basics

$u(x,t)$: velocity , $x = (x,y,z)$, $u = u(u,v,w)$

$p(x,t)$: pressure

$\rho(x,t)$: density

Navier-Stokes equation : $u_t + (u \cdot \nabla)u = -\frac{\nabla p}{\rho} + \nu \Delta u$

incompressible flow : $\nabla \cdot u = 0$, $\rho = \text{cnst}$

initial conditions , boundary conditions

vorticity : $\omega = \nabla \times u$

$\omega = 0$: irrotational flow $\Rightarrow$ $u = \nabla \phi$, ϕ : potential function

2D flow

$\nabla \cdot u = 0$ $\Rightarrow$ there exists ψ : stream function st $u = \psi_y$, $v = -\psi_x$

$\omega = v_x - u_y$, $\Delta \psi = -\omega$, $\omega_t + (u \cdot \nabla)\omega = \nu \Delta \omega$

1. pipe flow experiment

Reynolds (1883)

a : pipe radius (length) , U : maximum fluid velocity (length/time)

laminar flow $\to$ transition $\to$ turbulence

nondimensionalization $\Rightarrow$ $u_t + (u \cdot \nabla)u = -\nabla p + \frac{1}{R}\Delta u$

$R = \frac{Ua}{\nu}$: Reynolds number

R_c : critical value

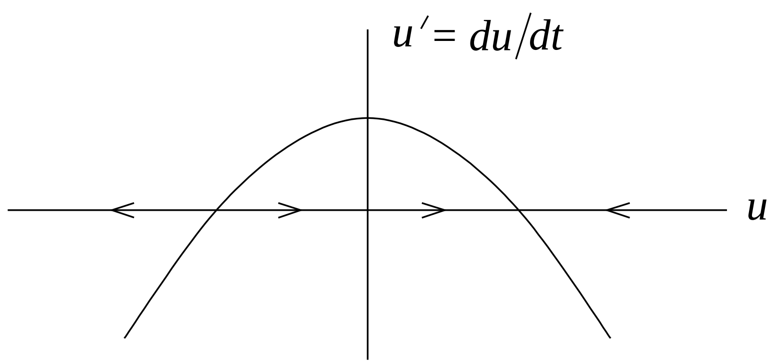
2. bifurcations

ex : turning point , saddle-node bifurcation

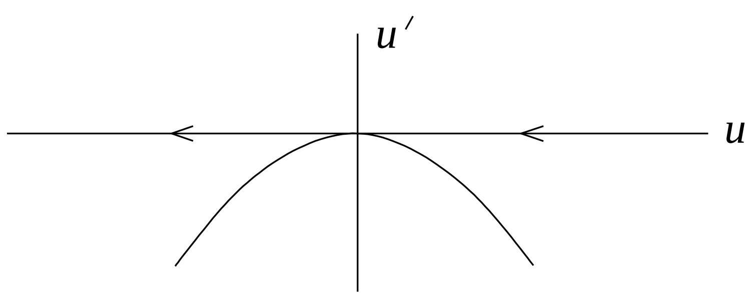
$$\frac{du}{dt} = a - u^2 \quad , \quad u(0) = u_0 \quad , \quad a = R - R_c \ : \ \text{control parameter}$$

$$\text{equilibrium} \ : \ a - u^2 = 0 \quad \Rightarrow \quad U = \begin{cases} \pm\sqrt{a} & , \quad a > 0 \\ 0 & , \quad a = 0 \\ \text{none} & , \quad a < 0 \end{cases}$$

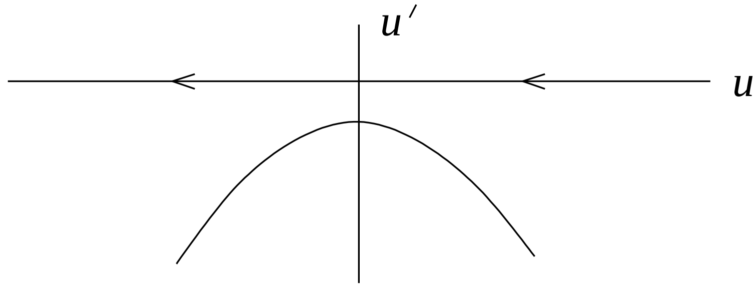
case 1 : $a > 0 \quad \Rightarrow \quad U = \sqrt{a}$ is stable , $U = -\sqrt{a}$ is unstable



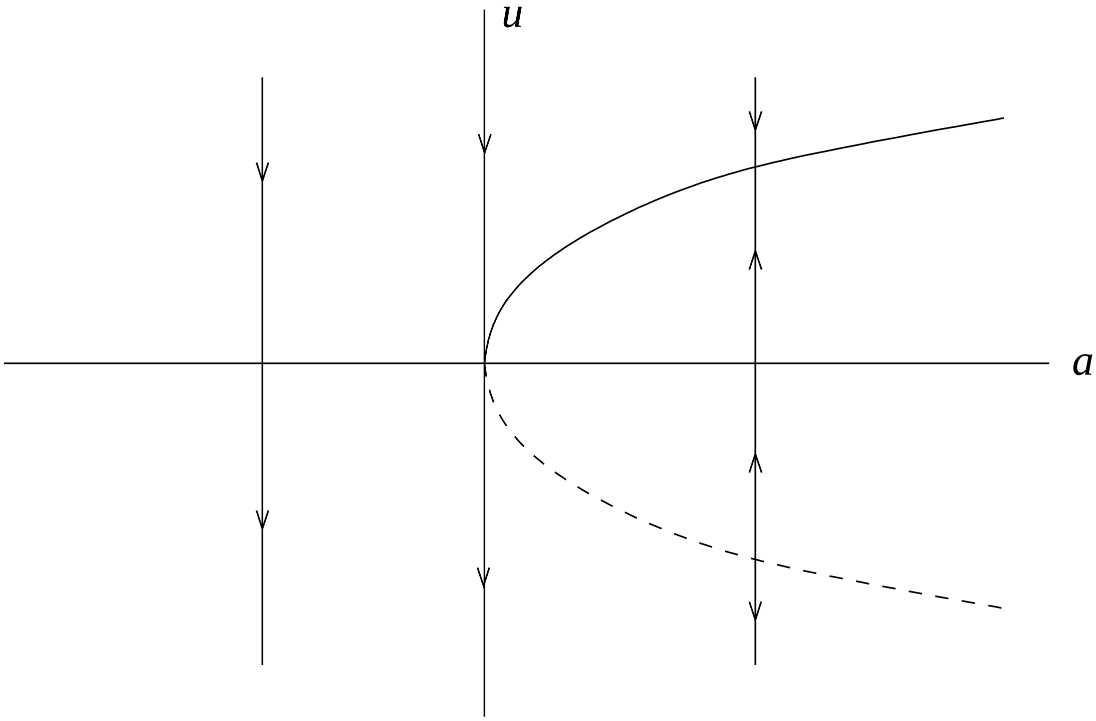
case 2 : $a = 0 \quad \Rightarrow \quad U = 0$ is unstable



case 3 : $a < 0$



<u>bifurcation diagram</u> : turning point



<u>linear stability analysis</u>

$u' \;=\; u - U$: <u>perturbation</u>

$$\frac{du'}{dt} \;=\; \frac{du}{dt} \;=\; a - u^2 \;=\; a - (u' + U)^2 \;=\; a - \big(\,(u')^2 + 2u'U + U^2\,\big)$$

$$\frac{du'}{dt} \;=\; -2\,Uu' - (u')^2$$

<u>linearized equation</u>

$$\frac{du'}{dt} = -2\,Uu' \quad\Rightarrow\quad u'(t) \;=\; u'(0)e^{st} \quad , \quad s \;=\; -2U$$: <u>growth rate</u>

$U = \sqrt{a} \quad\Rightarrow\quad s < 0 \quad\Rightarrow\quad \lim_{t\to\infty} u'(t) = 0$: stable

$U = -\sqrt{a} \quad\Rightarrow\quad s > 0 \quad\Rightarrow\quad \lim_{t\to\infty} u'(t) = \pm\infty$: unstable

$U = 0 \quad\Rightarrow\quad s = 0$: <u>marginal stability</u> , need to look at nonlinear terms to determine stability

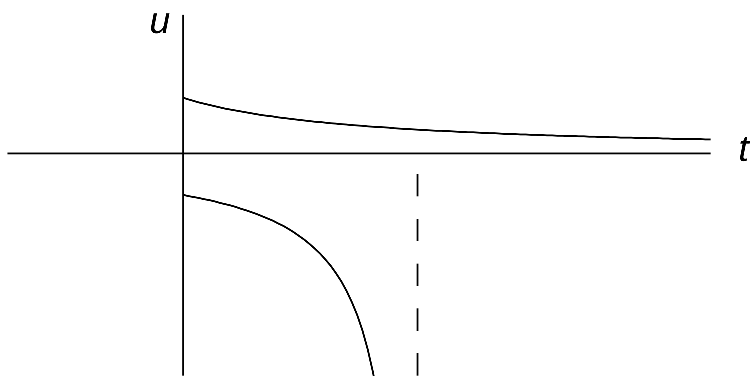
<u>explicit solution</u>

<u>case 1</u> : $a = 0$, $\dfrac{du}{dt} = -u^2 \quad \Rightarrow \quad u(t) = \dfrac{u_0}{u_0 t + 1}$

$u_0 = 0 \quad \Rightarrow \quad u(t) = 0$ for all t

$u_0 > 0 \quad \Rightarrow \quad \lim_{t\to\infty} u(t) = 0$

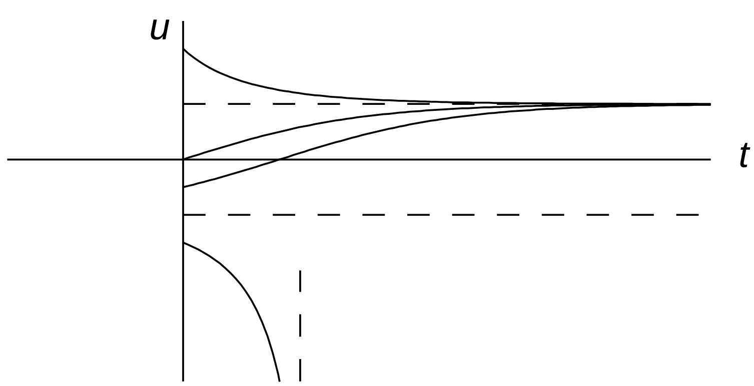
$u_0 < 0 \quad \Rightarrow \quad u(t) \to -\infty$ as $t \to t_c = \dfrac{-1}{u_0}$: <u>blow-up</u>



<u>case 2</u> : $a > 0$, $\dfrac{du}{dt} = a - u^2 \quad \Rightarrow \quad u(t) = \sqrt{a}\left(\dfrac{u_0 + \sqrt{a}\tanh\sqrt{a}\,t}{\sqrt{a} + u_0\tanh\sqrt{a}\,t}\right)$

$u_0 = \pm\sqrt{a} \quad \Rightarrow \quad u(t) = \pm\sqrt{a}$ for all t

$u_0 > -\sqrt{a} \quad \Rightarrow \quad \lim_{t\to\infty} u(t) = \sqrt{a}$, $u_0 < -\sqrt{a} \quad \Rightarrow$ blow-up



