frac{1}{2\pi i} \sum_{j=1}^{N} \Gamma_j \log(z - z_j)$$

$$\frac{dW}{dz} = u - iv = \frac{1}{2\pi i} \sum_{j=1}^{N} \frac{\Gamma_j}{z - z_j}$$

$$\omega(z) = \sum_{j=1}^{N} \Gamma_j \, \delta(z - z_j)$$

Consider a fluid model in which $z_j = z_j(t)$ and each point vortex is convected in the velocity field induced by the other vortices.

point vortex equations

$$\overline{\frac{dz_j}{dt}} = \frac{1}{2\pi i} \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{\Gamma_k}{z_j - z_k} \quad , \quad z_j(0) \ : \ \text{given}$$

note

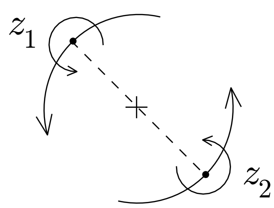
The model is consistent with Kelvin's theorem (in ideal flow, the circulation around a material curve is invariant in time).

<u>ex</u> : $N = 2$, $\Gamma_1 = \Gamma_2 = \Gamma$

$$\overline{\frac{dz_1}{dt}} = \frac{1}{2\pi i}\,\frac{\Gamma}{z_1 - z_2} \quad , \quad \overline{\frac{dz_2}{dt}} = \frac{1}{2\pi i}\,\frac{\Gamma}{z_2 - z_1} = -\overline{\frac{dz_1}{dt}} \quad \Rightarrow \quad z_1 + z_2 = 0$$

$$\overline{\frac{dz_1}{dt}} = \frac{1}{2\pi i}\,\frac{\Gamma}{2z_1} \quad , \quad z_1 = re^{i\theta} \quad \Rightarrow \quad \dot{z}_1 = \dot{r}e^{i\theta} + re^{i\theta}i\dot{\theta}$$

$$\dot{r}e^{-i\theta} - ire^{-i\theta}\dot{\theta} = \frac{-i\Gamma}{4\pi}\,\frac{1}{re^{i\theta}} \quad \Rightarrow \quad \begin{cases} \dot{r} = 0 \quad \Rightarrow \quad r(t) = r_0 \\ \dot{\theta} = \dfrac{\Gamma}{4\pi r^2} \quad \Rightarrow \quad \theta(t) = \theta_0 + \dfrac{\Gamma t}{4\pi r_0^2} \end{cases}$$

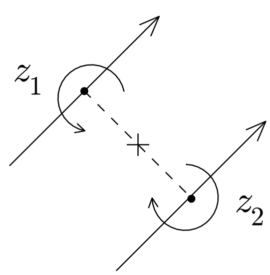


The vortices rotate about the midpoint of the line connecting them, with angular velocity $\Gamma/\pi d^2$, where d is the distance between them.

<u>ex</u> : $N = 2$, $\Gamma_1 = -\Gamma_2 = \Gamma$

$$\overline{\frac{dz_1}{dt}} = \frac{1}{2\pi i}\,\frac{-\Gamma}{z_1 - z_2} \quad , \quad \overline{\frac{dz_2}{dt}} = \frac{1}{2\pi i}\,\frac{\Gamma}{z_2 - z_1} = \overline{\frac{dz_1}{dt}} \quad \Rightarrow \quad z_1 - z_2 = de^{i\alpha}$$

$$\overline{\frac{dz_1}{dt}} = \frac{1}{2\pi i}\,\frac{-\Gamma}{de^{i\alpha}} = \frac{\Gamma}{2\pi d}\,e^{i(\frac{\pi}{2} - \alpha)} \quad \Rightarrow \quad z_1(t) = z_{10} + \frac{\Gamma t}{2\pi d}\,e^{i(\alpha - \frac{\pi}{2})}$$



The vortices move on parallel straight lines orthogonal to the line connecting them, with velocity $\Gamma/2\pi d$, where d is the distance between them.

<u>note</u>

1. contrast this with point charges, point masses
2. $N = 3$, 4 , ...

<u>vortex method</u>

flow map : $x(\alpha,\beta,t)$, $y(\alpha,\beta,t)$

α , β : <u>Lagrangian coordinates</u>

$$\frac{\partial x}{\partial t}(\alpha,\beta,t) = u(x(\alpha,\beta,t), y(\alpha,\beta,t), t)$$

$$\frac{\partial y}{\partial t}(\alpha,\beta,t) = v(x(\alpha,\beta,t), y(\alpha,\beta,t), t)$$

How to determine $u\,,v$?

$$\Delta\psi = -\omega$$

$$g(x,y) = -\frac{1}{2\pi}\log\sqrt{x^2+y^2} \quad\Rightarrow\quad \Delta g = -\delta \;\Rightarrow\; \psi = g*\omega$$

$$\psi(x,y,t) = (g*\omega)(x,y,t) = \int_{\mathbb{R}^2} g(x-\tilde{x}, y-\tilde{y})\,\omega(\tilde{x},\tilde{y},t)\,d\tilde{x}\,d\tilde{y}$$

<u>check</u>

$$\begin{aligned}\Delta\psi(x,y,t) &= \int_{\mathbb{R}^2} \Delta g(x-\tilde{x}, y-\tilde{y})\,\omega(\tilde{x},\tilde{y},t)\,d\tilde{x}\,d\tilde{y}\\ &= \int_{\mathbb{R}^2} -\delta(x-\tilde{x}, y-\tilde{y})\,\omega(\tilde{x},\tilde{y},t)\,d\tilde{x}\,d\tilde{y} = -\omega(x,y,t) \qquad \underline{\text{ok}}\end{aligned}$$

change variables : $\tilde{x} = x(\alpha,\beta,t)$, $\tilde{y} = y(\alpha,\beta,t)$

$$\psi(x,y,t) = \int_{\mathbb{R}^2} g(x-\tilde{x}, y-\tilde{y})\,\omega(x(\alpha,\beta,t), y(\alpha,\beta,t), t)\,J(\alpha,\beta,t)\,d\alpha\,d\beta$$

<u>recall</u>

1. incompressible flow $\Rightarrow$ $J=1$
2. ideal 2D flow $\Rightarrow$ $\omega(x(\alpha,\beta,t), y(\alpha,\beta,t), t) = \omega(\alpha,\beta,0) = \omega_0(\alpha,\beta)$

$$\Rightarrow\; \psi(x,y,t) = \int_{\mathbb{R}^2} g(x-\tilde{x}, y-\tilde{y})\,\omega_0(\alpha,\beta)\,d\alpha\,d\beta \quad,\quad u=\psi_y \quad,\quad v=-\psi_x$$

<u>Lagrangian form of 2D incompressible Euler</u>

unknowns : $x(\alpha,\beta,t)$, $y(\alpha,\beta,t)$

$$\frac{\partial x}{\partial t} = \int_{\mathbb{R}^2} g_y(x-\tilde{x}, y-\tilde{y})\,\omega_0(\tilde{\alpha},\tilde{\beta})\,d\tilde{\alpha}\,d\tilde{\beta}$$

$$\frac{\partial y}{\partial t} = \int_{\mathbb{R}^2} -g_x(x-\tilde{x}, y-\tilde{y})\,\omega_0(\tilde{\alpha},\tilde{\beta})\,d\tilde{\alpha}\,d\tilde{\beta}$$

$$g_y(x,y) = \frac{-y}{2\pi(x^2+y^2)} \quad,\quad -g_x(x,y) = \frac{x}{2\pi(x^2+y^2)}$$

discretization

$(\alpha,\beta) \to (\alpha_j,\beta_j)$, $j = 1,\dots,N$, $x(\alpha_j,\beta_j,t) = x_j(t)$, $y(\alpha_j,\beta_j,t) = y_j(t)$

$$\frac{dx_j}{dt} = \sum_{\substack{k=1\\k\neq j}}^{N} \frac{-(y_j-y_k)}{2\pi((x_j-x_k)^2+(y_j-y_k)^2)}\Gamma_k$$

$$\frac{dy_j}{dt} = \sum_{\substack{k=1\\k\neq j}}^{N} \frac{x_j-x_k}{2\pi((x_j-x_k)^2+(y_j-y_k)^2)}\Gamma_k$$

$\Gamma_k = \omega_0(\alpha_k,\beta_k)h^2$, h : meshsize

note

1. These are the point vortex equations in Cartesian coordinates.

check : $u - iv = \dfrac{1}{2\pi i z} = \dfrac{-i(x-iy)}{2\pi(x^2+y^2)} = \dfrac{-y}{2\pi(x^2+y^2)} - i\,\dfrac{x}{2\pi(x^2+y^2)}$ ok

2. This is the exact evolution equation if $\omega_0(\alpha,\beta) = \sum_{j=1}^{N} \delta(\alpha-\alpha_j,\beta-\beta_j)\Gamma_j$.

3. The following quantities are invariant in time.

$$X = \sum_{j=1}^{N}\Gamma_j\,x_j \quad , \quad Y = \sum_{j=1}^{N}\Gamma_j\,y_j \quad , \quad R^2 = \sum_{j=1}^{N}\Gamma_j\,(x_j^2+y_j^2)$$

$$H = -\frac{1}{2\pi}\sum_{j=1}^{N}\sum_{k>j}\Gamma_j\,\Gamma_k\,\log\sqrt{(x_j-x_k)^2+(y_j-y_k)^2}$$

pf

X , Y , R^2 : hw

define $p_j = x_j$, $q_j = -\Gamma_j y_j$, consider $H = H(p_1,\dots,p_N,q_1,\dots,q_N)$

claim : $\dfrac{dp_j}{dt} = -\dfrac{\partial H}{\partial q_j}$, $\dfrac{dq_j}{dt} = \dfrac{\partial H}{\partial p_j}$: Hamiltonian system

$$\frac{dH}{dt} = \sum_{j=1}^{N}\Big(\frac{\partial H}{\partial p_j}\frac{dp_j}{dt} + \frac{\partial H}{\partial q_j}\frac{dq_j}{dt}\Big) = \sum_{j=1}^{N}\Big(\frac{\partial H}{\partial p_j}\Big(-\frac{\partial H}{\partial q_j}\Big) + \frac{\partial H}{\partial q_j}\frac{\partial H}{\partial p_j}\Big) = 0$$

check : $N = 3$

$$H = -\frac{1}{4\pi}\big(\Gamma_1\Gamma_2\log((x_1-x_2)^2+(y_1-y_2)^2) + \Gamma_1\Gamma_3\log((x_1-x_3)^2+(y_1-y_3)^2)$$
$$+ \Gamma_2\Gamma_3\log((x_2-x_3)^2+(y_2-y_3)^2)\big)$$

$$\frac{\partial H}{\partial p_1} = \frac{\partial H}{\partial x_1} = \Gamma_1 \cdot -\frac{dy_1}{dt} = \frac{dq_1}{dt}$$

$$\frac{\partial H}{\partial q_1} = \frac{\partial H}{\partial y_1} \cdot \frac{1}{-\Gamma_1} = \Gamma_1 \cdot \frac{dx_1}{dt} \cdot \frac{1}{-\Gamma_1} = -\frac{dp_1}{dt} \qquad \underline{\text{ok}}$$

<u>note</u> : If the Γ_j all have the same sign, then the point vortices cannot collide.

<u>boundary condition</u>

$u \cdot n|_{\partial D} = 0$

<u>method 1</u> : add a suitable potential flow

Let u be the velocity induced by the point vortex method and assume $u \cdot n|_{\partial D} \neq 0$.

solve $\Delta\phi = 0$, $\left.\frac{\partial \phi}{\partial n}\right|_{\partial D} = -u \cdot n\Big|_{\partial D}$

solvability : ok from $\nabla \cdot u = 0$ check ...

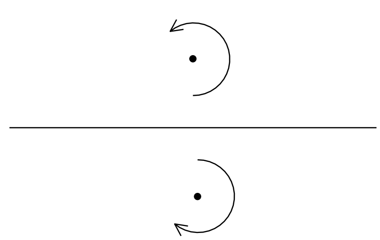
replace u by $u + \nabla\phi$

<u>method 2</u> : images

If D is simple, we can modify $g(x, y)$ by adding image vortices.

