Math 655 Topics in Fluid Dynamics - Hydrodynamic Stability Winter 2004

hw#3 , due: Thursday, April 1

1. Show that the z-component of vorticity in polar coordinates is $\omega_z = \dfrac{\partial u_\theta}{\partial r} + \dfrac{u_\theta}{\partial r} - \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}$.

Problems $2-5$ are from chapter 3 of Drazin & Reid

2. *Oscillations of a swirl flow in a cylinder.* The velocity field $(u_r,u_\theta,u_z)=(0,r\Omega_0,0)$ defines an inviscid equilibrium swirl flow with constant angular velocity Ω_0 in a cylinder of radius R_0. In class we showed that the pressure perturbation $p(r)$ satisfies the equation $(D_*D-(n^2/r^2))p=k^2(1+(4\Omega_0^2/\gamma^2))p$, where n and k are the azimuthal and axial wavenumbers, $\gamma=s+in\Omega_0$, and s is the growth rate. The boundary conditions are that $p(r)$ has a finite value at $r=0$ and $\gamma Dp+(2in\Omega_0/r)p=0$ at $r=R_0$. Show that $s=\pm 2i\Omega_0/(1+\alpha^2/a^2)^{1/2}-in$, where $a=k/R_0$ and α is any root of the equation $\alpha J_n'(\alpha)\pm n(1+\alpha^2/a^2)^{1/2}J_n(\alpha)=0$. Hence the flow is marginally stable and imag(s) gives the oscillation frequency of the normal mode. Show that when $n=0$, this becomes $s=\pm 2i\Omega_0/(1+j_{1,m}^2/a^2)$, where $j_{1,m}$ is the mth positive zero of $J_1(\alpha)$.

3. *Rayleigh's theorem for swirl flow.* Consider a general inviscid swirl flow with velocity $(u_r,u_\theta,u_z)=(0,V(r),0)$ between two cylinders, $R_1\le r\le R_2$. In class we showed that a 2D perturbation satisfies the equation $(s+in\Omega)(D_*D-n^2/r^2)\phi-inr^{-1}(DD_*V)\phi=0$, where $\phi=ru$ and $u=u(r)$ is the radial perturbation velocity. Multiply the equation by $r\phi^*/(s+in\Omega)$ (where ϕ^* is the complex conjugate of ϕ), integrate from $r=R_1$ to R_2, apply the boundary conditions $\phi(R_1)=\phi(R_2)=0$, take the imaginary part of the result, and hence derive the relation real$(s)\cdot n\int_{R_1}^{R_2}\left((DD_*V)|\phi|^2/|s+in\Omega|^2\right)r\,dr=0$. This yields Rayleigh's theorem for 2D perturbations of a swirl flow, i.e. a necessary condition for inviscid instability is that the basic vorticity profile should have a local maximum or local minimum somewhere in the flow domain.

4. *Oscillations of a columnar vortex* (Kelvin modes). Consider the basic inviscid swirl flow given by $\Omega(r)=\Omega_0$ for $r<R_0$, $\Omega(r)=\Omega_0(R_0/r)^2$ for $r>R_0$. Show that the flow has constant vorticity for $r<R_0$ and is irrotational for $r>R_0$. Show that the flow is marginally stable with respect to both axisymmetric and 2D perturbations, and that in the latter case the oscillation frequency is imag$(s)=-\Omega_0(n-1)$. Assume that the equation of the boundary of the perturbed vortex is $r=R_0(1+\epsilon\exp(st+in\theta))$ and show that the perturbation represents a sequence of waves traveling around the vortex with angular velocity $\omega(n)=\Omega_0(1-n^{-1})$. Note that $\omega(1)=0$ and $|\omega(n)|<|\Omega_0|$ for $n\ge 2$; explain what this means geometrically.

5. *Instability of a cylindrical vortex sheet* (Rotunno). Consider the basic inviscid swirl flow given by $\Omega(r)=0$ for $r<R_0$, $\Omega(r)=\Omega_0(R_0/r)^2$ for $r>R_0$, which represents a cylindrical vortex sheet of radius R_0. Show that the flow is marginally stable with respect to axisymmetric perturbations, but unstable with respect to 2D perturbations with growth rate $s=\pm(\Omega_0/2)((n^2-2n)^{1/2}-in)$.