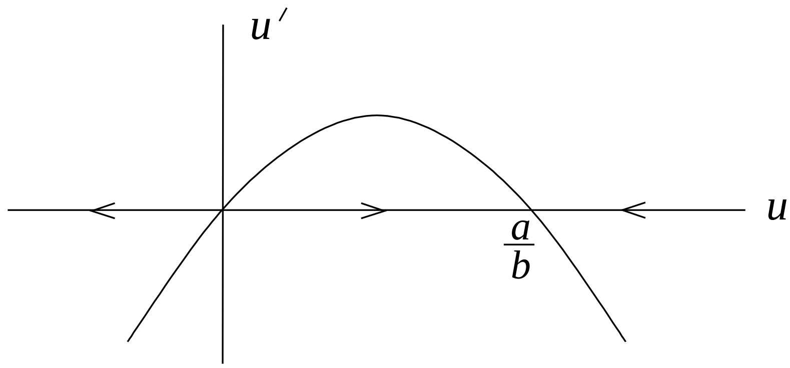
<u>case 3</u> : $a < 0$, $u(t) = \sqrt{-a}\left(\dfrac{u_0 - \sqrt{-a}\,\tan\sqrt{-a}\,t}{\sqrt{-a} + u_0\,\tan\sqrt{-a}\,t}\right) \quad \Rightarrow$ blow-up for all u_0

<u>ex</u> : transcritical bifurcation

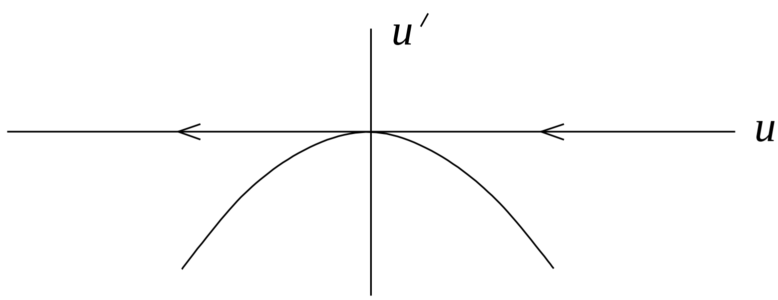
$$\frac{du}{dt} = au - bu^2 \quad , \quad b > 0$$

equilibrium : $au - bu^2 = 0 \quad \Rightarrow \quad u(a - bu) = 0 \quad \Rightarrow \quad U = 0 \, , \, \frac{a}{b}$

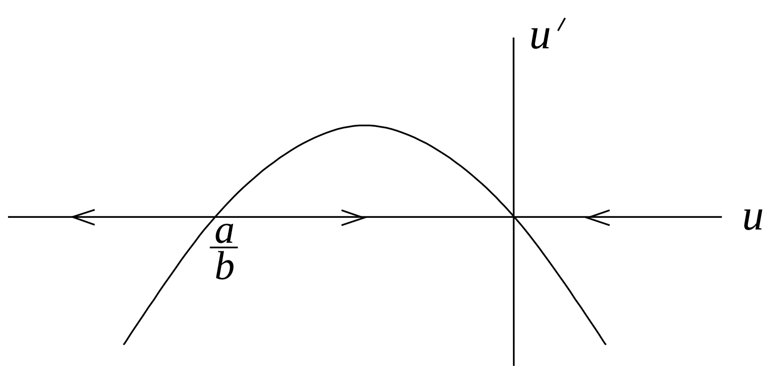
<u>case 1</u> : $a > 0 \quad \Rightarrow \quad U = 0$ is unstable , $U = \frac{a}{b}$ is stable



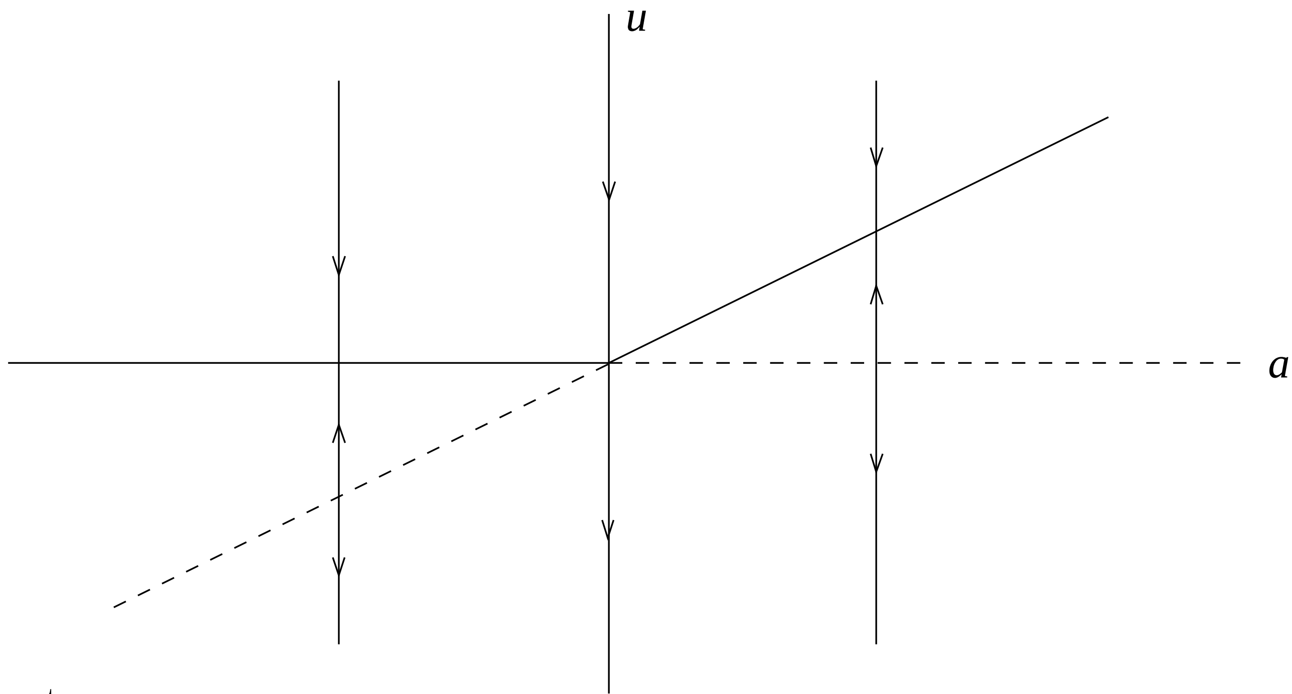
<u>case 2</u> : $a = 0 \quad \Rightarrow \quad U = 0$ is unstable



<u>case 3</u> : $a < 0 \quad \Rightarrow \quad U = 0$ is stable , $U = \frac{a}{b}$ is unstable



<u>bifurcation diagram</u> : transcritical bifurcation



<u>note</u>

For $a < 0$ the zero solution is stable and the nonzero solution is unstable, and for $a > 0$ the zero solution is unstable and the nonzero solution is stable; there is an <u>exchange of stability</u> at the bifurcation point $a = 0$.

<u>linear stability</u>

$$u' \;=\; u - U$$

$$\frac{du'}{dt} \;=\; \frac{du}{dt} \;=\; au - bu^2 \;=\; a(u' + U) - b(u' + U)^2$$

$$\frac{du'}{dt} \;=\; (a - 2bU)u' \quad \Rightarrow \quad u'(t) \;=\; u'(0)e^{st} \quad , \quad s \;=\; a - 2bU$$

$$U = 0 \quad \Rightarrow \quad s = a \;:\; \begin{cases} \text{stable} & \text{if } a < 0 \\ \text{unstable} & \text{if } a > 0 \end{cases}$$

$$U = \frac{a}{b} \quad \Rightarrow \quad s = -a \;:\; \begin{cases} \text{unstable} & \text{if } a < 0 \\ \text{stable} & \text{if } a > 0 \end{cases}$$

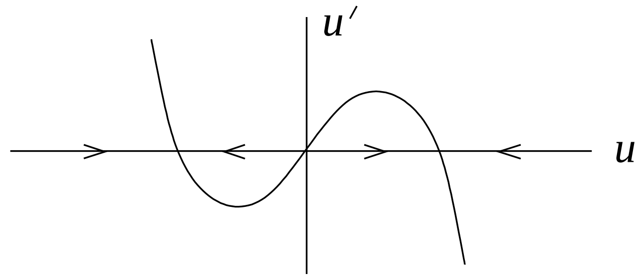
<u>explicit solution</u> : hw

pitchfork bifurcation

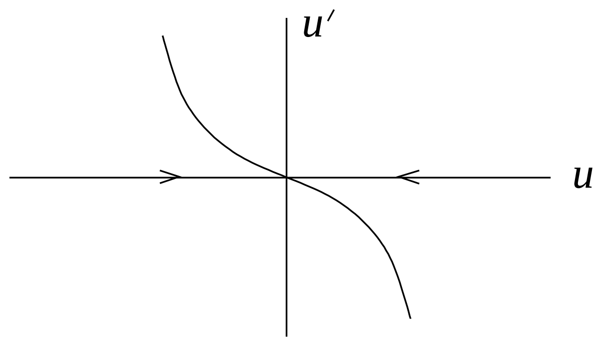
$$\frac{du}{dt} = au - bu^3 \quad : \quad \underline{\text{Landau equation}}$$

$$\text{equilibrium} \quad : \quad u(a - bu^2) = 0 \quad \Rightarrow \quad U = \begin{cases} 0 \;\; , \;\; \pm\sqrt{\dfrac{a}{b}} & \text{if } \dfrac{a}{b} > 0 \\ 0 & \text{if } \dfrac{a}{b} < 0 \end{cases}$$