<u>ex</u>

$D = \{(x, y) : y > 0\}$: upper half-plane



(x_j, y_j) , Γ_j

$(x_j, -y_j)$, $-\Gamma_j$

$$g(x,y) \;=\; -\frac{\Gamma_j}{2\pi}\log\sqrt{(x-x_j)^2+(y-y_j)^2} + \frac{\Gamma_j}{2\pi}\log\sqrt{(x-x_j)^2+(y+y_j)^2}$$

$$= \quad \text{point vortex} \quad + \quad \text{potential flow on } D$$

<u>method 3</u> : analytical expression

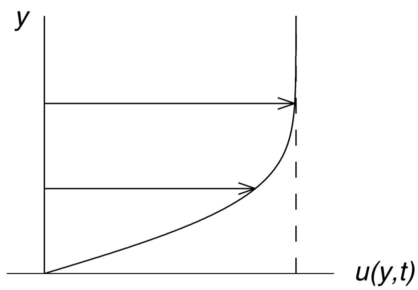
periodic BC in x $\Rightarrow$ $W(z) = \dfrac{1}{2\pi i} \log \sin \pi z$

2.2 boundary layers

ex 1 : impulsively started viscous flow along an infinite plate

D = upper half-plane = $\{(x,y) : y > 0\}$, $\partial D = x$-axis

$t = 0 \Rightarrow u = 1 , v = 0$, $t > 0 \Rightarrow u = u(y,t) , v = 0$: parallel shear flow



$u_x + v_y = 0$: ok

$u_t + uu_x + vu_y = -p_x + \nu(u_{xx} + u_{yy})$

$v_t + uv_x + vv_y = -p_y + \nu(v_{xx} + v_{yy}) \Rightarrow p_y = 0 \Rightarrow p = p(x,t)$

$u_t - \nu u_{yy} = -p_x = f_1(t) \Rightarrow p(x,t) = f_1(t)x + f_2(t)$

set $f_1(t) = 0$ to avoid infinite pressure at $x = \pm\infty$

$u_t = \nu u_{yy}$: diffusion equation

$t = 0 \Rightarrow u(y,0) = 1$

$y = 0 \Rightarrow u(0,t) = 0$

method 1 : Fourier transform (Math 556)

method 2 : dimensional analysis

$[y] = L , [t] = T , [\nu] = \dfrac{L^2}{T} \Rightarrow \eta = \dfrac{y}{\sqrt{\nu t}}$: non-dimensional

look for a similarity solution of the form $u(y,t) = f(\eta)$

$$f' \cdot \frac{y}{\sqrt{\nu}} \cdot \frac{-1}{2} \frac{1}{t^{3/2}} = \nu f'' \cdot \frac{1}{\nu t} \Rightarrow f'' + \frac{1}{2}\eta f' = 0$$

$$\Rightarrow e^{\eta^2/4} f'' + e^{\eta^2/4} \cdot \tfrac{1}{2}\eta f' = (e^{\eta^2/4} f')' = 0 \Rightarrow e^{\eta^2/4} f' = A \Rightarrow f' = Ae^{-\eta^2/4}$$

$$f(\eta) = A\int_0^\eta e^{-\alpha^2/4} d\alpha + B , \alpha = 2\beta \Rightarrow f(\eta) = 2A\int_0^{\eta/2} e^{-\beta^2} d\beta + B$$

$y = 0 \;\Rightarrow\; f(0) = 0 \;\Rightarrow\; B = 0$

$t = 0 \;\Rightarrow\; f(\infty) = 1 \;\Rightarrow\; 2A\int_0^\infty e^{-\beta^2}d\beta = 1 \;\Rightarrow\; A = \dfrac{1}{\sqrt{\pi}}$

$f(\eta) \;=\; \dfrac{2}{\sqrt{\pi}}\int_0^{\eta/2} e^{-\beta^2}d\beta \;=\; \mathrm{erf}\Big(\dfrac{\eta}{2}\Big)$, $\mathrm{erf}(x) \;=\; \dfrac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}ds$: <u>error function</u>

$u(y,t) \;=\; \mathrm{erf}\Big(\dfrac{y}{\sqrt{4\nu t}}\Big)$, check …

<u>note</u>

$t > 0$, $\lim_{y\to\infty} u(y,t) \;=\; \mathrm{erf}(\infty) \;=\; 1 \;\Rightarrow$ far from the plate the flow is undisturbed

$y > 0$, $\lim_{t\to\infty} u(y,t) \;=\; \mathrm{erf}(0) \;=\; 0 \;\Rightarrow$ eventually the flow comes to rest

define $y = \delta$ by $u(\delta,t) = 0.99$: <u>boundary layer thickness</u>

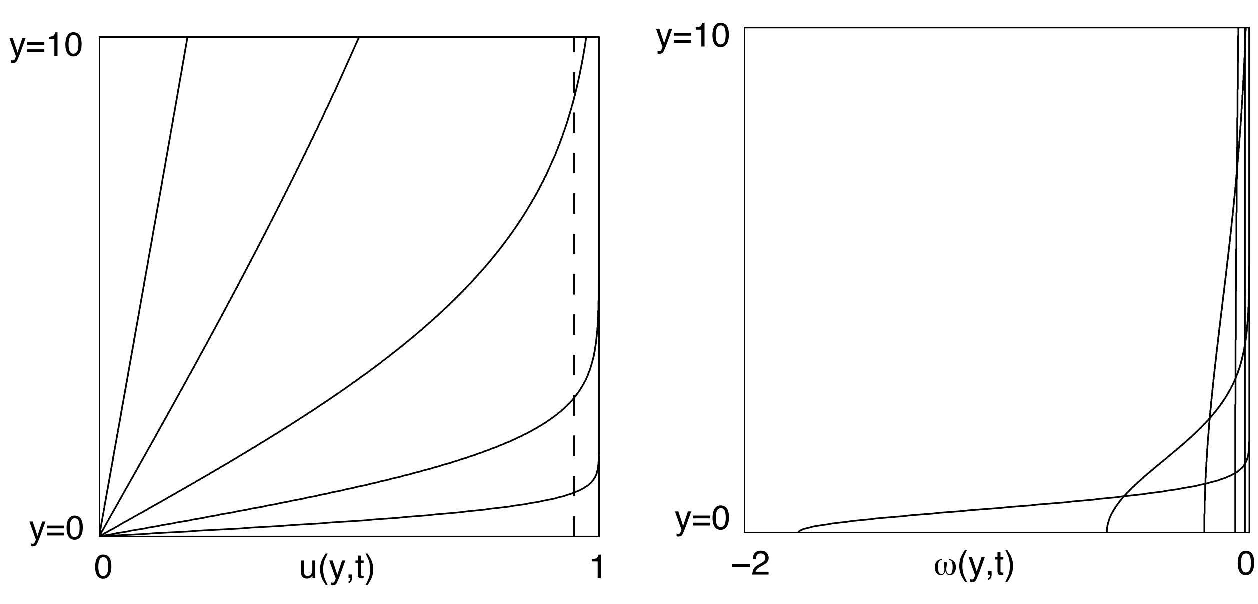
$f\Big(\dfrac{\delta}{\sqrt{\nu t}}\Big) \;=\; 0.99 \;\Rightarrow\; \delta = f^{-1}(0.99)\cdot\sqrt{\nu t} \;=\; O(\sqrt{\nu t})$

$\omega = v_x - u_y = -\mathrm{erf}'\Big(\dfrac{y}{\sqrt{4\nu t}}\Big)\dfrac{1}{\sqrt{4\nu t}} = -\dfrac{2}{\sqrt{\pi}}\,e^{-y^2/4\nu t}\,\dfrac{1}{\sqrt{4\nu t}} = -\dfrac{1}{\sqrt{\pi\nu t}}\,e^{-y^2/4\nu t}$

$u_t = \nu u_{yy} \quad\Rightarrow\quad \omega_t = \nu\omega_{yy}$

For $t = 0$, the vorticity is a bound vortex sheet on the plate, and for $t > 0$, the sheet diffuses into the flow domain. Hence the solid boundary is a source of vorticity. The vorticity is concentrated in a boundary layer of width $O(\sqrt{\nu t}\,)$.

<u>ex</u> : $\nu = 10^{-1}$, $t = 1, 10, 10^2, 10^3, 10^4$



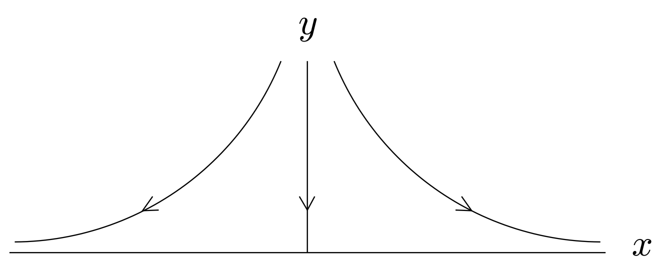
<u>ex 2</u> : steady viscous flow into a stagnation point on a wall

reference : "Theoretical Hydrodynamics" , Milne-Thomson (chap. 23)

<u>potential flow</u>

$$W(z) = \frac{\gamma}{2}z^2 \Rightarrow \frac{dW}{dz} = u_p - iv_p = \gamma z = \gamma(x+iy) \quad , \quad \gamma > 0$$

$u_p(x,y) = \gamma x$, $v_p(x,y) = -\gamma y$: strain flow



on the wall : $u_p(x,0) = \gamma x$, $v_p(x,0) = 0$: slip flow , bound vortex sheet

<u>viscous flow</u> : u , v

guess : $v(x,y) = -f(y)$, note : $v_p(x,y) = -f_p(y)$, $f_p(y) = \gamma y$

$$u_x + v_y = 0 \Rightarrow u_x = -v_y = f'(y) \Rightarrow u(x,y) = xf'(y)$$

$$uu_x + vu_y = -p_x + \nu(u_{xx} + u_{yy}) \Rightarrow x(f')^2 - f\cdot xf'' = -p_x + \nu\cdot xf'''$$

$$uv_x + vv_y = -p_y + \nu(v_{xx} + v_{yy}) \Rightarrow \qquad ff' = -p_y + \nu\cdot -f''$$

$$p_y = -ff' - \nu f'' \Rightarrow p = -\tfrac{1}{2}f^2 - \nu f' + g(x)$$

$$\frac{p_x}{x} = \frac{g'(x)}{x} = -(f')^2 + ff'' + \nu f''' = \text{ constant } = -\gamma^2 \quad , \text{ by analogy with } f_p$$

no-slip bc $\Rightarrow$ $f(0) = f'(0) = 0$; $\lim_{y\to\infty}(u,v) = (u_p\,,v_p) \Rightarrow f'(\infty) = \gamma$

$$\nu f''' + ff'' - (f')^2 = -\gamma^2 \quad , \quad f(0) = f'(0) = 0 \ , \ f'(\infty) = \gamma$$

<u>non-dimensionalization</u>

$$[y] = L \quad , \quad [f] = \frac{L}{T} \quad , \quad [\nu] = \frac{L^2}{T} \quad , \quad [\gamma] = \frac{1}{T}$$

$$\eta = y\Big(\frac{\gamma}{\nu}\Big)^{1/2} \ , \quad f(y) = (\nu\gamma)^{1/2}F(\eta)$$

