Math 655 Topics in Fluid Dynamics - Hydrodynamic Stability Winter 2004

hw#1 , due: Tuesday, February 10

0. Give a brief description of your academic background and research interests. If you work in a lab or research group, give your supervisor's name and describe briefly the topic you're working on. (1 paragraph is fine)

1. page 36 , problem 2.4 , bifurcation from infinity

2. page 36 , problem 2.5 , secondary instability

3. page 41 , problem 2.18 , instability due to linear resonance

4. page 44 , problem 2.22 , Jeans instability of a self-gravitating gas

5. page 57 , problem 3.2 , effect of surface tension on Kelvin-Helmholtz instability

Hints: You may assume 2D flow, as in class, i.e. $\widetilde{k} = k$ in Drazin's notation. You may assume that the dynamic boundary condition is $p_2 - p_1 = \gamma\kappa$, where γ is the surface tension and κ is the curvature of the interface.

6. Consider the IVP, $du/dt = au - bu^2, u(0) = u_0$, where $b > 0$. In class we saw that a transcritical bifurcation occurs at $a = 0$.

a) Find the explicit solution $u(t)$. Treat the cases $a = 0$ and $a \neq 0$ separately.

b) Verify the following asymptotic properties using the explicit solution.

$$a = 0\ ,\ u_0 > 0 \ \Rightarrow\ \lim_{t\to\infty} u(t) = 0 \quad , \quad u_0 < 0 \ \Rightarrow\ \text{blow-up (for } a = 0 \text{ and } a \neq 0)$$

$$a > 0\ ,\ u_0 > 0 \ \Rightarrow\ \lim_{t\to\infty} u(t) = \frac{a}{b} \quad , \quad a < 0\ ,\ u_0 > 0 \ \Rightarrow\ \lim_{t\to\infty} u(t) = 0$$

c) The results show that blow-up occurs if and only if $u_0 < 0$. Sketch the graph of the critical time t_c as a function of the initial amplitude $|u_0|$. As the initial amplitude increases, does the critical time increase or decrease?

7. Consider the BVP, $y_2'' + n^2\pi^2 y_2 = -\lambda_1 \sin n\pi x$, $y_2(0) = y_2(1) = 0$. In class it was claimed that the solution y_2 exists if and only if $\lambda_1 = 0$. Prove this by direct calculation.

8. Consider the IVP, $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \big(\lambda - \frac{1}{2}\int_0^1 \big(\frac{\partial u}{\partial x}(x,t)\big)^2 dx\big)u$, $u(x,0) = f(x)$, $u(0,t) = u(1,t) = 0$, where $\lambda \geq 0$. In class we found the equilibrium solutions; $u_0(x) = 0$ is a solution for all λ and for each $n = 1, 2, \ldots$, there is a branch of nonzero solutions given by $u_n(x) = A \sin n\pi x$, $\lambda = n^2\pi^2\big(1 - A^2/4\big)$.

a) Perform a linear stability analysis of the equilibrium solutions and plot the results in a bifurcation diagram, indicating stability/instability by a solid/dashed line.

b) Perform a weakly nonlinear stability analysis of the zero solution close to the first bifurcation point, $\lambda = \lambda_c = \pi^2$, on the supercritical side.