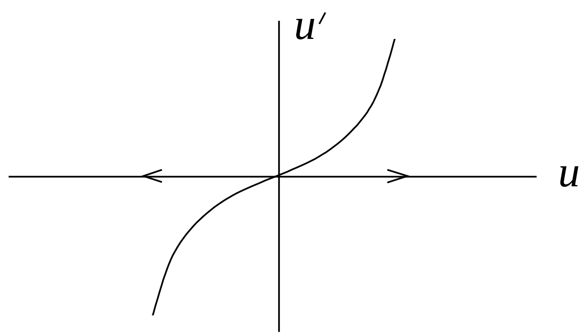
case 1 : $b > 0$, $a > 0$



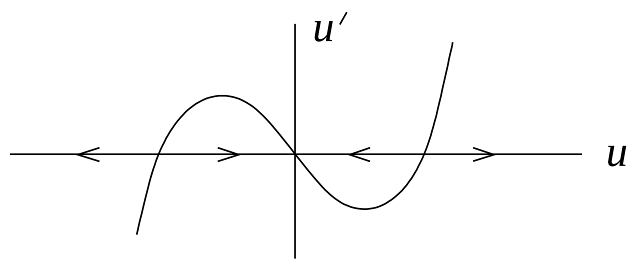
case 2 : $b > 0$, $a < 0$



case 3 : $b < 0$, $a > 0$

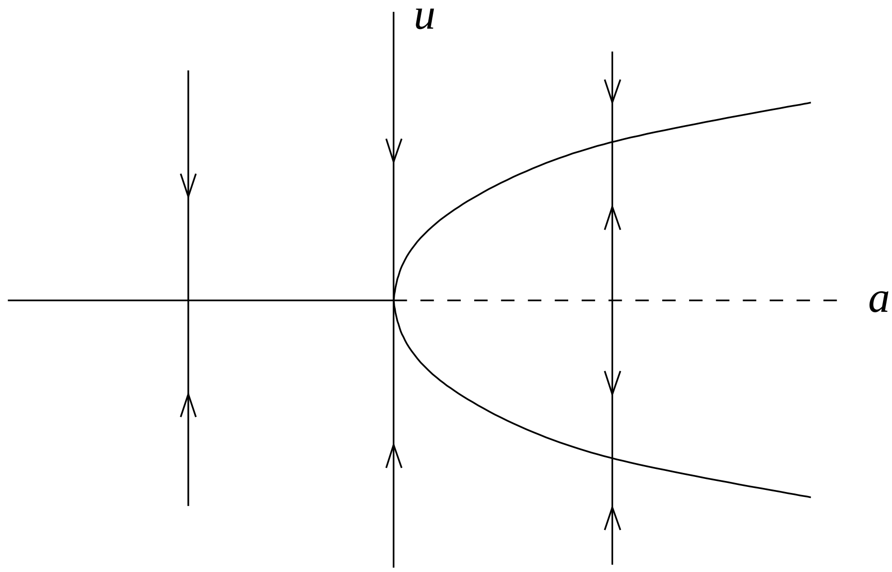


case 4 : $b < 0$, $a < 0$

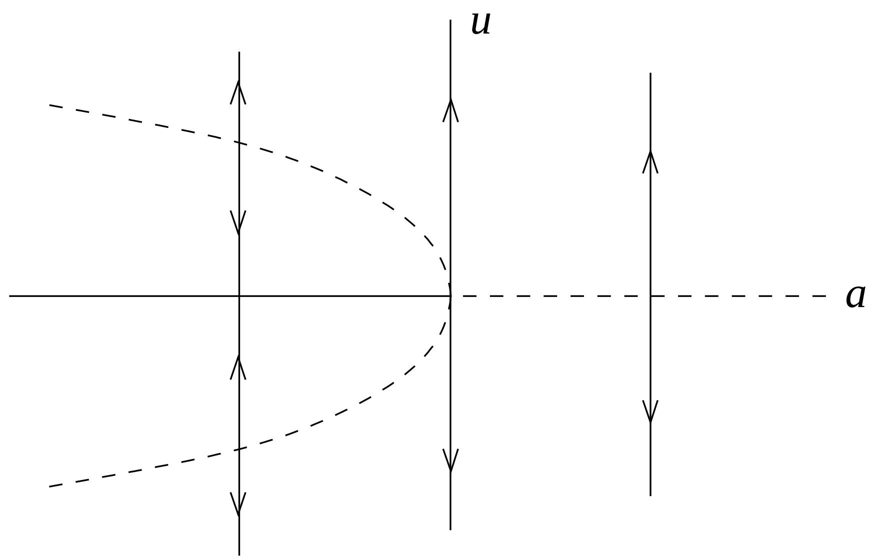


<u>bifurcation diagram</u>

$b > 0$: <u>supercritical</u> pitchfork bifurcation



$b < 0$: <u>subcritical</u> pitchfork bifurcation



<u>linear stability</u>

$$u' \;=\; u - U$$

$$\frac{du'}{dt} \;=\; \frac{du}{dt} \;=\; au - bu^3 \;=\; a(u'+U) - b(u'+U)^3$$

$$\frac{du'}{dt} \;=\; (a - 3bU^2)u' \quad \Rightarrow \quad u'(t) \;=\; u'(0)e^{st} \quad , \quad s \;=\; a - 3bU^2$$

$$U = 0 \quad \Rightarrow \quad s = a \;:\; \begin{cases} \text{stable} & \text{if } a < 0 \\ \text{unstable} & \text{if } a > 0 \end{cases} \quad \Big(\text{as in transcritical bifurcation}\Big)$$

$$U = \pm\sqrt{a/b} \quad \Rightarrow \quad s = -2a \;:\; \begin{cases} \text{unstable} & \text{if } a > 0 \\ \text{stable} & \text{if } a < 0 \end{cases}$$

<u>explicit solution</u>

$$\frac{du}{dt} = au - bu^3 \quad \Rightarrow \quad u^2(t) = \begin{cases} \dfrac{au_0^2}{(a - bu_0^2)e^{-2at} + bu_0^2} & \text{if } \; a \neq 0 \\[2ex] \dfrac{u_0^2}{2bu_0^2 t + 1} & \text{if } \; a = 0 \end{cases}$$

<u>case 1</u> : $b > 0 \;,\; a > 0$

$$\lim_{t\to\infty} u(t) = \operatorname{sign}(u_0)\sqrt{a/b}$$

The system is <u>bistable</u>, i.e. there are two stable equilibrium points. A perturbation of $U = 0$ grows due to linear instability, but eventually <u>equilibrates</u> due to nonlinearity.

<u>case 2</u> : $b > 0 \;,\; a < 0$

$$\lim_{t\to\infty} u(t) = 0 \quad \text{for all } u_0$$

Nonlinearity reinforces the linear stability of $U = 0$.

<u>case 3</u> : $b < 0 \;,\; a > 0$

$$(a - bu_0^2)e^{-2at} + bu_0^2 = \begin{cases} a & \text{if } \; t = 0 \\ bu_0^2 & \text{if } \; t \to \infty \end{cases} \quad \Rightarrow \quad \text{blow-up}$$

Nonlinearity reinforces the linear instability of $U = 0$.

<u>case 4</u> : $b < 0$, $a < 0$

$(a - bu_0^2)e^{-2at} + bu_0^2 = \big($ pos or neg $\big)$ + neg

$|u_0| > \sqrt{a/b} \quad \Rightarrow \quad u_0^2 > a/b \quad \Rightarrow \quad bu_0^2 < a \quad \Rightarrow \quad 0 < a - bu_0^2 \quad \Rightarrow$ blow-up

$|u_0| > \sqrt{a/b} \quad \Rightarrow \quad \lim\limits_{t\to\infty} u(t) = 0$

$|u_0| = \sqrt{a/b}$ is a <u>threshold</u> for instability.

$U = 0$ is subject to a <u>finite amplitude instability</u>.

<u>ex</u> : Hopf bifurcation

$$\frac{dx}{dt} = -y + \big(a - x^2 - y^2\big)x$$

$$\frac{dy}{dt} = \quad x + \big(a - x^2 - y^2\big)y$$

$$\text{equilibrium} \ : \quad \left.\begin{aligned} -y + \big(a - x^2 - y^2\big)x = 0 \\ x + \big(a - x^2 - y^2\big)y = 0 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} -y^2 + \big(a - x^2 - y^2\big)xy = 0 \\ x^2 + \big(a - x^2 - y^2\big)xy = 0 \end{aligned}$$

$$\Rightarrow \quad x^2 + y^2 = 0 \quad \Rightarrow \quad X = Y = 0$$

<u>linear stability</u>

$x' = x$, $y' = y$

$$\left.\begin{aligned} \frac{dx'}{dt} &= -y' + ax' \\ \frac{dy'}{dt} &= \quad x' + ay' \end{aligned}\right\} \quad \Rightarrow \quad \frac{d}{dt}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

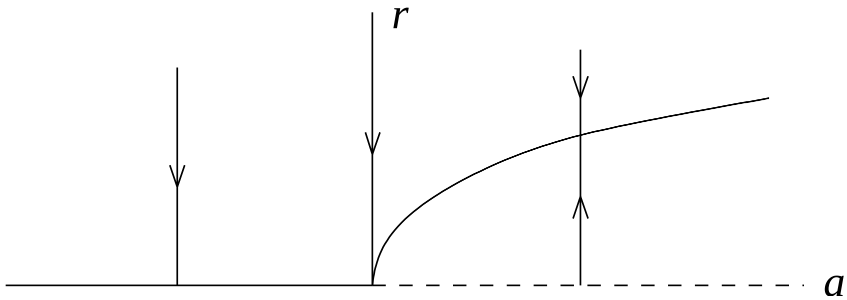
$$\det\begin{pmatrix} a - s & -1 \\ 1 & a - s \end{pmatrix} = \big(a - s\big)^2 + 1 = 0 \quad \Rightarrow \quad s = a \pm i$$

$$x'(t)\,, y'(t) \in \operatorname{span}\big\{e^{at}\sin t\,, e^{at}\cos t\big\} \quad \Rightarrow \quad \big(X, Y\big) = (0,0) \ \text{ is} \begin{cases} \text{stable} & \text{if } a < 0 \\ \text{unstable} & \text{if } a > 0 \end{cases}$$

explicit solution

$x = r\cos\theta \ , \ y = r\sin\theta$

$$\frac{dr}{dt} = ar - r^3 \quad , \ \text{equilibrium}: \ R = \begin{cases} 0 \ , \sqrt{a} & \text{if } a > 0 \\ 0 & \text{if } a < 0 \end{cases} \ , \ \text{supercritical pb at } a = 0$$



$$\frac{d\theta}{dt} = 1 \quad \Rightarrow \quad \theta(t) = t + \theta_0$$

For $a > 0$, the equilibrium $R = \sqrt{a}$ yields a time-dependent periodic solution of the original system given by $x(t) = \sqrt{a}\cos(t + \theta_0)$, $y(t) = \sqrt{a}\sin(t + \theta_0)$.

In general, $(x(t)\,,y(t))$ defines an orbit in the xy-plane.

$a < 0 \quad \Rightarrow \quad (0\,,0)$ is a stable focus, i.e. all orbits approach $(0\,,0)$ as $t \to \infty$

$a > 0 \quad \Rightarrow \quad (0\,,0)$ is a unstable focus and $x^2 + y^2 = a$ is a stable limit cycle, i.e. all orbits with $(x_0\,,y_0) \neq (0\,,0)$ approach the lc as $t \to \infty$