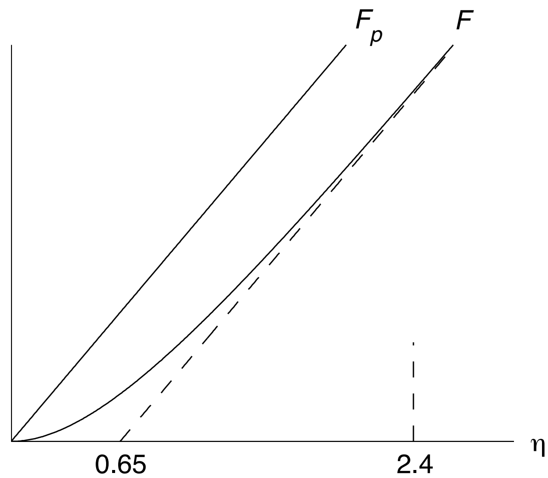
$$\nu(\nu\gamma)^{1/2}F'''\left(\frac{\gamma}{\nu}\right)^{3/2} + (\nu\gamma)^{1/2}F(\nu\gamma)^{1/2}F''\left(\frac{\gamma}{\nu}\right) - \left((\nu\gamma)^{1/2}F'\left(\frac{\gamma}{\nu}\right)^{1/2}\right)^2 = -\gamma^2$$

$$F''' + FF'' - (F')^2 = -1 \quad , \quad F(0) = F'(0) = 0 \ , \ F'(\infty) = 1$$

<u>note</u>

$$f_p(y) = \gamma y = \gamma\eta\left(\frac{\nu}{\gamma}\right)^{1/2} = (\nu\gamma)^{1/2}F_p(\eta)$$

$\Rightarrow \ F_p(\eta) = \eta$: satisfies ODE and 2 bc, but not the 3rd bc : $F_p'(0) \neq 0$



can be shown : $F(\eta) \geq 0$, $0 \leq F'(\eta) \leq 1$ and $\eta > 2.4 \Rightarrow \begin{cases} F(\eta) \sim \eta - 0.65 \\ 0.99 < F'(\eta) < 1 \end{cases}$

<u>note</u>

1. $y > 2.4\left(\frac{\nu}{\gamma}\right)^{1/2} \Rightarrow \dfrac{u_p - u}{u_p} = \dfrac{\gamma x - xf'(y)}{\gamma x} = 1 - F'(\eta) < 0.01$

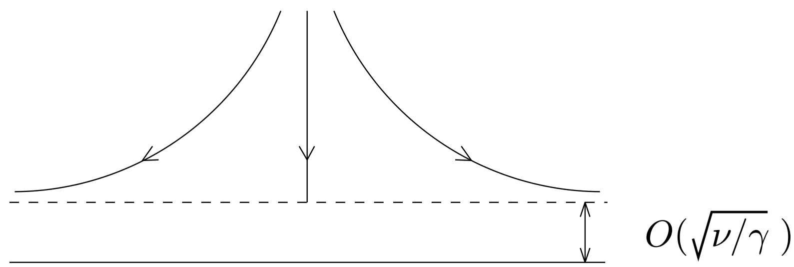
$$\Rightarrow \text{ boundary layer thickness } = O(\sqrt{\nu/\gamma}\ )$$

2. $v(y) - v_p(y) = -f(y) + \gamma y = -(\nu\gamma)^{1/2}F + \gamma\eta\left(\frac{\nu}{\gamma}\right)^{1/2} = (\nu\gamma)^{1/2}(\eta - F(\eta))$

$$y > 2.4\left(\frac{\nu}{\gamma}\right)^{1/2} \quad \Rightarrow \quad v(y) - v_p(y) \sim 0.65(\nu\gamma)^{1/2}$$

$$\Rightarrow \ v(y) \sim v_p(y) + 0.65(\nu\gamma)^{1/2} = -\gamma y + 0.65(\nu\gamma)^{1/2} = -\gamma\left(y - 0.65\left(\frac{\nu}{\gamma}\right)^{1/2}\right)$$

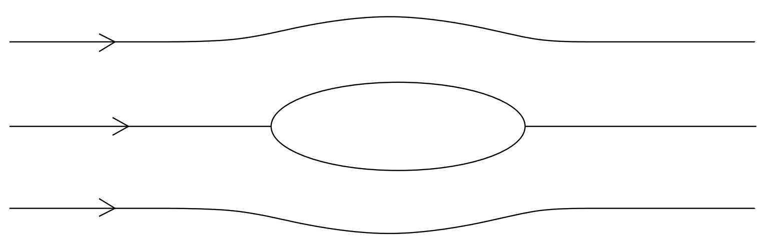
$$\Rightarrow \ v(y) \sim v_p\left(y - 0.65\left(\frac{\nu}{\gamma}\right)^{1/2}\right) \quad , \quad \text{similarly } \ u(y) \sim u_p\left(y - 0.65\left(\frac{\nu}{\gamma}\right)^{1/2}\right)$$



Outside the boundary layer, the viscous flow is well-approximated by the potential flow shifted upward by a distance $O(\sqrt{\nu/\gamma}\,)$, the displacement thickness.

Prandtl's boundary layer hypothesis

Experiments show that slightly viscous low-speed flow past a slender body stays attached to the body.

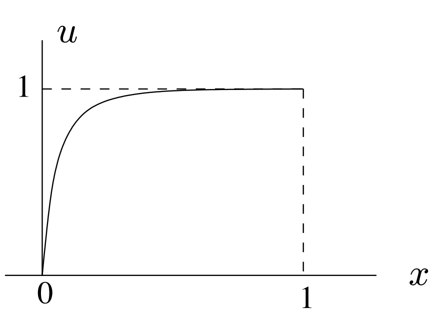


Prandtl asserted that in this case, (1) the effect of viscosity is important only in a thin boundary layer near the body, (2) outside the boundary layer there is essentially ideal potential flow, and (3) inside the boundary layer there is essentially viscous parallel shear flow in which the fluid velocity increases rapidly from zero on the body to the potential flow value at the edge of the boundary layer.

model problem

$\epsilon u'' + u' = 0 \quad , \quad 0 \le x \le 1 \quad , \quad u(0) = 0 \quad , \quad u(1) = 1 \quad , \quad 0 < \epsilon << 1$

exact solution : $u(x) = \dfrac{1 - e^{-x/\epsilon}}{1 - e^{-1/\epsilon}}$



<u>note</u>

1. $x > O(\epsilon) \;\Rightarrow\; u(x) \sim 1$: $u(x)$ has a boundary layer of width $O(\epsilon)$

2. $\lim_{\epsilon\to 0} u(x,\epsilon) \;=\; \overline{u}(x) \;=\; \begin{cases} 0 & \text{if } \; x = 0 \\ 1 & \text{if } \; 0 < x \leq 1 \end{cases}$, $\lim_{\epsilon\to 0} \sup_{0\leq x\leq 1} |u(x,\epsilon) - \overline{u}(x)| = 1$

$\Rightarrow$ non-uniform convergence : <u>singular perturbation problem</u>

<u>method of matched asymptotic expansions</u> (Math 557)

$u(x,\epsilon) \;\sim\; u_0(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \;\cdots$: <u>regular perturbation series</u>

$\epsilon u'' + u' = 0$

$\epsilon(u_0'' + \epsilon u_1'' + \cdots) + (u_0' + \epsilon u_1' + \cdots) = 0$

$u_0' + \epsilon(u_0'' + u_1') + \cdots = 0$

$\Rightarrow\; u_0' \;=\; 0 \;\Rightarrow\; u_0(x) \;=\; \text{constant} \;=\; 0 \;\text{ or }\; 1 \;?$

Try $u_0(x) = 1$. Then $u_0(x)$ satisfies the bc at $x = 1$, but the bc at $x = 0$ fails.

define $s = \dfrac{x}{\phi(\epsilon)}$: <u>stretched coordinate</u> , $u(x) = v(s)$

$$\frac{du}{dx} = \frac{dv}{ds}\cdot\frac{ds}{dx} \;\Rightarrow\; u' = \frac{v'}{\phi(\epsilon)} \quad , \quad u'' = \frac{v''}{\phi(\epsilon)^2}$$

$$\epsilon u'' + u' = 0 \;\Rightarrow\; \frac{\epsilon\, v''}{\phi(\epsilon)^2} + \frac{v'}{\phi(\epsilon)} = 0$$

$$\phi(\epsilon) = \epsilon \;\Rightarrow\; v'' + v' = 0 \;\;,\;\; v(0) = 0 \quad\Rightarrow\quad v(s) = c(1 - e^{-s})$$

$v(s)$: <u>inner solution</u> , valid near $x = 0$

$u_0(x)$: <u>outer solution</u> , valid away from $x = 0$

<u>matching condition</u>

outer limit of inner solution $=$ inner limit of outer solution

$\lim_{s\to\infty} v(s) = \lim_{x\to 0} u_0(x) \;\Rightarrow\; c = 1$

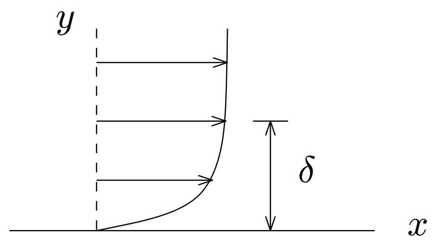
<u>uniform asymptotic approximation</u>

$u(x) \;\sim\; v(s) + u_0(x) - c \;=\; c(1 - e^{-s}) + 1 - 1 \;=\; 1 - e^{-x/\epsilon}$

1. $\sup_{0\leq x\leq 1} |u(x) - (v(s) + u_0(x) - c)| = e^{-1/\epsilon} \to 0$ as $\epsilon \to 0$: uniform convergence

2. The inner bc is satisfied exactly and the outer bc is satisfied asymptotically.

ex 3 : viscous flow past a wall



$u_x + v_y = 0$

$u_t + uu_x + vu_y = -p_x + \nu(u_{xx} + u_{yy})$

$v_t + uv_x + vv_y = -p_y + \nu(v_{xx} + v_{yy})$

note : NS equations + no-slip bc = singular perturbation problem

outer solution : potential flow

inner solution : valid in the boundary layer in the limit $\nu \to 0$

stretched coordinates : $y = \delta y^*$, $v = \delta v^*$, x , t , u , p : unscaled

$$u_x + \delta v^*_{y^*}\frac{1}{\delta} = 0$$

$$u_t + uu_x + \delta v^* u_{y^*}\frac{1}{\delta} = -p_x + \nu\Big(u_{xx} + u_{y^*y^*}\frac{1}{\delta^2}\Big)$$

$$\delta v^*_t + u\,\delta v^*_x + \delta v^*\delta v^*_{y^*}\frac{1}{\delta} = -p_{y^*}\frac{1}{\delta} + \nu\Big(\delta v^*_{xx} + \delta v^*_{y^*y^*}\frac{1}{\delta^2}\Big)$$

We expect that convection and diffusion are equally important in the boundary layer, so we take $\delta \sim \sqrt{\nu}$.

Prandtl boundary layer equations

$$u_x + v_y = 0$$

$$u_t + uu_x + vu_y = -p_x + \nu u_{yy}$$

$$0 = -p_y$$

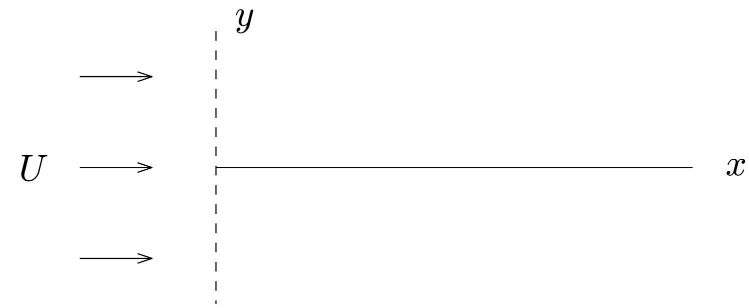
1. Diffusion acts only in the y-direction.
2. no-slip bc : $u(x,0,t) = v(x,0,t) = 0$
3. matching

$\lim_{y\to\infty} u(x,y,t) = \lim_{y\to 0} u_0(x,y,t)$: slip velocity of potential flow on ∂D

$p_y = 0 \;\Rightarrow\; p(x,y,t) = \lim_{y\to\infty} p(x,y,t) = \lim_{y\to 0} p_0(x,y,t) = p_0(x,t)$

While the pressure in the NS eqs is one of the unknown functions, the pressure in the Prandtl eqs is a known function given by the potential flow on ∂D.

<u>ex 4</u> : steady leading-edge flow past a semi-infinite plate



$D = \{(x,y)\ :\ x > 0\ ,\ y > 0\}$

<u>boundary conditions</u>

$y = 0 \ \Rightarrow\ u(x,0) = 0\ ,\ v(x,0) = 0$

$y = \infty \ \Rightarrow\ u(x,\infty) = U\ ,\ v(x,\infty)$: unspecified

$x = 0 \ \Rightarrow\ u(0,y) = U\ ,\ v(0,y)$: unspecified

pressure : $p_x = p_y = 0$

<u>steady boundary layer equations</u>

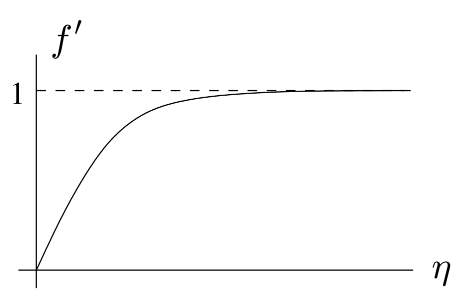
$u_x + v_y = 0$

$uu_x + vu_y = \nu u_{yy}$

similarity solution : $\eta = y\left(\dfrac{U}{\nu x}\right)^{1/2}$, $\psi(x,y) = (\nu U x)^{1/2} f(\eta)$

$u\ =\ \psi_y\ =\ (\nu U x)^{1/2} f'(\eta)\left(\dfrac{U}{\nu x}\right)^{1/2}\ =\ U f'(\eta)$: <u>Blasius profile</u>

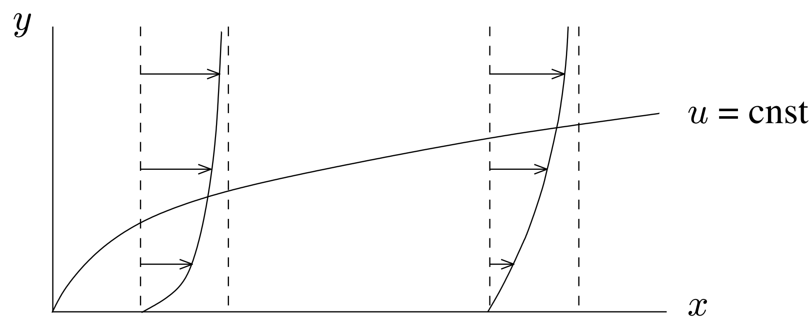
$\ldots\quad f''' + \frac{1}{2} f f'' = 0$, $f(0) = f'(0) = 0$, $f'(\infty) = 1$ check : hw



<u>note</u>

1. $\eta \to \infty \ \Rightarrow\ \begin{cases} x : \text{fixed}\ ,\ y \to \infty \\ y : \text{fixed}\ ,\ x \to 0 \end{cases}$

2. $\eta = \text{cnst} \;\Rightarrow\; u(x,y) = \text{cnst}$ for $x = cy^2$: parabolas



<u>note</u>

1. comparison of boundary layer thickness for various flows

a) steady leading-edge flow past a semi-infinite plate : $\delta \sim \left(\dfrac{\nu x}{U}\right)^{1/2}$

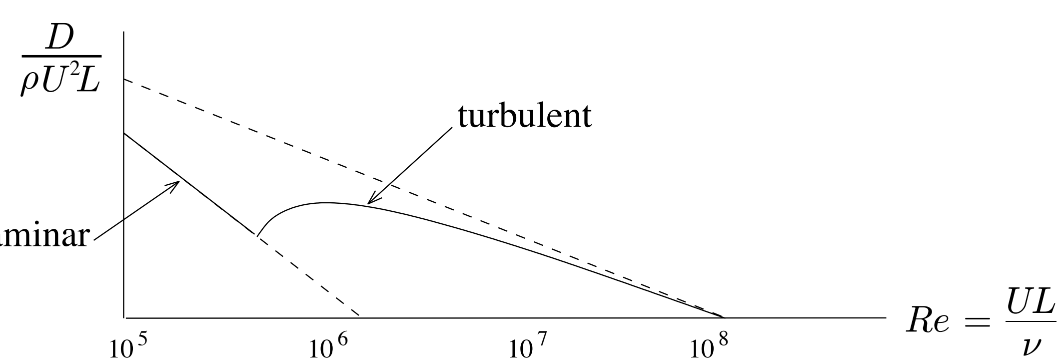
b) steady flow into a stagnation point on a wall : $\delta \sim \left(\dfrac{\nu}{\gamma}\right)^{1/2}$

c) impulsively started flow along an infinite plate : $\delta \sim (\nu t)^{1/2}$

2. drag on a portion of the plate

$$D = \int_0^L \mu\, u_y(x,0)\, dx \;=\; \int_0^L \rho\nu\, U f''(0) \left(\frac{U}{\nu x}\right)^{1/2} dx \;\sim\; \rho U^2 L \left(\frac{UL}{\nu}\right)^{-1/2} \sim\; Re^{-1/2}$$

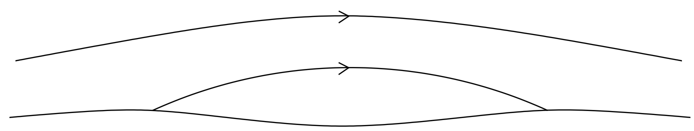
However, experiments show that for sufficiently large Re, the flow is unstable and undergoes a transition to turbulence.



For a given value of Re, the drag induced by a turbulent boundary layer is greater than the drag induced by a laminar boundary layer; this is attributed to <u>turbulent diffusion</u>.

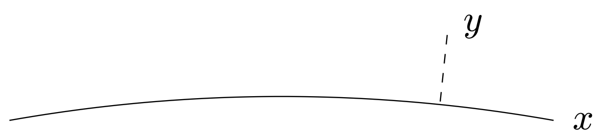
<u>separation</u>

Consider steady viscous flow past a curved wall.



<u>local analysis near separation point</u>

Let x, y be curvilinear coordinates in a neighborhood of the wall.



wall vorticity : $\omega(x,0) \;=\; (v_x - u_y)(x,0) \;=\; -u_y(x,0) \;=\; \omega_0(x)$

$$u(x,y) \;=\; u(x,0) + u_y(x,0)y + \frac{1}{2}\,u_{yy}(x,0)y^2 + \cdots \;=\; -\omega_0(x)y + A(x)y^2 + \cdots$$

$$\psi(x,y) \;=\; \int_0^y u(x,s)\,ds \;=\; -\frac{1}{2}\,\omega_0(x)y^2 + \frac{A(x)}{3}y^3 + \cdots$$

recall : θ-component of momentum equation in polar coordinates

$$\frac{\partial u_\theta}{\partial t} + (u\cdot\nabla)u_\theta \;=\; -\frac{1}{r}\frac{\partial p}{\partial \theta} + \nu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right)$$

substitute : $(u_r, u_\theta) \to (v, -u)$, $\left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}\right) \to \left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right)$

$$\begin{aligned} y = 0 \;\Rightarrow\; 0 \;&=\; p_x(x,0) + \nu(-u_{yy}(x,0) + \kappa(x)\cdot -u_y(x,0)) \\ &=\; p_x(x,0) + \nu(-2A(x) + \kappa(x)\cdot\omega_0(x)) \end{aligned}$$

$$A(x) = \frac{1}{2}\left(\kappa(x)\,\omega_0(x) + \frac{p_x(x,0)}{\nu}\right)$$

$$\psi(x,y) = -\frac{1}{2}\,y^2\left(\omega_0(x) - \frac{1}{3}\left(\kappa(x)\,\omega_0(x) + \frac{p_x(x,0)}{\nu}\right)y\right) + \cdots$$

The streamline $\psi = 0$ has 2 branches.

1. $y = 0$: the wall is a streamline
2. $\omega_0(x) - \frac{1}{3}\Big(\kappa(x)\,\omega_0(x) + \frac{p_x(x,0)}{\nu}\Big)y = 0$

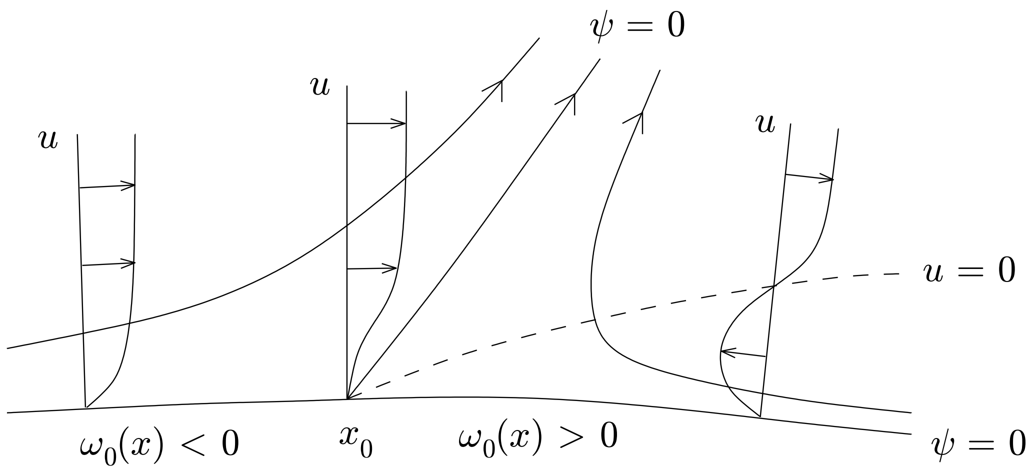
This defines a curve $y = y(x)$, the <u>separating streamline</u>, that intersects the wall at a point $(x,y) = (x_0, 0)$ st $\omega_0(x_0) = 0$.

<u>note</u>

1. $y(x) = \dfrac{3\,\omega_0(x)}{\kappa(x)\,\omega_0(x) + p_x(x,0)/\nu} \;\Rightarrow\; y'(x_0) = \dfrac{3\,\nu\,\omega_0'(x_0)}{p_x(x_0,0)}$

2. $u(x,y) = \psi_y = -\omega_0(x)y + \cdots \;,\; v(x,y) = -\psi_x = \frac{1}{2}\omega_0'(x)y^2 + \cdots$

<u>case 1</u> : $\omega_0'(x_0) > 0 \;,\; p_x(x_0,0) > 0 \;\Rightarrow\; y'(x_0) > 0 \;,\; v(x,y) > 0$: separation



The vorticity in the boundary layer leaves the wall near the separation point and enters the interior of the flow domain as a <u>free shear layer</u>.