note

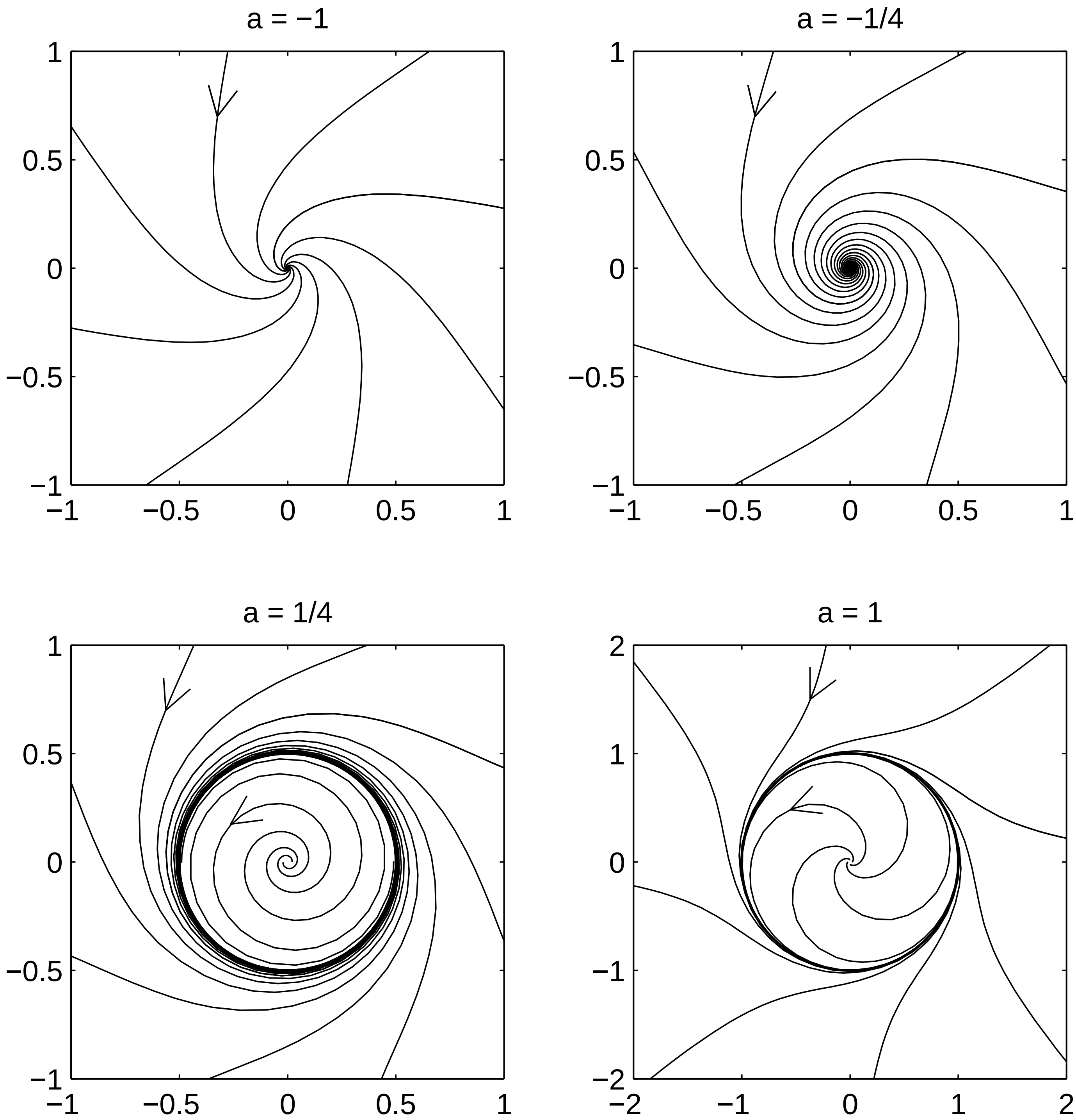
1. $du/dt = a - u^2$: turning point , $s = \pm\sqrt{a}$

2. $du/dt = au - bu^2$: transcritical , $s = \pm a$

3. $du/dt = au - bu^3$: pitchfork , $s = \{a\,, -2a\}$

4. $\begin{cases} dx/dt = -y + (a - x^2 - y^2)x \\ dy/dt = \ \ x + (a - x^2 - y^2)y \end{cases}$: Hopf , $s = a \pm i$

In all cases the bifurcation occurs at $a = 0$, i.e. when the real part of s changes sign, where s is an eigenvalue of the linearized problem. Cases 1-3 are called zero-crossing bifurcations. In case 4, the bifurcation occurs when a pair of complex conjugate eigenvalues crosses the imaginary axis.



a = −1
a = −1/4
a = 1/4
a = 1
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1
2
1
0
−1
−2
−2
−1
0
1
2

3. Kelvin-Helmholtz instability

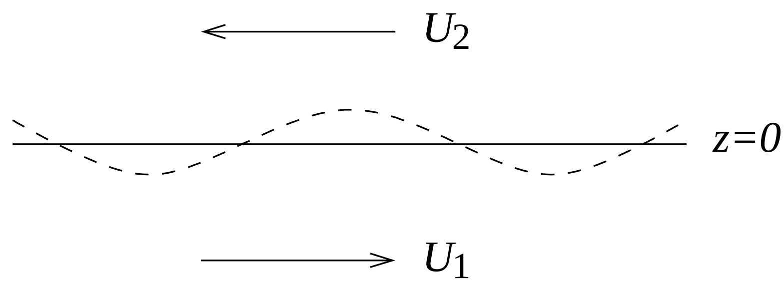
incompressible , inviscid , 2D flow

basic flow : two parallel streams moving at different speeds

$$u_0(z) = \begin{cases} U_1\,\vec{e}_x \ , \ z<0 \\ U_2\,\vec{e}_x \ , \ z>0 \end{cases}$$

$$\rho_0(z) = \begin{cases} \rho_1 \ , \ z<0 \\ \rho_2 \ , \ z>0 \end{cases}$$

$$p_0(z) = \begin{cases} p_0 - \rho_1 g z \ , \ z<0 \\ p_0 - \rho_2 g z \ , \ z>0 \end{cases}$$



The flat interface is in hydrostatic equilibrium, i.e. $\nabla p + \rho g \vec{e}_z = 0$.

perturbed interface : $z = \zeta(x,t)$

potential functions : $\phi_1(x,z,t)$ on $z < \zeta(x,t)$, $\phi_2(x,z,t)$ on $z > \zeta(x,t)$

$$u = \nabla\phi \quad , \quad \nabla\cdot u = 0 \quad \Rightarrow \quad \Delta\phi = 0$$

$$\Delta\phi_1 = 0 \ \text{ on } \ z < \zeta(x,t) \quad , \quad \nabla\phi_1 \to U_1\,\vec{e}_x \ \text{ as } \ z \to -\infty$$

$$\Delta\phi_2 = 0 \ \text{ on } \ z > \zeta(x,t) \quad , \quad \nabla\phi_2 \to U_2\,\vec{e}_x \ \text{ as } \ z \to +\infty$$

boundary conditions on interface

1. kinematic bc : interface moves with the fluid velocity

$$\frac{\partial\phi_i}{\partial z} = \frac{D\zeta}{Dt} = \frac{\partial\zeta}{\partial t} + \frac{\partial\phi_i}{\partial x}\cdot\frac{\partial\zeta}{\partial x} \ \text{ on } \ z = \zeta(x,t) \quad , \quad i = 1,2$$

2. dynamic bc : pressure is continuous across interface

$$\rho_1\Big(c_1 - \frac{1}{2}\big|\nabla\phi_1\big|^2 - \frac{\partial\phi_1}{\partial t} - gz\Big) = \rho_2\Big(c_2 - \frac{1}{2}\big|\nabla\phi_2\big|^2 - \frac{\partial\phi_2}{\partial t} - gz\Big) \ \text{ on } \ z = \zeta(x,t)$$

basic flow $\Rightarrow$ $\rho_1\Big(c_1 - \frac{1}{2}U_1^2\Big) = \rho_2\Big(c_2 - \frac{1}{2}U_2^2\Big)$

linear stability

$\phi_1 = U_1 x + \phi_1'$ on $x < \zeta'$ $\Rightarrow$ $\Delta\phi_1' = 0$ on $z < 0$, $\phi_1' \to 0$ as $z \to -\infty$

$\phi_2 = U_2 x + \phi_2'$ on $x > \zeta'$ $\Rightarrow$ $\Delta\phi_2' = 0$ on $z > 0$, $\phi_2' \to 0$ as $z \to +\infty$

$$\frac{\partial\phi_i'}{\partial z} = \frac{\partial\zeta'}{\partial t} + U_i\frac{\partial\zeta'}{\partial x} \quad \text{on } z = 0 \quad , \quad i = 1\,,2$$

$$\nabla\phi_i = U_i\,\vec{e}_x + \nabla\phi_i' \quad \Rightarrow \quad \left|\nabla\phi_i\right|^2 = U_i^2 + \left|\nabla\phi_i'\right|^2 + 2U_i\frac{\partial\phi_i'}{\partial x}$$

$$\rho_1\Big(-U_1\frac{\partial\phi_1'}{\partial x} - \frac{\partial\phi_1'}{\partial t} - g\zeta'\Big) = \rho_2\Big(-U_2\frac{\partial\phi_2'}{\partial x} - \frac{\partial\phi_2'}{\partial t} - g\zeta'\Big) \quad \text{on } z = 0$$

look for normal mode solutions

$\phi_1'(x,z,t) = \widehat{\phi}_1(z)e^{st+ikx}$, k : wavenumber , assume $k > 0$

note : s is an eigenvalue , $\phi_1'(x,z,t)$ is an eigenfunction

$s = s_r + is_i \quad \Rightarrow \quad e^{st+ikx} = e^{s_r t}e^{ik(x-ct)}$

s_r : growth rate , $c = s_i/k$: phase speed

$$\Delta\phi_1' = 0 \quad \Rightarrow \quad \widehat{\phi}_1(z)\cdot(ik)^2 e^{st+ikx} + \frac{d^2\widehat{\phi}_1}{dz^2}\,e^{st+ikx} = 0$$

$$\frac{d^2\widehat{\phi}_1}{dz^2} - k^2\widehat{\phi}_1 = 0 \quad , \quad \widehat{\phi}_1 \to 0 \text{ as } z \to -\infty \quad \Rightarrow \quad \widehat{\phi}_1(z) = A_1 e^{kz}$$

similarly $\widehat{\phi}_2(z) = A_2 e^{-kz}$

$\zeta'(x,t) = Be^{st+ikx}$

kinematic bc $\Rightarrow$ $kA_1 = sB + U_1 ikB = (s + ikU_1)B$

minus sign missing -> $kA_2 = sB + U_2 ikB = (s + ikU_2)B$

dynamic bc $\Rightarrow$ $\rho_1(U_1 ikA_1 + sA_1 + gB) = \rho_1(U_2 ikA_2 + sA_2 + gB)$

$$\rho_1\big((ikU_1+s)A_1+gB\big) \;=\; \rho_2\big((ikU_2+s)A_2+gB\big)$$

$$\rho_1\big((ikU_1+s)^2+kg\big) \;=\; \rho_2\big(-(ikU_2+s)^2+kg\big) \;\;:\;\; \underline{\text{dispersion relation}}$$

$$s \;=\; -ik\,\frac{\rho_1U_1+\rho_2U_2}{\rho_1+\rho_2} \;\pm\; \Bigg(\underbrace{k^2\,\frac{\rho_1\rho_2\big(U_1-U_2\big)^2}{(\rho_1+\rho_2)^2}}_{\uparrow\ \text{inertia}} \;-\; \underbrace{kg\,\frac{\rho_1-\rho_2}{\rho_1+\rho_2}}_{\uparrow\ \text{bouyancy}}\Bigg)^{1/2}$$

ex : surface gravity waves

$U_1=U_2=0$, $\rho_2=0$ $\Rightarrow$ $s=\pm i\big(kg\big)^{1/2}$: marginally stable traveling waves

speed $= \pm\left(\dfrac{g}{k}\right)^{1/2}$ $\Rightarrow$ short waves move slowly , long waves move rapidly

ex : internal gravity waves

$U_1=U_2=0$, $\rho_1\neq\rho_2$: stratified fluid $\Rightarrow$ $s=\pm i\left(kg\,\dfrac{\rho_1-\rho_2}{\rho_1+\rho_2}\right)^{1/2}$

$\rho_1>\rho_2$ (light fluid over heavy fluid) : marginally stable traveling waves

$\rho_1<\rho_2$: Rayleigh-Taylor instability , $s=s_r\sim\sqrt{k}$

ex : Kelvin-Helmholtz instability

$U_1\neq U_2$: shear flow

case 1 : $\rho_1=\rho_2$ $\Rightarrow$ $s=-ik\,\dfrac{U_1+U_2}{2}\;\pm\;k\,\dfrac{|U_1-U_2|}{2}$

phase speed $=\dfrac{U_1+U_2}{2}$, $s_r\sim k$: more unstable than R-T

case 2 : $\rho_1\neq\rho_2$ $\Rightarrow$ condition for instability : $k\rho_1\rho_2\big(U_1-U_2\big)^2>g\big(\rho_1^2-\rho_2^2\big)$