<u>note</u>

1. The u profile has an inflectgion point at $x = x_0$.

2. pressure gradient on the wall : $p_x(x,0) = \nu(u_{yy}(x,0) + \kappa(x)u_y(x,0))$

$x << x_0$: far upstream of separation point

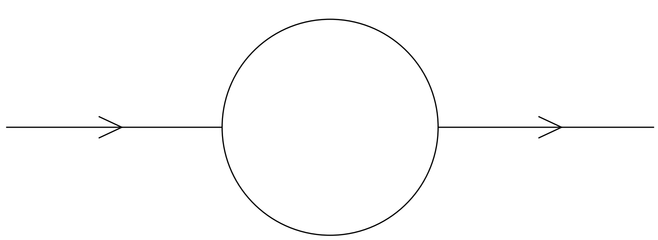
$\Rightarrow \quad u_y > 0 \;,\; u_{yy} < 0$ but $|u_y| \sim \dfrac{1}{\delta} \;,\; |u_{yy}| \sim \dfrac{1}{\delta^2} \;,\; \kappa \sim 1$

$\Rightarrow \quad p_x < 0$: p decreases in flow direction , <u>favorable</u> pressure gradient

$x \to x_0$: approaching separation point from upstream $\quad\Rightarrow\quad u_y \to 0 \;,\; u_{yy} > 0$

$\Rightarrow \quad p_x > 0$: p increases in flow direction , <u>adverse</u> pressure gradient

<u>recall</u>



front : favorable pressure gradient

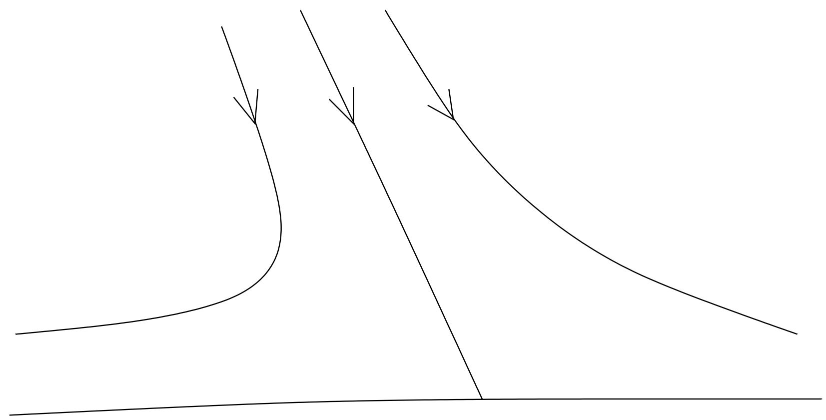
back : adverse pressure gradient

3. $\psi(x,y) \sim -\frac{1}{2}\,\omega_0(x)y^2 + \cdots \sim -\frac{1}{2}\,\omega_0'(x_0)(x-x_0)y^2 + \cdots$

$u = \psi_y \sim -\omega_0'(x_0)(x-x_0)y \quad , \quad v = -\psi_x \sim \frac{1}{2}\,\omega_0'(x_0)y^2$

$\frac{v}{u} \sim \frac{y}{x-x_0} \sim \frac{\eta}{\sqrt{x-x_0}}$: boundary layer scaling fails , Goldstein singularity

case 2 : $\omega_0'(x_0) < 0$, $p_x(x_0,0) > 0 \;\Rightarrow\; y'(x_0) < 0$, $v(x,y) < 0$: reattachment



hydrodynamic stability

warm-up

1. $u_t = u_x$

look for solutions of the form $u(x,t) = Ae^{ik(x-ct)}$

k : wavenumber , wavelength : $\lambda = \frac{2\pi}{k}$, c : wave speed

$-ikcAe^{ik(x-ct)} = ikAe^{ik(x-ct)} \quad \Rightarrow \quad c = -1$

$u(x,t) = Ae^{ik(x+t)}$: traveling wave

2. $u_t = u_{xx}$

$-ikcAe^{ik(x-ct)} = (ik)^2Ae^{ik(x-ct)} \quad \Rightarrow \quad c = -ik$

$u(x,t) = Ae^{-k^2t}e^{ikx}$: stationary wave , amplitude decays in time

in general , $c = c(k) = c_r + ic_i$

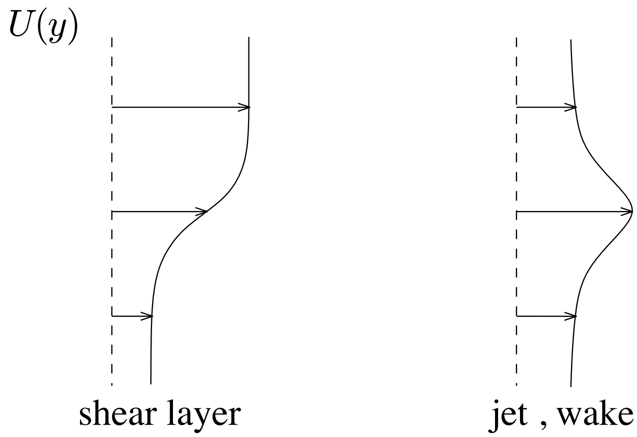
$u(x,t) = Ae^{ik(x-ct)} = Ae^{kc_it}e^{ik(x-c_rt)}$

c_r : wave speed , kc_i : growth rate , $kc_i > 0$: amplification

$kc_i < 0$: decay

$kc_i = 0$: marginally stable

<u>parallel shear flow</u> : $u(x,y,t) = U(y)$, $v(x,y,t) = 0$



vorticity : $\omega(x,y,t) = (v_x - u_y)(x,y,t) = -U'(y) \;\Rightarrow\; \begin{cases} \omega_t + u\omega_x + v\omega_y = 0 \\ u_x + v_y = 0 \end{cases}$

Consider a time-dependent perturbation of the steady base flow.

$u(x,y,t) = U(y) + \tilde{u}(x,y,t)$

$v(x,y,t) = \qquad\qquad \tilde{v}(x,y,t)$

$\omega(x,y,t) = -U'(y) + \tilde{\omega}(x,y,t)$

$\omega_t + u\omega_x + v\omega_y = 0 \quad\Rightarrow\quad \tilde{\omega}_t + (U+\tilde{u})\tilde{\omega}_x + \tilde{v}(-U'' + \tilde{\omega}_y) = 0$

<u>linear stability</u>

$\tilde{\omega}_t + U\tilde{\omega}_x - U''\tilde{v} = 0$, $\tilde{u}_x + \tilde{v}_y = 0$

$\tilde{\psi}(x,y,t) = \phi(y)\, e^{ik(x-ct)}$

$\tilde{u} = \tilde{\psi}_y = \phi' e^{ik(x-ct)}$, $\tilde{v} = -\tilde{\psi}_x = -ik\,\phi\, e^{ik(x-ct)}$

$\tilde{\omega} = -\Delta\tilde{\psi} = -((ik)^2\phi + \phi'')\, e^{ik(x-ct)} = -(\phi'' - k^2\phi)\, e^{ik(x-ct)}$

$-ikc \cdot -(\phi'' - k^2\phi) + U \cdot ik \cdot -(\phi'' - k^2\phi) - U'' \cdot -ik\,\phi = 0$

<u>Rayleigh equation</u>

$(U-c)(\phi'' - k^2\phi) - U''\phi = 0$ + bc

given : $U(y)$, k , find : $\phi(y)$, c : eigenvalue problem

<u>jump relation</u>

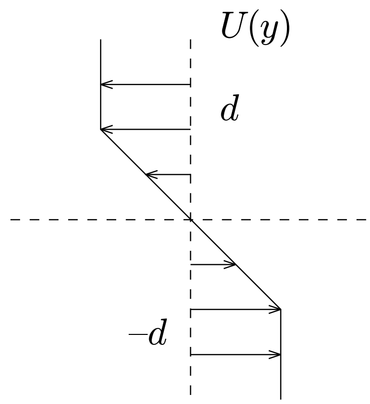
Assume U , ϕ are continuous, but U', ϕ' have a jump discontinuity at $y = y_0$.

$$((U-c)\phi' - U'\phi)' = (U-c)k^2\phi$$

$$\int_{y_0-\epsilon}^{y_0+\epsilon} ((U-c)\phi' - U'\phi)'dy = ((U-c)\phi' - U'\phi)\Big|_{y_0-\epsilon}^{y_0+\epsilon} = \int_{y_0-\epsilon}^{y_0+\epsilon} (U-c)k^2\phi\,dy$$

$$\epsilon \to 0 \;\Rightarrow\; (U(y_0)-c)[\phi'] = [U']\phi(y_0) \;\text{ where }\; [f] = f(y_0^+) - f(y_0^-)$$

ex : layer of constant vorticity



$$U(y) = \begin{cases} -\omega_0\, d & \text{if} \quad y > d \\ -\omega_0\, y & \text{if} \quad -d \le y \le d \\ \omega_0\, d & \text{if} \quad y < -d \end{cases}$$

$$U'(y) = \begin{cases} 0 & \text{if} \quad y > d \\ -\omega_0 & \text{if} \quad -d \le y \le d \\ 0 & \text{if} \quad y < -d \end{cases}$$

$$U''(y) = \omega_0(\delta(y-d) - \delta(y+d))$$

$$\Rightarrow\; \phi'' - k^2\phi = 0 \;\text{ on each interval (assuming } U(y) - c \neq 0)$$

bc : $\lim_{y\to\pm\infty} \phi(y) = 0$, $\phi(y)$ is continuous at $y = \pm d$, jump relation at $y = \pm d$

$$y > d \;\Rightarrow\; \phi(y) = (ae^{2kd} + b)e^{-ky}$$

$$-d < y < d \;\Rightarrow\; \phi(y) = ae^{ky} + be^{-ky}$$

$$y < -d \;\Rightarrow\; \phi(y) = (a + be^{2kd})e^{ky}$$

$$y = d \;\Rightarrow\; [U'] = \omega_0$$

$$y = -d \;\Rightarrow\; [U'] = -\omega_0$$

$$y > d \;\Rightarrow\; \phi'(y) = -k(ae^{2kd} + b)e^{-ky}$$

$$-d < y < d \;\Rightarrow\; \phi'(y) = kae^{ky} - kbe^{-ky}$$

$$y < -d \;\Rightarrow\; \phi'(y) = k(a + be^{2kd})e^{ky}$$

$$y = d \;\Rightarrow\; [\phi'] = -2kae^{kd}$$

$$y = -d \;\Rightarrow\; [\phi'] = -2kbe^{kd}$$

$$(U(y_0) - c)[\phi'] = [U']\phi(y_0)$$

$$y_0 = d \;\Rightarrow\; (-\omega_0 d - c) \cdot -2kae^{kd} = \omega_0(ae^{kd} + be^{-kd})$$

$$y_0 = -d \;\Rightarrow\; (\omega_0 d - c) \cdot -2kbe^{kd} = -\omega_0(ae^{-kd} + be^{kd})$$

$$\begin{pmatrix} (2k(\omega_0\, d + c) - \omega_0)e^{kd} & -\omega_0 e^{-kd} \\ \omega_0 e^{-kd} & (\omega_0 - 2k(\omega_0\, d - c))e^{kd} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

det $= 0$

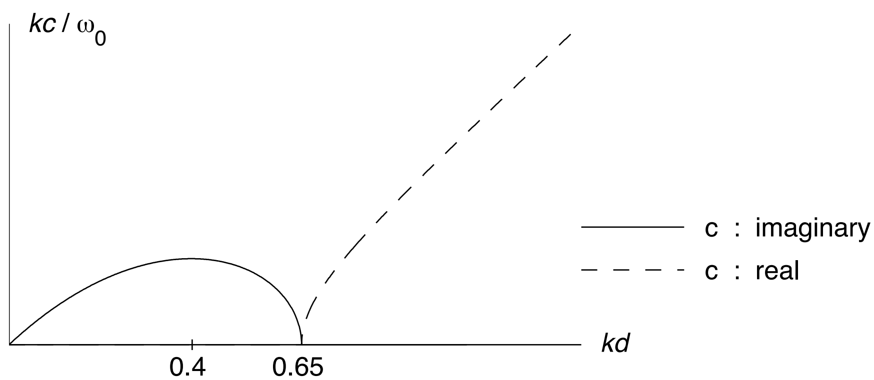
$$\Rightarrow\ (2k(\omega_0 d + c) - \omega_0)(\omega_0 - 2k(\omega_0 d - c))e^{2kd} + \omega_0^2 e^{-2kd} = 0$$

$$\Rightarrow\ (2k(\omega_0 d + c)\omega_0 - 4k^2(\omega_0^2 d^2 - c^2) - \omega_0^2 + 2k(\omega_0 d - c)\omega_0) = -\omega_0^2 e^{-4kd}$$

$$\Rightarrow\ (4kd\omega_0^2 - 4k^2 d^2 \omega_0^2 - \omega_0^2 + 4k^2 c^2) = -\omega_0^2 e^{-4kd}$$

$$\Rightarrow\ \ -\omega_0^2(1 - 2kd)^2 + 4k^2c^2 = -\omega_0^2 e^{-4kd}$$

$$\Rightarrow\ k^2c^2 = \frac{\omega_0^2}{4}((1 - 2kd)^2 - e^{-4kd})$$: <u>dispersion relation</u>



check : $kd \to 0 \ \Rightarrow\ k^2c^2 < 0$, $kd \to \infty \ \Rightarrow\ k^2c^2 > 0$

recall : $\tilde{\psi}(x, y, t) \ =\ \phi(y)e^{ik(x-ct)} \ =\ \phi(y)e^{kc_i t}e^{ik(x-c_r t)}$

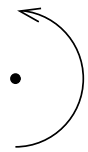
$\Rightarrow$ { long waves are unstable : <u>Kelvin-Helmholtz instability</u>
short waves are marginally stable

<u>note</u>

1. max growth rate occurs for $kd = 0.4 \ \Rightarrow\ \lambda = \dfrac{2\pi}{k} \ \sim\ 8 \times$ layer thickness
2. questions : 3D effects , viscosity , nonlinearity (picture)
3. long wave limit : $kd \to 0$, $\omega_0 d \to U \ \Rightarrow$ vortex layer $\to$ vortex sheet

<u>vortex sheet</u> : model for a thin shear layer

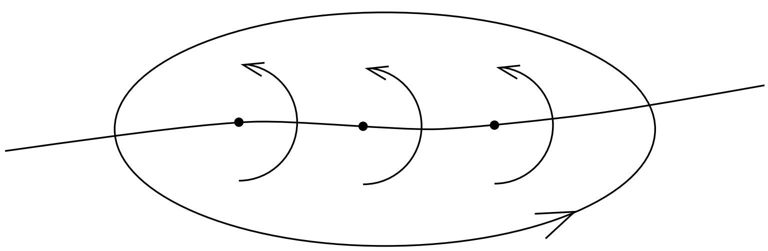
<u>recall</u> : point vortex



z_0 : position , Γ : circulation

induced velocity : $(u - iv)(z) = \dfrac{\Gamma}{2\pi i(z - z_0)}$

vortex sheet = superposition of point vortices on a curve



induced velocity : $(u - iv)(z) = \dfrac{1}{2\pi i}\displaystyle\int \dfrac{d\Gamma(s)}{z - z(s)} = \dfrac{1}{2\pi i}\displaystyle\int \dfrac{d\Gamma}{z - z(\Gamma)}$

s : arclength

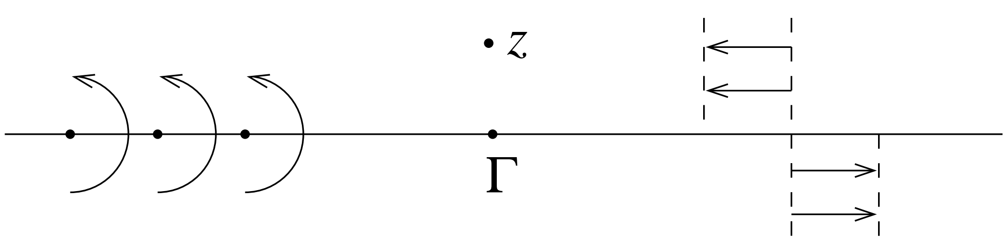
$z(s)$: parametrization

$\Gamma(s)$: total circulation between $z(0)$ and $z(s)$

$\Gamma'(s) = \sigma(s)$: <u>vortex sheet strength</u> $\Rightarrow$ $d\Gamma = \sigma(s)\, ds$

<u>ex</u>

$z(\Gamma) = \Gamma$: flat vortex sheet of constant strength ($\sigma = \frac{d\Gamma}{ds} = 1$)



$$
\begin{aligned}
(u - iv)(z) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\Gamma}{z - \Gamma} \;=\; \frac{-1}{2\pi i} \log(z - \Gamma)\Big|_{-\infty}^{\infty} \\
&= \frac{-1}{2\pi i} \lim_{\Gamma \to \infty} \left(\log\left|\frac{z - \Gamma}{z + \Gamma}\right| + i(\arg(z - \Gamma) - \arg(z + \Gamma)) \right) \\
&= \frac{-1}{2\pi} \begin{cases} \pi - 0 & \text{if } \; y > 0 \\ \\ -\pi - 0 & \text{if } \; y < 0 \end{cases} \qquad = \qquad \begin{cases} -\frac{1}{2} & \text{if } \; y > 0 \\ \\ \;\;\frac{1}{2} & \text{if } \; y < 0 \end{cases}
\end{aligned}
$$

<u>general case</u>

$$(u - iv)(z) \;=\; \frac{1}{2\pi i} \int_a^b \frac{d\Gamma}{z - z(\Gamma)} \quad : \quad \text{velocity induced by a vortex sheet}$$

$$F(z) \;=\; \frac{1}{2\pi i} \int_C \frac{f(w)\, dw}{z - w} \quad : \quad \text{Cauchy integral}$$

C : curve in complex w-plane

$z(\Gamma) = w \;\;,\;\; d\Gamma = f(w)\, dw$

$$f(w) = \frac{d\Gamma}{dw} = \frac{d\Gamma}{ds} \frac{ds}{dw} = \sigma \, e^{-i\theta} \;\; \text{where} \;\; e^{i\theta} = \frac{dw}{ds} \;\; : \;\; \text{unit tangent vector to curve}$$

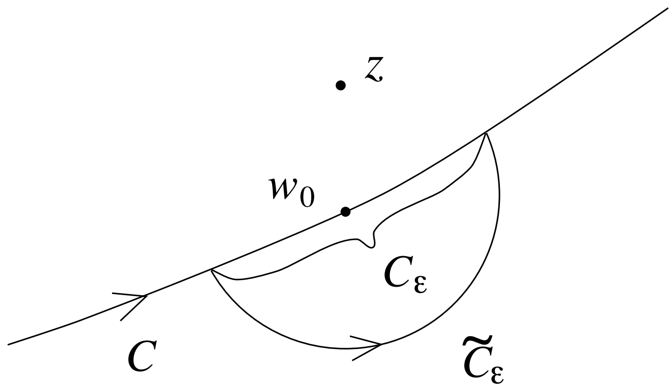
$z \notin C \quad \Rightarrow \quad F(z)$ is analytic (with mild assumptions on $f(w)$)

$\Rightarrow$ away from the sheet, the induced velocity is incompressible and irrotational

$$z \in C \quad \Rightarrow \quad F(z) \;=\; \frac{1}{2\pi i} \int_C \frac{f(w)\, dw}{z - w} \quad : \quad \text{improper integral}$$

reference : Carrier, Krook & Pearson, p. 412

Let $z \notin C$, $w_0 \in C$.



$$F_+(w_0) = \lim_{z \to w_0} F(z) \quad : \quad z \text{ on top of } C$$

$$= \lim_{z \to w_0} \frac{1}{2\pi i} \int_C \frac{f(w)\, dw}{z - w}$$

$$= \lim_{z \to w_0} \frac{1}{2\pi i} \Big(\int_{C - C_\epsilon} \frac{f(w)\, dw}{z - w} + \int_{\widetilde{C}_\epsilon} \frac{f(w)\, dw}{z - w} \Big) \quad : \quad \text{assuming } f(w) \text{ is analytic}$$

$$= \frac{1}{2\pi i} \int_{C - C_\epsilon} \frac{f(w)\, dw}{w_0 - w} + \frac{1}{2\pi i} \int_{\widetilde{C}_\epsilon} \frac{f(w)\, dw}{w_0 - w} \quad : \quad \text{true for all } \epsilon > 0$$

1. $\displaystyle \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{C - C_\epsilon} \frac{f(w)\, dw}{w_0 - w} = \frac{1}{2\pi i} \, \mathrm{pv} \int_C \frac{f(w)\, dw}{w_0 - w} = F_p(w_0) \quad : \quad \underline{\text{principal value}}$

2. $\displaystyle \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\widetilde{C}_\epsilon} \frac{f(w)\, dw}{w_0 - w} = \frac{1}{2} \mathrm{Res}\Big(\frac{f(w)}{w_0 - w}; w = w_0 \Big) = -\frac{1}{2} f(w_0)$

<u>Plemelj formulas</u>

$$\left. \begin{aligned} F_+(w_0) &= F_p(w_0) - \tfrac{1}{2} f(w_0) \\ F_-(w_0) &= F_p(w_0) + \tfrac{1}{2} f(w_0) \end{aligned} \right\} \quad \Rightarrow \quad \begin{aligned} F_p(w_0) &= \tfrac{1}{2}(F_+(w_0) + F_-(w_0)) \\ f(w_0) &= \quad F_-(w_0) - F_+(w_0) \end{aligned}$$

<u>application</u>

$$(u-iv)(z) = \frac{1}{2\pi i}\int_a^b \frac{d\Gamma}{z-z(\Gamma)}$$

$$F(z) = \frac{1}{2\pi i}\int_C \frac{f(w)\,dw}{z-w} \quad , \quad f(w) = \sigma\, e^{-i\theta}$$

$$[u-iv]\big|_{z(\Gamma)} = F_-(z(\Gamma)) - F_+(z(\Gamma)) = f(z(\Gamma)) = \sigma(\Gamma)\, e^{-i\theta(\Gamma)}$$

$$[u+iv]\big|_{z(\Gamma)} = \sigma(\Gamma)\, e^{i\theta(\Gamma)}$$

<u>conclusions</u>

The velocity induced by a vortex sheet is discontinuous; the normal component is continuous, but the tangential component has a jump. The magnitude of the jump is the vortex sheet strength. The vorticity is a delta function with support on the sheet.

<u>recall</u>

$$z(\Gamma) = \Gamma$$

$$(u-iv)(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\Gamma}{z-\Gamma} = \begin{cases} -\frac{1}{2} & \text{if } \ y>0 \\ \frac{1}{2} & \text{if } \ y<0 \end{cases}$$

$$\frac{1}{2\pi i}\,\mathrm{pv}\int_{-\infty}^{\infty} \frac{d\Gamma}{\Gamma_0-\Gamma} = \lim_{\epsilon\to 0}\frac{1}{2\pi i}\Big(\int_{-\infty}^{\Gamma_0-\epsilon} + \int_{\Gamma_0+\epsilon}^{\infty}\Big)\frac{d\Gamma}{\Gamma_0-\Gamma} = 0 \ = \tfrac{1}{2}\big(-\tfrac{1}{2}+\tfrac{1}{2}\big) = 0$$

<u>ok</u>

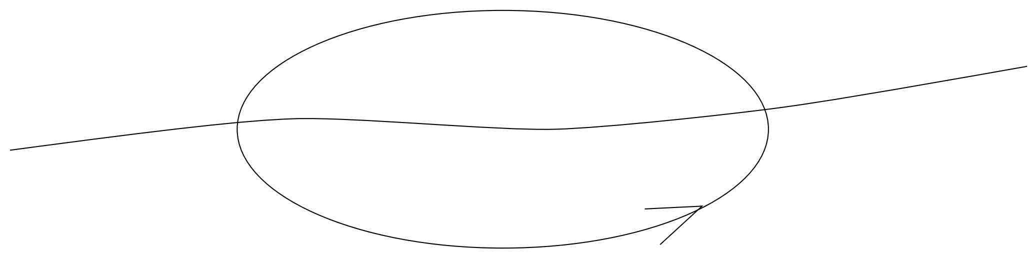
$$\sigma(\Gamma) = \frac{d\Gamma}{ds} = \frac{d\Gamma}{dx} = \frac{1}{2} - \Big(-\frac{1}{2}\Big) = 1 \qquad \underline{\text{ok}}$$