$\rho_1>\rho_2$: stable stratification , short/long waves are unstable/stable

$\rho_1<\rho_2$: unstable stratification , all wavelengths are unstable

physical mechanism : pressure , vorticity other issues

viscous effects

finite thickness

surface tension

solid walls

nonlinear effects

temporal vs. spatial stability

three-dimensionality

4. capillary instability of a jet

5. miscellaneous

5.1 temporal/spatial instability

$u' = e^{st+ikx}$

k : real $\Rightarrow$ periodic in space

s : real $\Rightarrow$ growth or decay in time

$k = i\kappa$, κ : real $\Rightarrow$ growth or decay in space

$s = i\omega$, ω : real $\Rightarrow$ periodic in time

note

The solution of the linearized initial value problem is obtained by superposition of normal modes corresponding to different wavenumbers.

bounded domain $\Rightarrow$ $u'(x,z,t) = \sum_k c_k \widehat{u}_k(z) e^{s_k t + ikx}$

unbounded domain $\Rightarrow$ $u'(x,z,t) = \int c(k)\, \widehat{u}_k(z) e^{s(k)t+ikx} dk$

<u>5.2 weakly nonlinear analysis</u>

<u>ex</u> : nonlinear eigenvalue problem

$$y'' + \left(\lambda - \frac{1}{2}\int_0^1 \big(y'(x)\big)^2 dx\right) y = 0 \quad , \quad y(0) = y(1) = 0$$

$y(x)$: steady-state shape of a beam

λ : forcing amplitude

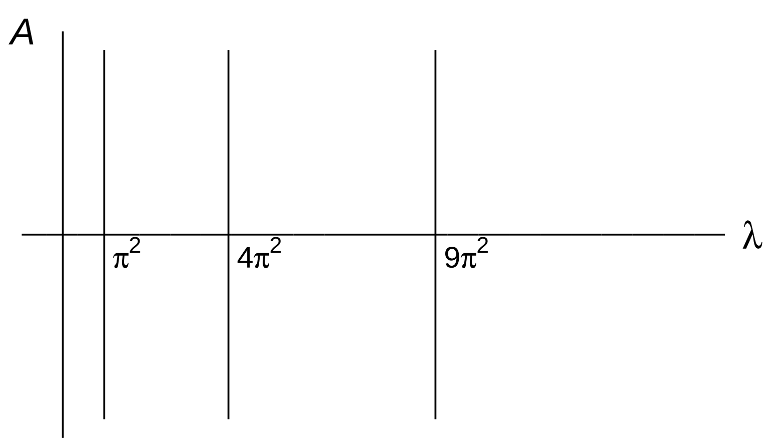
note : $y(x) = 0$ is a solution for all $\lambda > 0$

<u>linearized problem</u>

$y'' + \lambda y = 0 \quad , \quad y(0) = y(1) = 0$

$y(x) = A \sin n\pi x \quad , \quad \lambda = n^2\pi^2 \quad , \quad n = 1\,, 2\,, 3\,, \ldots$

<u>bifurcation diagram</u>



<u>note</u>

1. The amplitude is undetermined in the linear analysis.

2. There are no nonzero solutions for $\lambda \neq n^2\pi^2$.

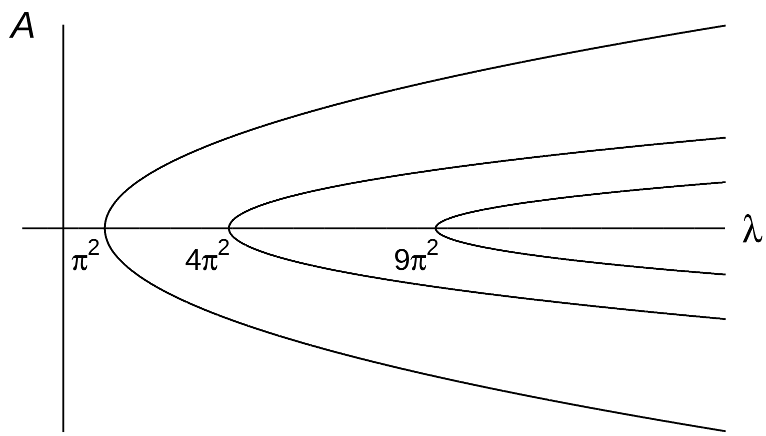
<u>nonlinear problem</u>

$$\mu = \lambda - \frac{1}{2}\int_0^1 \big(y'(x)\big)^2 dx$$

$$y'' + \mu y = 0 \quad , \quad y(0) = y(1) = 0 \quad \Rightarrow \quad y(x) = A\sin n\pi x \quad , \quad \mu = n^2\pi^2$$

$$\mu = \lambda - \frac{1}{2}\int_0^1 A^2 n^2\pi^2 \cos^2 n\pi x\, dx = \lambda - \frac{A^2n^2\pi^2}{4} = n^2\pi^2$$

$$\Rightarrow \quad \lambda = n^2\pi^2\left(1 + \frac{A^2}{4}\right)$$



<u>note</u>

1. The amplitude is determined by nonlinear effects.

2. There are a finite number of nonzero solutions for any $\lambda > \pi^2$. At the values $\lambda = n^2\pi^2$, the number of nonzero solutions increases from $2n-2$ to $2n$; these are supercritical pitchfork bifurcations.

<u>perturbation theory</u>

$$y = \epsilon y_1 + \epsilon^2 y_2 + \cdots \quad , \quad \lambda = \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \cdots$$

$$y'' + \left(\lambda - \frac{1}{2}\int_0^1 \big(y'(x)\big)^2 dx\right) y = \epsilon y_1'' + \epsilon^2 y_2'' + \cdots$$
$$+ \left(\lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \cdots - \frac{1}{2}\int_0^1 \epsilon^2 \big(y_1'\big)^2 dx\right)\left(\epsilon y_1 + \epsilon^2 y_2 + \cdots\right)$$

$\epsilon \;:\; y_1'' + \lambda_0 y_1 = 0 \;\;,\;\; y_1(0) = y_1(1) = 0 \quad \Rightarrow \quad y_1(x) = \sin n\pi x \;\;,\;\; \lambda_0 = n^2\pi^2$

$\epsilon^2 \;:\; y_2'' + \lambda_0 y_2 + \lambda_1 y_1 = 0$

$y_2'' + n^2\pi^2 y_2 = -\lambda_1 \sin n\pi x \quad , \quad y_2(0) = y_2(1) = 0$

<u>claim</u> : The solution y_2 exists if and only if $\lambda_1 = 0$. <u>pf</u> : hw

$$\epsilon^3 \;:\; y_3'' + \lambda_0 y_3 + \lambda_1 y_2 + \left(\lambda_2 - \frac{1}{2}\int_0^1 (y_1')^2 dx\right) y_1 \;=\; 0$$

$$y_3'' + n^2\pi^2 y_3 = -\left(\lambda_2 - \frac{n^2\pi^2}{4}\right)\sin n\pi x \;\;,\;\; y_3(0) = y_3(1) = 0 \;\;\Rightarrow\;\; \lambda_2 = \frac{n^2\pi^2}{4}$$

$$\Rightarrow \quad y(x) \;=\; \epsilon \sin n\pi x + \cdots \quad , \quad \lambda \;=\; n^2\pi^2\left(1 + \frac{\epsilon^2}{4} + \cdots\right)$$

<u>note</u>

1. Perturbation theory yields the exact solution, in this example.

2. The claim is a special case of a general result.

<u>claim</u>

Let L be a linear operator on a Hilbert space (st range$L$ is closed). Then the equation $Lu = v$ has a solution $\Leftrightarrow$ $<v\,,w> = 0$ for all w st $L^*w = 0$, i.e. v is orthogonal to all solutions of the homogeneous adjoint equation (<u>solvability condition</u>). (In a finite dimensional space this says that $Ax = b$ has a solution $\Leftrightarrow$ $b^Tw = 0$ for all w st $A^Tw = 0$.)

<u>ex</u>

$L^2(0\,,1)$ is a Hilbert space with inner product $<u\,,v> = \displaystyle\int_0^1 u(x)\,v(x)\,dx$

$$L = \frac{d^2}{dx^2} + \lambda_0 \quad , \quad D(L) = \left\{u \in L^2(0\,,1) \;:\; u' \in L^2(0\,,1)\;,\; u(0) = u(1) = 0\right\}$$

$L^* = L \quad \big(L$ is self-adjoint$\big)$

$y_2'' + \lambda_0 y_2 = -\lambda_1 y_1 \quad , \quad y_2(0) = y_2(1) \quad \Rightarrow \quad Ly_2 = -\lambda_1 y_1 \;\;,\;\; Ly_1 = 0$

solvability condition : $<-\lambda_1 y_1\,, y_1> = 0 \quad \Rightarrow \quad \lambda_1 = 0$

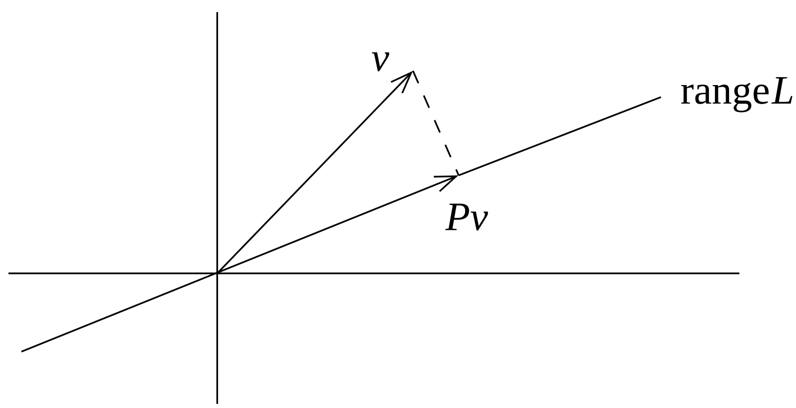
pf

$\Rightarrow$) suppose $Lu = v$ has a solution and consider w st $L^* w = 0$

then $<v\,,w> = <Lu\,,w> = <u\,,L^*w> = <v\,,0> = 0$ ok

$\Leftarrow$) suppose $<v\,,w> = 0$ for all w st $L^*w = 0$

Let P be the orthogonal projector onto range$L$ (which exists because range$L$ is assumed to be closed).