<u>ex</u> : compute the induced velocity of a circular vortex sheet

a) constant strength

b) strength defined by potential flow past a cylinder

<u>vortex sheet motion</u>



Γ : circulation parameter , Lagrangian variable

$z(\Gamma,t)$: flow map of curve

<u>Birkhoff-Rott equation</u>

$$\overline{\frac{\partial z}{\partial t}}(\Gamma,t) = \frac{1}{2\pi i}\,\mathrm{pv}\int_a^b \frac{d\tilde{\Gamma}}{z(\Gamma,t)-z(\tilde{\Gamma},t)}$$

$$z = x+iy \quad , \quad \overline{z} = x-iy \quad , \quad \frac{1}{iz} = \frac{-i(x-iy)}{x^2+y^2}$$

$$\frac{\partial x}{\partial t} = \frac{-1}{2\pi}\,\mathrm{pv}\int_a^b \frac{(y-\tilde{y})\,d\tilde{\Gamma}}{(x-\tilde{x})^2+(y-\tilde{y})^2} \quad , \quad x = x(\Gamma,t) \quad , \quad \tilde{x} = x(\tilde{\Gamma},t) \quad , \quad \ldots$$

$$\frac{\partial y}{\partial t} = \frac{1}{2\pi}\,\mathrm{pv}\int_a^b \frac{(x-\tilde{x})\,d\tilde{\Gamma}}{(x-\tilde{x})^2+(y-\tilde{y})^2}$$

<u>equilibrium</u>

$z(\Gamma,t) = \Gamma$: flat vortex sheet of constant strength , $-\infty < \Gamma < \infty$

<u>linear stability</u>

$$x(\Gamma,t) = \Gamma + p(\Gamma,t)$$

$$y(\Gamma,t) = \qquad q(\Gamma,t)$$

$$\begin{aligned}(x-\tilde{x})^2+(y-\tilde{y})^2 &= (\Gamma+p-(\tilde{\Gamma}+\tilde{p}))^2+(q-\tilde{q})^2 \sim ((\Gamma-\tilde{\Gamma})+(p-\tilde{p}))^2 \\ &\sim (\Gamma-\tilde{\Gamma})^2 + 2(\Gamma-\tilde{\Gamma})(p-\tilde{p})\end{aligned}$$

$$\frac{1}{(x-\tilde{x})^2+(y-\tilde{y})^2} \sim \frac{1}{(\Gamma-\tilde{\Gamma})^2\Big(1+2\Big(\frac{p-\tilde{p}}{\Gamma-\tilde{\Gamma}}\Big)\Big)} \sim \frac{1}{(\Gamma-\tilde{\Gamma})^2}\Big(1-2\Big(\frac{p-\tilde{p}}{\Gamma-\tilde{\Gamma}}\Big)\Big)$$

$$\frac{y-\tilde{y}}{(x-\tilde{x})^2+(y-\tilde{y})^2} \sim \frac{q-\tilde{q}}{(\Gamma-\tilde{\Gamma})^2}$$

$$\frac{x-\tilde{x}}{(x-\tilde{x})^2+(y-\tilde{y})^2} \sim \frac{(\Gamma-\tilde{\Gamma})+(p-\tilde{p})}{(\Gamma-\tilde{\Gamma})^2}\Big(1-2\Big(\frac{p-\tilde{p}}{\Gamma-\tilde{\Gamma}}\Big)\Big) \sim \frac{1}{\Gamma-\tilde{\Gamma}} - \frac{p-\tilde{p}}{(\Gamma-\tilde{\Gamma})^2}$$

$$\frac{\partial p}{\partial t} = \frac{-1}{2\pi}\,\mathrm{pv}\int_{-\infty}^{\infty}\frac{q-\tilde{q}}{(\Gamma-\tilde{\Gamma})^2}\,d\tilde{\Gamma}$$

$$\frac{\partial q}{\partial t} = \frac{-1}{2\pi}\,\mathrm{pv}\int_{-\infty}^{\infty}\frac{p-\tilde{p}}{(\Gamma-\tilde{\Gamma})^2}\,d\tilde{\Gamma}$$

look for solutions of the form $\quad p = Pe^{\omega t+ik\Gamma} \quad , \quad q = Qe^{\omega t+ik\Gamma} \quad , \quad k>0$

$$\omega P e^{\omega t+ik\Gamma} = \frac{-1}{2\pi}\,\mathrm{pv}\int_{-\infty}^{\infty}\frac{Qe^{\omega t}(e^{ik\Gamma}-e^{ik\tilde{\Gamma}})}{(\Gamma-\tilde{\Gamma})^2}\,d\tilde{\Gamma}$$

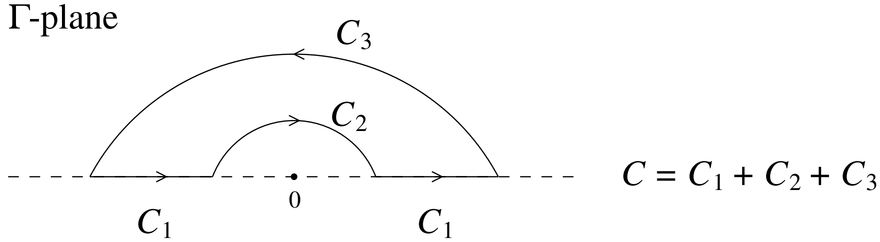
$$\omega Q e^{\omega t+ik\Gamma} = \frac{-1}{2\pi}\,\mathrm{pv}\int_{-\infty}^{\infty}\frac{Pe^{\omega t}(e^{ik\Gamma}-e^{ik\tilde{\Gamma}})}{(\Gamma-\tilde{\Gamma})^2}\,d\tilde{\Gamma}$$

$$\omega P = -I(k)\,Q$$

$$\omega Q = -I(k)\,P$$

$$I(k) = \frac{1}{2\pi}\,\mathrm{pv}\int_{-\infty}^{\infty}\frac{1-e^{ik(\tilde{\Gamma}-\Gamma)}}{(\Gamma-\tilde{\Gamma})^2}\,d\tilde{\Gamma} = \frac{1}{2\pi}\,\mathrm{pv}\int_{-\infty}^{\infty}\frac{1-e^{ik\Gamma}}{\Gamma^2}\,d\Gamma = \frac{k}{2}$$

pf



$$\frac{1}{2\pi i}\int_C \frac{1-e^{ik\Gamma}}{\Gamma^2}\,d\Gamma = 0$$

$$\Rightarrow \quad \frac{1}{2\pi i}\,\mathrm{pv}\int_{-\infty}^{\infty} \frac{1-e^{ik\Gamma}}{\Gamma^2}\,d\Gamma - \frac{1}{2}\,\mathrm{Res}\Big(\frac{1-e^{ik\Gamma}}{\Gamma^2};\Gamma=0\Big) = 0$$

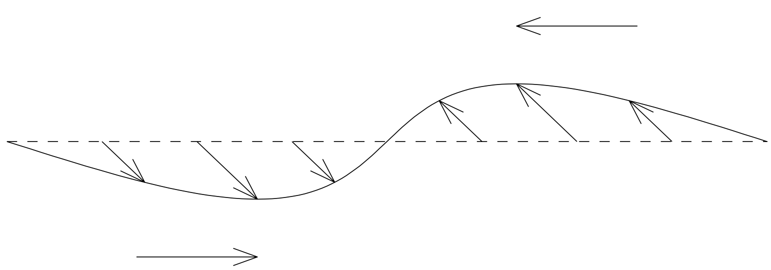
$$\Rightarrow \quad I(k) = i\cdot\frac{1}{2}\lim_{\Gamma\to 0}\frac{1-e^{ik\Gamma}}{\Gamma} = i\cdot\frac{1}{2}\cdot -ik = \frac{k}{2} \qquad \underline{\text{ok}}$$

$$\omega^2 PQ = I(k)^2 PQ \quad \Rightarrow \quad \omega = \pm\, I(k) = \pm\frac{k}{2}$$

Short wavelength perturbations ($k \to \infty$) have arbitrarily large growth rates ($\omega \to \infty$). : Kelvin-Helmholtz instability

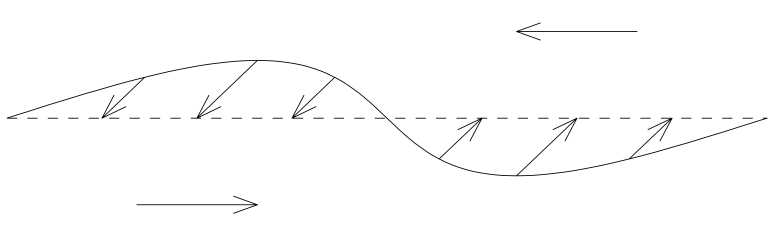
growing mode

$$z(\Gamma,t) = \Gamma + \epsilon(1-i)\,e^{\frac{k}{2}t}\sin k\Gamma$$



decaying mode

$$z(\Gamma,t) = \Gamma + \epsilon(1+i)\,e^{-\frac{k}{2}t}\sin k\Gamma$$



note : Material points travel on straight lines inclined at 45° to the x-axis.

<u>note</u>

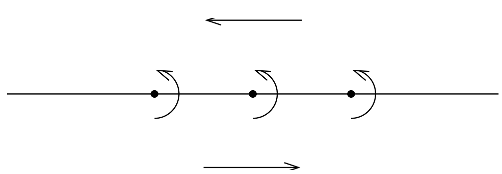
Consider a sequence of growing modes with $\epsilon = \frac{1}{k}$.

$z_k(\Gamma, t) = \Gamma + \frac{1}{k}(1-i)e^{\frac{k}{2}t}\sin k\Gamma$

Then $\lim_{k\to\infty} |z_k(\Gamma, 0)| = 0$, but for any $t > 0$, $\lim_{k\to\infty} |z_k(\Gamma, t)| = \infty$ (if $\Gamma \neq n\pi$).

$\Rightarrow$ The IVP for vortex sheet motion is <u>ill-posed</u> in the sense of Hadamard.

<u>discretization</u>



$z(\Gamma, t) \quad , \quad -\infty < \Gamma < \infty$

$$\Delta\Gamma = \frac{2\pi}{N} \quad , \quad \Gamma_j = j\,\Delta\Gamma \quad , \quad j = 0, \pm1, \pm2\ldots \quad , \quad z(\Gamma_j, t) \sim z_j(t)$$

$$\overline{\frac{\partial z}{\partial t}}(\Gamma, t) = \frac{1}{2\pi i}\,\mathrm{pv}\int_{-\infty}^{\infty} \frac{d\tilde{\Gamma}}{z(\Gamma, t) - z(\tilde{\Gamma}, t)}$$

$$\overline{\frac{dz_j}{dt}}(t) = \frac{1}{2\pi i}\sum_{\substack{k=-\infty \\ k\neq j}}^{\infty} \frac{\Delta\Gamma}{z_j(t) - z_k(t)}$$: <u>point vortex approximation</u>

<u>equilibrium</u>

$z_j(t) = \Gamma_j = j\Delta\Gamma$

check : $\overline{\frac{dz_j}{dt}} = \frac{1}{2\pi i}\sum_{k\neq j}\frac{\Delta\Gamma}{\Gamma_j - \Gamma_k} = \frac{1}{2\pi i}\sum_{k\neq j}\frac{1}{j-k} = \frac{1}{2\pi i}\sum_{k\neq 0}\frac{1}{k} = 0$ <u>ok</u>

<u>note</u>

1. The Birkhoff-Rott equation is ill-posed. The point vortex approximation is consistent, but unstable.

2. If $z(\Gamma, 0)$ is analytic, then $z(\Gamma, t)$ is analytic for a finite time.

3. The analytic solution breaks down at a finite critical time $t = t_c$ (Moore, 1979). The singularity is a blow-up in the curvature; the sheet remains continuously differentiable.

4. For $t < t_c$, the PVA converges as $N \to \infty$, but for $t > t_c$, the PVA does not converge.

<u>regularization</u>

$$\frac{\partial x}{\partial t} = \frac{-1}{2\pi} \int_{-\infty}^{\infty} \frac{(y - \tilde{y})\, d\tilde{\Gamma}}{(x - \tilde{x})^2 + (y - \tilde{y})^2 + \delta^2}$$

$$\frac{\partial y}{\partial t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(x - \tilde{x})\, d\tilde{\Gamma}}{(x - \tilde{x})^2 + (y - \tilde{y})^2 + \delta^2}$$

δ : smoothing parameter , point vortex $\rightarrow$ <u>vortex blob</u> (Chorin, 1973)

<u>note</u>

1. For $\delta > 0$, the regularized equation is well-posed.

2. For any $t > 0$ and $\delta > 0$, the vortex blob method converges as $N \to \infty$ (denote the limit by $z(\Gamma, t; \delta)$).