$Pv \in \text{range}L \quad , \quad v - Pv \perp \text{range}L$

$z = v - Pv \;\;,\;\; ||L^*z||^2 = <L^*z\,,L^*z> = <z\,,LL^*z> = 0 \quad \big(z \perp \text{range}L\big)$

$\Rightarrow \quad L^*z = 0 \quad \Rightarrow \quad <v\,,z> = 0 \quad \big(\text{by assumption}\big)$

$\Rightarrow \quad 0 = <v\,,z> = \;<z + Pv\,,z> = \;<z\,,z> + <Pv\,,z> = ||z||^2$

$\big(Pv \in \text{range}L \;,\; z \perp \text{range}L\big)$

$\Rightarrow \quad z = 0 \quad \Rightarrow \quad v = Pv \in \text{range}L$ ok

<u>general nonlinear eigenvalue problem</u>

$Lu = \lambda f(u) \quad , \quad u(0) = u(1) = 0$

L : 2nd order , self-adjoint , linear differential operator

$L\phi_j = \mu_j \phi_j \quad , \quad j = 1\,, 2\,, \ldots$

μ_j : eigenvalue , simple , $0 < \mu_1 < \mu_2 < \cdots$

ϕ_j : eigenfunction

<u>ex</u> : $Lu = -u''$, $u(0) = u(1) = 0$, $\mu_j = j^2\pi^2$, $\phi_j(x) = \sin j\pi x$

λ : physical parameter

$f(u) = a_1 u + a_2 u^2 + a_3 u^3 + \cdots \quad , \quad a_1 > 0$

<u>note</u> : $u(x) = 0$ is a solution for all λ

<u>perturbation theory</u>

$u \;=\; \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots$

$Lu \;=\; \epsilon L u_1 + \epsilon^2 L u_2 + \epsilon^3 L u_3 + \cdots$

$\lambda \;=\; \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \epsilon^3\lambda_3 + \cdots$

$f(u) \;=\; a_1\big(\epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots\big) \;+\; a_2\big(\epsilon^2 u_1^2 + \epsilon^3 2u_1u_2 + \cdots\big) \;+\; a_3\epsilon^3 u_1^3 \;+\; \cdots$

$$\begin{aligned} \lambda f(u) \;=\;& \epsilon\lambda_0 a_1 u_1 \\ &+ \epsilon^2\big(\lambda_0\big(a_1u_2 + a_2u_1^2\big) + \lambda_1 a_1 u_1\big) \\ &+ \epsilon^3\big(\lambda_0\big(a_1u_3 + 2a_2u_1u_2 + a_3u_1^3\big) + \lambda_1\big(a_1u_2 + a_2u_1^2\big) + \lambda_2 a_1 u_1\big) \;+\; \cdots \end{aligned}$$

$Lu \;=\; \lambda f(u)$

ϵ : $Lu_1 \;=\; \lambda_0 a_1 u_1 \quad \Rightarrow \quad \lambda_0 a_1 \;=\; \mu_j$ for some $j \quad , \quad u_1 \;=\; \phi_j$

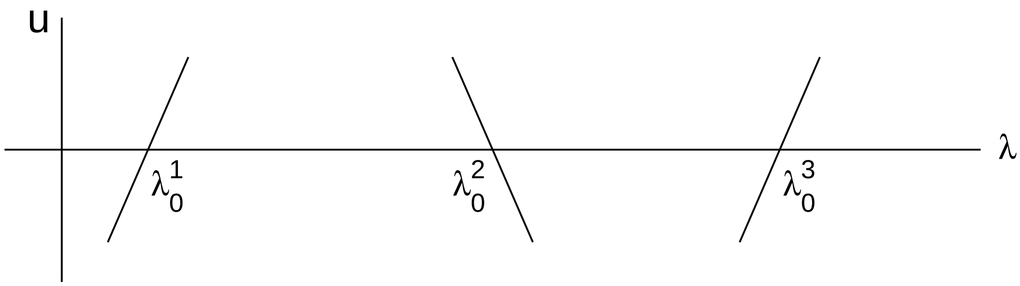
ϵ^2 : $Lu_2 \;=\; \lambda_0\big(a_1u_2 + a_2u_1^2\big) + \lambda_1 a_1 u_1$

solvability : $<\lambda_0 a_2 u_1^2 + \lambda_1 a_1 u_1\,,\phi_j> = 0$, uniqueness : $<u_2\,,\phi_j> = 0$

$$\Rightarrow \quad \lambda_1 \;=\; -\,\frac{\lambda_0 a_2 <\phi_j^2\,,\phi_j>}{a_1 <\phi_j\,,\phi_j>}$$

$$\Rightarrow \quad u \;=\; \epsilon\phi_j \;+\; \cdots \quad , \quad \lambda \;=\; \frac{\mu_j}{a_1}\left(1 - \epsilon\,\frac{a_2 <\phi_j^2\,,\phi_j>}{a_1 <\phi_j\,,\phi_j>} \;+\; \cdots\right)$$

<u>case 1</u> : $a_2 <\phi_j^2\,,\phi_j> \neq 0 \quad \Rightarrow$ transcritical bifurcation at $\lambda \;=\; \dfrac{\mu_j}{a_1} \;=\; \lambda_0^j$



<u>case 2</u> : $a_2 <\phi_j^2\,,\phi_j> = 0 \quad \Rightarrow \quad \lambda_1 = 0$

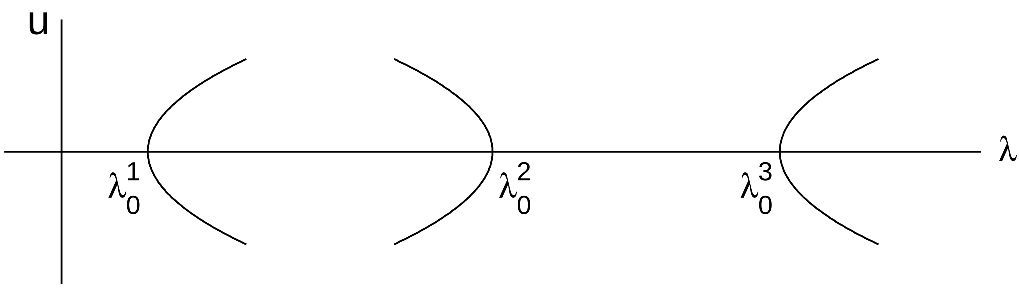
assume $a_2 = 0$, we take $u_2 = 0$

$$\epsilon^3 \;:\; Lu_3 \;=\; \lambda_0\big(a_1u_3 + 2a_2u_1u_2 + a_3u_1^3\big) + \lambda_1\big(a_1u_2 + a_2u_1^2\big) + \lambda_2 a_1 u_1$$

$$<\lambda_0 a_3 u_1^3 + \lambda_2 a_1 u_1\,,\phi_j> = 0 \quad \Rightarrow \quad \lambda_2 \;=\; -\,\frac{\lambda_0 a_3 <\phi_j^3\,,\phi_j>}{a_1 <\phi_j\,,\phi_j>}$$

$$\Rightarrow \quad u \;=\; \epsilon\phi_j \;+\; \cdots \quad , \quad \lambda \;=\; \frac{\mu_j}{a_1}\left(1 - \epsilon^2\,\frac{a_3 <\phi_j^3\,,\phi_j>}{a_1 <\phi_j\,,\phi_j>} \;+\; \cdots\right)$$

$a_3 <\phi_j^3\,,\phi_j> \neq 0 \quad \Rightarrow$ pitchfork bifurcation at $\lambda = \lambda_0^j$



time-dependent problem

$$\frac{\partial u}{\partial t} + Lu \;=\; \lambda f(u) \quad , \quad u(x,0) = g(x) \quad , \quad u(0,t) = u(1,t) = 0$$

linear stability (of the zero solution)

$$\frac{\partial u}{\partial t} + Lu \;=\; \lambda a_1 u \quad , \quad f(u) \;=\; a_1 u + a_2 u^2 + \cdots \quad , \quad \text{assume } a_1 > 0$$

$$u(x,t) \;=\; \sum_{j=1}^{\infty} c_j(t)\phi_j(x) \quad , \quad L\phi_j = \mu_j\phi_j \quad , \quad j = 1\,,2\,,\ldots$$

$$\sum_{j=1}^{\infty} c_j'(t)\phi_j(x) + \sum_{j=1}^{\infty} c_j(t)L\phi_j(x) \;=\; \lambda a_1 \sum_{j=1}^{\infty} c_j(t)\phi_j(x)$$

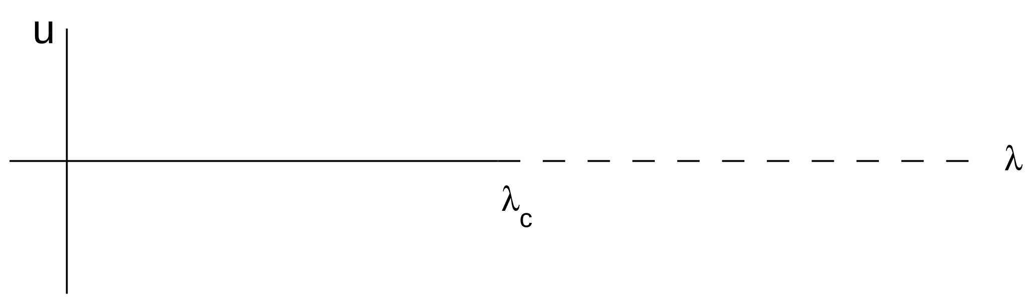
$$\sum_{j=1}^{\infty} \big(c_j'(t) + \big(\mu_j - \lambda a_1\big)c_j(t)\big)\phi_j(x) \;=\; 0$$

$$c_j' + \big(\mu_j - \lambda a_1\big)c_j \;=\; 0 \quad \Rightarrow \quad c_j(t) \;=\; c_j(0)e^{-(\mu_j - \lambda a_1)t}$$

$$t = 0 \quad \Rightarrow \quad u(x,0) \;=\; g(x) \;=\; \sum_{j=1}^{\infty} c_j(0)\phi_j(x) \quad \Rightarrow \quad c_j(0) \;=\; \frac{<g\,,\phi_j>}{<\phi_j\,,\phi_j>}$$

$$u(x,t) \;=\; \sum_{j=1}^{\infty} c_j(0)e^{-(\mu_j - \lambda a_1)t}\phi_j(x)$$

linear stability $\Leftrightarrow$ $\mu_j - \lambda a_1 > 0$ for all j $\Leftrightarrow$ $\lambda < \lambda_c = \dfrac{\mu_1}{a_1}$



<u>supercritical case</u> : $\lambda - \lambda_c = \epsilon^2$

$-(\mu_1 - \lambda a_1)t \,=\, a_1(\lambda - \lambda_c)t \,=\, a_1\epsilon^2 t$, $\epsilon^2 t \,=\, \tau$: slow time , t : fast time

$$u(x,t) \,=\, c_1(0)e^{a_1\tau}\phi_1(x) \,+\, \sum_{j=2}^{\infty} c_j(0)e^{-(\mu_j - \lambda a_1)t}\phi_j(x) \,+\, \cdots$$

= slowly growing + rapidly decaying : eventually becomes invalid

<u>nonlinear stability</u> (Keller , Matkowsky)

$$\frac{\partial u}{\partial t} + Lu \,=\, \lambda f(u) \quad , \quad u(x,0) = \epsilon g(x) \quad , \quad u(0,t) = u(1,t) = 0$$

assume $f(u) = a_1 u + a_3 u^3$: steady-state pitchfork bifurcation at $\lambda = \dfrac{\mu_j}{a_1}$

<u>method of multiple scales</u>

$$u(x,t) \,=\, \epsilon u_1(x,t,\tau) \,+\, \epsilon^3 u_3(x,t,\tau) \,+\, \cdots$$

$$\frac{\partial u}{\partial t} \,=\, \epsilon\left(\frac{\partial u_1}{\partial t} \,+\, \frac{\partial u_1}{\partial \tau}\,\epsilon^2\right) \,+\, \epsilon^3\frac{\partial u_3}{\partial t} \,+\, \cdots \,=\, \epsilon\frac{\partial u_1}{\partial t} \,+\, \epsilon^3\left(\frac{\partial u_1}{\partial \tau} \,+\, \frac{\partial u_3}{\partial t}\right) \,+\, \cdots$$

$$f(u) \,=\, a_1\big(\epsilon u_1 \,+\, \epsilon^3 u_3 \,+\, \cdots\big) \,+\, a_3\big(\epsilon u_1 \,+\, \cdots\big)^3$$

$$= \epsilon a_1 u_1 \,+\, \epsilon^3\big(a_1 u_3 \,+\, a_3 u_1^3\big) \,+\, \cdots$$

$$\lambda f(u) \,=\, \big(\lambda_c \,+\, \epsilon^2\big)\big(\,\epsilon a_1 u_1 \,+\, \epsilon^3\big(a_1 u_3 \,+\, a_3 u_1^3 \,+\, \cdots\big)\,\big)$$

$$= \epsilon\lambda_c a_1 u_1 \,+\, \epsilon^3\big(\lambda_c\big(a_1 u_3 \,+\, a_3 u_1^3\big) \,+\, a_1 u_1\big) \,+\, \cdots$$