3. For any $t > 0$, $z(\Gamma, t; \delta)$ converges as $\delta \to 0$. For $t > t_c$, the limit is a spiral with an infinite number of turns.

<u>chapter 3 : gas flow in one dimension</u>

take fluid compressibility into account

<u>3.1 characteristics</u>

recall : isentropic Euler equations

$\rho_t + \nabla \cdot (\rho u) = 0$

$\rho u_t + \rho(u \cdot \nabla)u = -\nabla p + \rho b$

$e_t + \nabla \cdot ((e + p)u) = \rho u \cdot b$

$e = \frac{1}{2}\rho|u|^2 + \rho\epsilon$

$\epsilon = \epsilon(\rho, p)$

assumptions : $b = 0$, $p = p(\rho)$, 1D

$\rho = \rho(x,t)$, $u = u(x,t)$

$\rho_t + (\rho u)_x = 0$

$\rho u_t + \rho u u_x = -p'(\rho)\rho_x$

note : $p'(\rho) = c^2(\rho)$, $c(\rho)$: <u>local sound speed</u>

<u>linear stability</u>

equilibrium : $\rho = \rho_0$, $u = 0$

$\rho = \rho_0 + \tilde{\rho}(x,t)$, $u = \tilde{u}(x,t)$

$\tilde{\rho}_t + \rho_0 \tilde{u}_x = 0 \ \Rightarrow \ \tilde{\rho}_{tt} + \rho_0 \tilde{u}_{xt} = 0$

$\rho_0 \tilde{u}_t = -c_0^2 \tilde{\rho}_x$, $c_0 = c(\rho_0) \ \Rightarrow \ \rho_0 \tilde{u}_{tx} = -c_0^2 \tilde{\rho}_{xx}$

$\Rightarrow \ \tilde{\rho}_{tt} = c_0^2 \tilde{\rho}_{xx}$: wave equation , $\tilde{\rho}(x,t) = F(x - c_0 t) + G(x + c_0 t)$

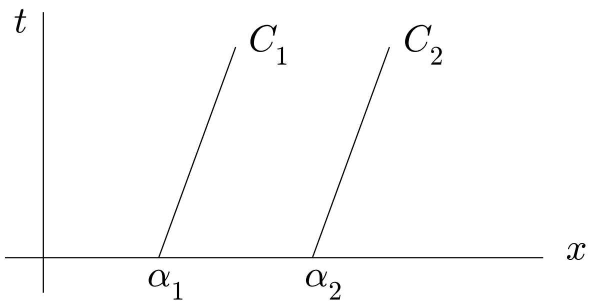
The fluid density (and velocity) undergo small amplitude variations which propagate in space at speed $\pm c_0$. This justifies calling $c(\rho)$ the local sound speed.

Before considering the coupled nonlinear system for ρ and u, we will consider some examples of linear and nonlinear scalar wave equations.

ex 1

$u_t + cu_x = 0 \ , \ u(x,0) = u_0(x)$

characteristics : $\dfrac{dx}{dt}(\alpha, t) = c \ , \ x(\alpha, 0) = \alpha \ \Rightarrow \ x(\alpha,t) = \alpha + ct$



1. The charateristics are straight lines with slope $\frac{-1}{c}$ in (x,t)-space.
2. The solution $u(x,t) = u_0(x + ct)$ is constant on characteristics.

ex 2

$u_t + uu_x = 0 \ , \ u(x,0) = u_0(x)$: inviscid Burger's equation

characteristics : $\dfrac{dx}{dt}(\alpha,t) = u(x(\alpha,t),t) \ , \ x(\alpha,0) = \alpha$

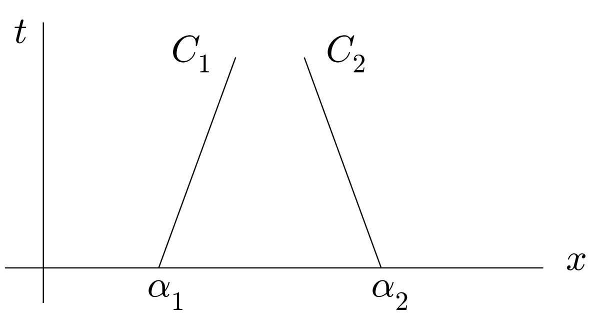
claim

1. The solution $u(x,t)$ is constant on characteristics.
2. The charateristics are straight lines with slope $\frac{-1}{u_0(\alpha)}$ in (x,t)-space.

pf

1. $\dfrac{d}{dt}u(x(\alpha,t),t) = u_x(x(\alpha,t),t) \cdot \dfrac{dx}{dt}(\alpha,t) + u_t(x(\alpha,t),t) = 0$ ok
2. $u(x(\alpha,t),t) = u(x(\alpha,0),0) = u(\alpha,0) = u_0(\alpha)$

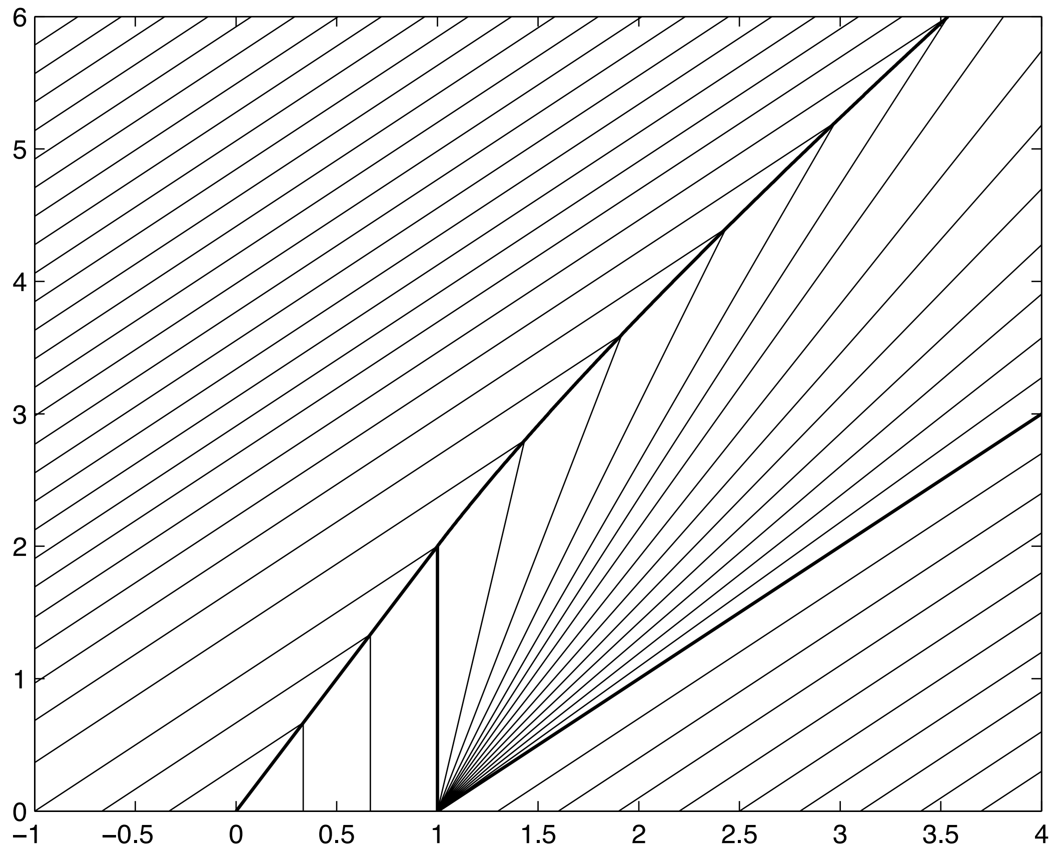
$\Rightarrow \ \dfrac{dx}{dt}(\alpha,t)u_0(\alpha) \ \Rightarrow \ x(\alpha,t) = \alpha + u_0(\alpha)t$ ok



3. Since characteristics can intersect, we need to consider solutions $u(x,t)$ which are discontinuous.

ex : $u_0(x) = \begin{cases} 1\ ,\ x<0 \\ 0\ ,\ 0<x<1 \\ 1\ ,\ 1<x \end{cases}$

The initial data implies the presence of a shock at $x = 0$ and a rarefaction fan at $x = 1$. The shock speed is $s = \frac{1}{2}(u_L + u_R) = \frac{1}{2}$ and the solution in the rarefaction fan is $u(x,t) = \frac{x-1}{t}$. The shock and rarefaction fan propagate independently for $0 \le t \le 2$, but for $t > 2$ they interact and the shock speed changes to $s = \frac{1}{2}(u_L + u_R) = \frac{1}{2}(1 + \frac{x-1}{t})$.



path of shock in (x,t)-plane : $\dfrac{dx}{dt} = \dfrac{x-1}{2t} + \dfrac{1}{2} \ \Rightarrow\ \dfrac{d(x-1)}{dt} - \dfrac{(x-1)}{2t} = \dfrac{1}{2}$

$$\Rightarrow\ t^{-1/2}\,\frac{d(x-1)}{dt} - t^{-1/2}\frac{(x-1)}{2t} = \frac{t^{-1/2}}{2} \ \Rightarrow\ \frac{d}{dt}(t^{-1/2}(x-1)) = \frac{t^{-1/2}}{2}$$

$$\Rightarrow\ t^{-1/2}(x-1) = t^{1/2} + a \ \Rightarrow\ x = 1 + t + at^{1/2}$$

$$(x,t) = (1,2) \ \Rightarrow\ 1 = 1 + 2 + a2^{1/2} \ \Rightarrow\ a = -\sqrt{2} \ \Rightarrow\ x = 1 + t - \sqrt{2t}$$

shock strength : $u_L - u_R = 1 - \left(\dfrac{x-1}{t}\right) = 1 - \left(\dfrac{1 + t - \sqrt{2t} - 1}{t}\right) = \sqrt{\dfrac{2}{t}}$

The solution approaches the constant state $u = 1$ in the limit $t \to \infty$.

<u>1D isentropic Euler equations</u>

$\rho_t + (\rho u)_x = 0$

$\rho u_t + \rho u u_x = -p_x$

assume : $p = p(\rho)$, $p_x = p'(\rho)\rho_x = c^2(\rho)\rho_x$, $c(\rho) = \sqrt{p'(\rho)}$: <u>local sound speed</u>

<u>linear stability</u>

equilibrium : $\rho = \rho_0$, $u = 0$

$\rho = \rho_0 + \tilde{\rho}(x,t)$, $u = \tilde{u}(x,t)$

$\tilde{\rho}_t + \rho_0 \tilde{u}_x = 0 \;\Rightarrow\; \tilde{\rho}_{tt} + \rho_0 \tilde{u}_{xt} = 0$

$\rho_0 \tilde{u}_t = -c_0^2 \tilde{\rho}_x$, $c_0 = c(\rho_0) \;\Rightarrow\; \rho_0 \tilde{u}_{tx} = -c_0^2 \tilde{\rho}_{xx}$

$\Rightarrow\; \tilde{\rho}_{tt} = c_0^2 \tilde{\rho}_{xx}$: wave equation , $\tilde{\rho}(x,t) = F(x - c_0 t) + G(x + c_0 t)$

The fluid density (and velocity) undergo small amplitude variations which propagate at speed $\pm c_0$. This justifies calling $c(\rho)$ the local sound speed.

<u>analysis of nonlinear system</u>

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u & \rho \\ c^2\rho^{-1} & u \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad , \quad c = c(\rho)$$

$A = \begin{pmatrix} u & \rho \\ c^2\rho^{-1} & u \end{pmatrix}$ is not symmetric, but it still has real e-values

$$\det(A - \lambda I) = \begin{vmatrix} u - \lambda & \rho \\ c^2\rho^{-1} & u - \lambda \end{vmatrix} = (u - \lambda)^2 - c^2 = 0 \;\Rightarrow\; \lambda = u \pm c$$