ϵ : $\dfrac{\partial u_1}{\partial t} \,+\, Lu_1 \,=\, \lambda_c a_1 u_1$: equation of linear stability

$$\Rightarrow \quad u_1(x,t,\tau) \,=\, \sum_{j=1}^{\infty} c_j(\tau)e^{-(\mu_j - \lambda_c a_1)t}\phi_j(x) \quad , \quad c_j(0) \,=\, \frac{<g\,,\phi_j>}{<\phi_j\,,\phi_j>}$$

$$= c_1(\tau)\phi_1(x) \,+\, \sum_{j=2}^{\infty} c_j(\tau)e^{-(\mu_j - \lambda_c a_1)t}\phi_j(x)$$

$$\epsilon^3 \;:\; \frac{\partial u_1}{\partial \tau} + \frac{\partial u_3}{\partial t} + Lu_3 \;=\; \lambda_c\big(a_1 u_3 + a_3 u_1^3\big) + a_1 u_1$$

$$\Rightarrow \quad \frac{\partial u_3}{\partial t} + Lu_3 - \lambda_c a_1 u_3 \;=\; -\frac{\partial u_1}{\partial \tau} + a_1 u_1 + \lambda_c a_3 u_1^3$$

$$\sim\; -c_1'\phi_1 + a_1 c_1 \phi_1 + \lambda_c a_3 c_1^3 \phi_1^3 \;=\; \text{rhs}$$

$$u_3(x,t,\tau) \;=\; \sum_{j=1}^{\infty} \alpha_j(t,\tau)\phi_j(x) \quad \Rightarrow \quad \frac{\partial \alpha_1}{\partial t} + \mu_1 \alpha_1 - \lambda_c a_1 \alpha_1 \;=\; \frac{<\text{rhs}\,,\phi_1>}{<\phi_1\,,\phi_1>}$$

$$\Rightarrow \quad \alpha_1(t,\tau) \;=\; \frac{<\text{rhs}\,,\phi_1>}{<\phi_1\,,\phi_1>} t + \alpha_1(0,\tau) \;:\; \underline{\text{secular growth}}$$

$$<\text{rhs}\,,\phi_1> \;=\; 0 \quad \Rightarrow \quad -c_1' + a_1 c_1 + \lambda_c a_3 c_1^3 \frac{<\phi_1^3\,,\phi_1>}{<\phi_1\,,\phi_1>} \;=\; 0$$

$$\Rightarrow \quad c_1' \;=\; a_1 c_1 - b c_1^3 \quad , \quad b \;=\; -\lambda_c a_3 \frac{<\phi_1^3\,,\phi_1>}{<\phi_1\,,\phi_1>} \;:\; \text{Landau equation}$$

supercritical case : $b > 0 \;\Leftrightarrow\; a_3 < 0$, $\lim_{\tau\to\infty} c_1(\tau) = \pm c_\infty$, $c_\infty^2 = \dfrac{a_1}{b}$

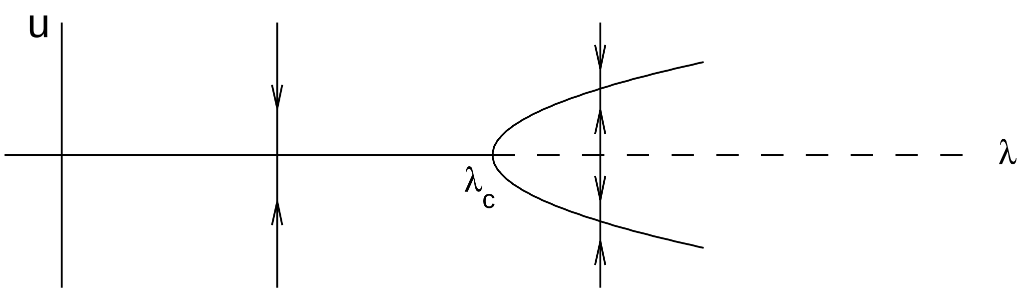
<u>summary</u>

$$\frac{\partial u}{\partial t} + Lu \;=\; \lambda\big(a_1 u + a_3 u^3\big) \quad , \quad u(x,0) = \epsilon g(x) \quad , \quad u(0,t) = u(1,t) = 0$$

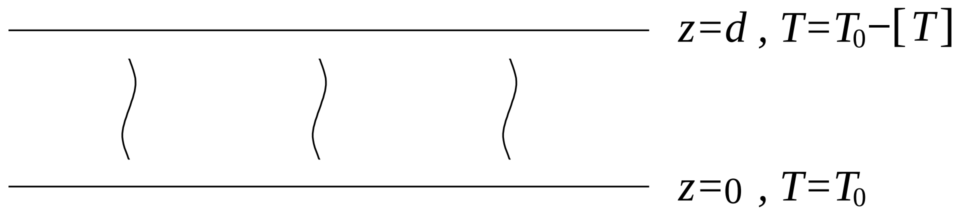
assume $a_1 > 0$, $a_3 < 0$, $L \ldots$

$$\lambda \;=\; \lambda_c + \epsilon^2 \quad , \quad u(x,t) \;=\; \epsilon \frac{c_\infty\, c_1(0)\, e^{a_1 \epsilon^2 t}}{\big(c_\infty^2 - c_1(0)^2 + c_1(0)^2 e^{2a_1\epsilon^2 t}\big)^{1/2}} \phi_1(x) + \cdots$$

$$c_1(0) \;=\; \frac{<g\,,\phi_1>}{<\phi_1\,,\phi_1>} \quad , \quad c_\infty^2 \;=\; -\frac{a_1 <\phi_1\,,\phi_1>}{\lambda_c\, a_3 <\phi_1^3\,,\phi_1>} \quad , \quad \lambda_c \;=\; \frac{\mu_1}{a_1}$$



6. Rayleigh-Bénard convection (lecture notes of John Neu)



temperature : T

density : $\rho = \rho_0(1 - \alpha(T - T_0))$, $0 < \alpha << 1$

Boussinesq approximation : valid for small density variations

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p - \underset{\substack{\uparrow \\ \text{bouyancy force}}}{\rho g\, \vec{e}_z} + \rho\nu\Delta\vec{u} \quad \to \quad \rho_0 \frac{D\vec{u}}{Dt} = -\nabla p - \rho g\, \vec{e}_z + \rho_0\nu\Delta\vec{u}$$

Boussinesq equations I

$$\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} = -\frac{\nabla p}{\rho_0} - \frac{\rho}{\rho_0}\, g\, \vec{e}_z + \nu\Delta\vec{u}$$

$$\nabla\cdot\vec{u} = 0$$

$T_t + (\vec{u}\cdot\nabla)T = \kappa\Delta T$: convection , conduction

$$\rho = \rho_0(1 - \alpha(T - T_0))$$

parameters

ρ_0 : constant background density

T_0 : constant temperature on $z = 0$

$[T]$: temperature difference across layer , $[T] > 0$

α : coefficient of volume expansion

ν : viscosity , κ : thermal diffusivity

d : layer thickness , g : acceleration due to gravity

<u>hydrostatic equilibrium</u>

$\vec{u} = 0 \quad , \quad T_t = 0 \quad , \quad T_x = 0$

$$-\frac{\nabla p}{\rho_0} - \frac{\rho}{\rho_0} g\,\vec{e}_z = 0 \quad \Rightarrow \quad \begin{cases} p_x = 0 \\ p_z = -\rho g = -\rho_0 g(1 - \alpha(T - T_0)) \end{cases}$$

$$T_{zz} = 0 \quad \Rightarrow \quad T = T_h = T_0 - \frac{[T]}{d}\,z$$

$$\Rightarrow \quad p_z = -\rho_0 g\left(1 + \frac{\alpha\,[T]}{d}\,z\right)$$

$$\Rightarrow \quad p = p_h = p_0 - \rho_0 g\left(z + \frac{\alpha\,[T]}{2d}\,z^2\right)$$

<u>note</u>

If $[T]$ is sufficiently large, the hydrostatic equilibrium becomes unstable.

define perturbations : $\theta = T - T_h \;,\; \pi = p - p_h$

$$\frac{\rho}{\rho_0} = 1 - \alpha(T - T_0) = 1 - \alpha(T - T_h + T_h - T_0) = 1 - \alpha\theta + \frac{\alpha\,[T]}{d}\,z$$

$$\nabla p = \nabla(\pi + p_h) = \nabla\pi - \rho_0 g\left(1 + \frac{\alpha\,[T]}{d}\,z\right)\vec{e}_z$$

$$-\frac{\nabla p}{\rho_0} - \frac{\rho}{\rho_0} g\,\vec{e}_z = -\frac{\nabla\pi}{\rho_0} + g\left(1 + \frac{\alpha\,[T]}{d}\,z\right)\vec{e}_z - \left(1 - \alpha\theta + \frac{\alpha\,[T]}{d}\,z\right)g\vec{e}_z$$

$$= -\frac{\nabla\pi}{\rho_0} + \alpha g\,\theta\,\vec{e}_z$$

$$(\vec{u}\cdot\nabla)T = (\vec{u}\cdot\nabla)\theta + (\vec{u}\cdot\nabla)T_h = (\vec{u}\cdot\nabla)\theta - \frac{[T]}{d}\,w$$

Boussineq eqs II

$$\nabla \cdot \vec{u} \;=\; 0$$

$$\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} \;=\; -\,\frac{\nabla \pi}{\rho_0} \;+\; \alpha g\, \theta\, \vec{e}_z \;+\; \nu \Delta \vec{u}$$

$$\theta_t + (\vec{u} \cdot \nabla)\theta \;=\; \frac{[T]}{d}\, w \;+\; \kappa \Delta \theta$$

note

$\nabla \cdot \vec{u} \;=\; u_x + w_z \;=\; 0 \;\;\Rightarrow\;\;$ there exists ψ : stream function st $u = \psi_z \,,\; w = -\psi_x$

eliminate pressure by taking curl of momentum equation

$$u_t + u u_x + w u_z \;=\; -\,\frac{\pi_x}{\rho_0} + \nu \Delta u$$

$$w_t + u w_x + w w_z \;=\; -\,\frac{\pi_z}{\rho_0} + \alpha g \theta + \nu \Delta w$$

$$(u_z - w_x)_t + u u_{xz} + u_z u_x + w u_{zz} + w_z u_z - u w_{xx} - u_x w_x - w w_{xz} - w_x w_z$$

$$= \; -\frac{\pi_{xz}}{\rho_0} + \nu \Delta u_z + \frac{\pi_{xz}}{\rho_0} - \alpha g \theta_x - \nu \Delta w_x$$

$$u_z - w_x \;=\; \Delta \psi \quad \Rightarrow \quad \Delta \psi_t \;+\; \psi_z \Delta \psi_x \;-\; \psi_x \Delta \psi_z \;=\; -\alpha g \theta_x \;+\; \nu \Delta^2 \psi$$

Boussinesq eqs III

$$\big(\partial_t - \nu \Delta\big) \Delta \psi \;+\; \alpha g \theta_x \;=\; -\psi_z \Delta \psi_x \;+\; \psi_x \Delta \psi_z$$

$$\big(\partial_t - \kappa \Delta\big) \theta \;+\; \frac{[T]}{d}\, \psi_x \;=\; -\psi_z \theta_x \;+\; \psi_x \theta_z$$

boundary conditions

$\psi = 0$ on $z = 0\,, d$ $\;\Rightarrow\;$ $w = 0$: zero normal velocity

$\psi_z = 0$ … " … $\;\Rightarrow\;$ $u = 0$: no-slip

$\psi_{zz} = 0$ … " … $\;\Rightarrow\;$ $u_z = 0$: zero stress , free surface

$\theta = 0$ … " …

<u>nondimensionalization</u>

$$x = \tilde{x}d \quad , \quad z = \tilde{z}d \quad , \quad t = \tilde{t}d^2/\kappa \quad , \quad \psi = \tilde{\psi}\kappa \quad , \quad \theta = \tilde{\theta}[T]$$

$$\left(\partial_{\tilde{t}}\,\frac{\kappa}{d^2} - \nu\frac{\tilde{\Delta}}{d^2}\right)\frac{\tilde{\Delta}}{d^2}\,\tilde{\psi}\kappa + \alpha g\tilde{\theta}_{\tilde{x}}\frac{[T]}{d} = -\tilde{\psi}_{\tilde{z}}\,\frac{\kappa}{d}\,\frac{\tilde{\Delta}}{d^2}\,\tilde{\psi}_{\tilde{x}}\,\frac{\kappa}{d} + \cdots$$

$$\left(\partial_{\tilde{t}}\,\frac{\kappa}{d^2} - \kappa\frac{\tilde{\Delta}}{d^2}\right)\tilde{\theta}[T] + \frac{[T]}{d}\,\tilde{\psi}_{\tilde{x}}\,\frac{\kappa}{d} = -\tilde{\psi}_{\tilde{z}}\,\frac{\kappa}{d}\,\tilde{\theta}_{\tilde{x}}\,\frac{[T]}{d} + \cdots$$

<u>Boussinesq eqs IV</u>

$$\big(\partial_t - Pr\Delta\big)\Delta\psi + Pr R\,\theta_x = -\psi_z\Delta\psi_x + \psi_x\Delta\psi_z$$

$$\big(\partial_t - \Delta\big)\theta + \psi_x = -\psi_z\theta_x + \psi_x\theta_z$$

$Pr = \dfrac{\nu}{\kappa}$: <u>Prandtl number</u> , $R = \dfrac{\alpha\, g\, d^3\,[T]}{\nu\,\kappa}$: <u>Rayleigh number</u>

bc : $\psi = \psi_{zz} = \theta = 0$ on $z = 0\,,1$

<u>equilibrium</u>

$\psi = \theta = 0$: no fluid motion , pure thermal conduction

<u>linear stability</u>

$$\big(\partial_t - Pr\Delta\big)\Delta\psi + Pr R\,\theta_x = 0$$

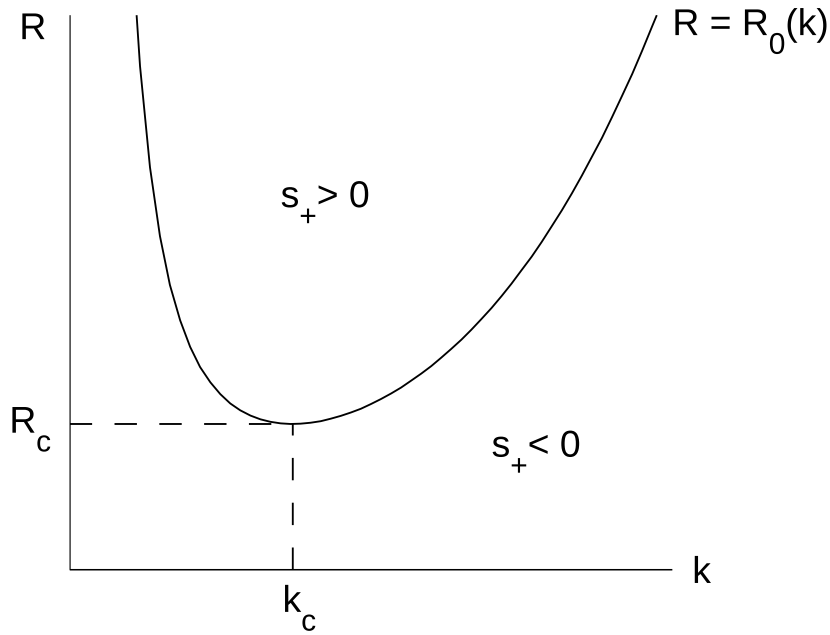
$$\big(\partial_t - \Delta\big)\theta + \psi_x = 0$$

look for $\psi = ae^{st}\sin kx \sin \pi z$, $\theta = be^{st}\cos kx \sin \pi z$

$$\begin{pmatrix} (s + Pr(k^2+\pi^2))(k^2+\pi^2) & Pr\,R\,k \\ k & s + k^2 + \pi^2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad (s + Pr(k^2+\pi^2))(k^2+\pi^2)(s + k^2 + \pi^2) - Pr\,R\,k^2 = 0$$

$$\Rightarrow \quad s = -\frac{(Pr+1)}{2}(k^2+\pi^2) \pm \left(Pr\,R\,\frac{k^2}{k^2+\pi^2} + \frac{(Pr-1)^2}{4}(k^2+\pi^2)^2\right)^{1/2}$$



note

1. the growth rates are real : $s_- < s_+$

2. $s = 0 \quad \Rightarrow \quad R_0(k) = \dfrac{(k^2+\pi^2)^3}{k^2}$: marginal stability curve

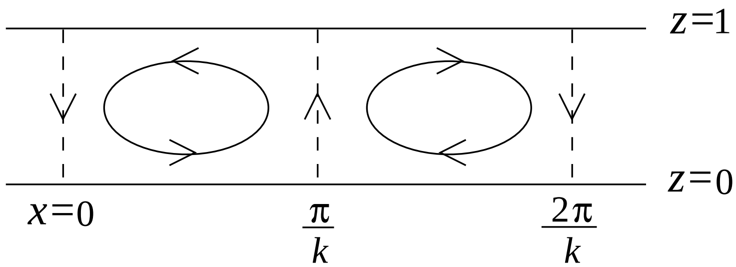
3. $R_c = \min\limits_{k>0} R_0(k) = \dfrac{27\pi^4}{4} = R_0(k_c) \quad , \quad k_c = \dfrac{\pi}{\sqrt{2}}$

4. $s_- < 0$ for all R , k ; $s_+ > 0 \Leftrightarrow R > R_0(k)$

5. for $R > R_c$, there is a band of unstable wavenumbers

6. $s = s(k)$ is an increasing function of $R \Rightarrow$ the hydrostatic equilibrium is stabilized by fluid viscosity and thermal diffusivity, and is destabilized by the adverse temperature gradient

7. $\psi \sim \sin kx \sin \pi z \quad \Rightarrow \quad u \sim \sin kx \cos \pi z \ , \quad w \sim -\cos kx \sin \pi z$



: convection cells , rolls

<u>weakly nonlinear analysis</u>

$R - R_c \,=\, c\,\epsilon^2 \quad , \quad \tau \,=\, \epsilon^2 t \;:\;$ slow time

$$\psi(t) \,=\, \epsilon\,\psi_1(\tau) \,+\, \epsilon^2\psi_2(\tau) \,+\, \cdots$$

$$\theta(t) \,=\, \epsilon\,\theta_1(\tau) \,+\, \epsilon^2\theta_2(\tau) \,+\, \cdots$$

$$\begin{aligned} \big(\partial_\tau\epsilon^2 - Pr\Delta\big)\Delta\psi \,+\, Pr\big(R_c + c\epsilon^2\big)\,\theta_x \,&=\, -\psi_z\Delta\psi_x \,+\, \psi_x\Delta\psi_z \\ \big(\partial_\tau\epsilon^2 - \Delta\big)\theta \qquad + \,\psi_x \,&=\, -\psi_z\theta_x \,+\, \psi_x\theta_z \end{aligned}$$

$$\begin{aligned} Pr\Delta^2\psi \,-\, PrR_c\theta_x \,&=\, \psi_z\Delta\psi_x \,-\, \psi_x\Delta\psi_z \,+\, \epsilon^2\big(\Delta\psi_\tau + Pr\,c\,\theta_x\big) \\ \Delta\theta \,-\, \psi_x \,&=\, \psi_z\theta_x \,-\, \psi_x\theta_z \,+\, \epsilon^2\theta_\tau \end{aligned}$$

$$\begin{aligned} \epsilon \;:\; Pr\Delta^2\psi_1 \,-\, PrR_c\theta_{1x} \,&=\, 0 \\ \Delta\theta_1 \,-\, \psi_{1x} \,&=\, 0 \end{aligned}$$

$$\begin{aligned} \epsilon^2 \;:\; Pr\Delta^2\psi_2 \,-\, PrR_c\theta_{2x} \,&=\, \psi_{1z}\Delta\psi_{1x} \,-\, \psi_{1x}\Delta\psi_{1z} \\ \Delta\theta_2 \,-\, \psi_{2x} \,&=\, \psi_{1z}\theta_{1x} \,-\, \psi_{1x}\theta_{1z} \end{aligned}$$

$$\begin{aligned} \epsilon^3 \;:\; Pr\Delta^2\psi_3 \,-\, PrR_c\theta_{3x} \,&=\, \psi_{2z}\Delta\psi_{1x} \,-\, \psi_{2x}\Delta\psi_{1z} \,+\, \psi_{1z}\Delta\psi_{2x} \,-\, \psi_{1x}\Delta\psi_{2z} \\ &\qquad + \Delta\psi_{1\tau} \,+\, Pr\,c\,\theta_{1x} \\ \Delta\theta_3 \,-\, \psi_{3x} \,&=\, \psi_{2z}\theta_{1x} \,-\, \psi_{2x}\theta_{1z} \,+\, \psi_{1z}\theta_{2x} \,-\, \psi_{1x}\theta_{2z} \,+\, \theta_{1\tau} \end{aligned}$$

<u>note</u>

The left hand side is the steady linear stability operator evaluated at $R = R_c$. To solve these equations we set $k = k_c$.

$$\epsilon \;:\; \psi_1(\tau) \,=\, a(\tau)\sin kx \sin\pi z$$

$$\theta_1(\tau) \,=\, b(\tau)\cos kx \sin\pi z \quad , \quad b(\tau) \,=\, -\,\frac{k\,a(\tau)}{k^2 + \pi^2}$$

$$\psi_{1z}\Delta\psi_{1x} - \psi_{1x}\Delta\psi_{1z} = 0$$

$$\psi_{1z}\theta_{1x} - \psi_{1x}\theta_{1z}$$

$$= a(\tau)\, b(\tau) k\pi \big(\sin kx \cos \pi z \cdot - \sin kx \sin \pi z - \cos kx \sin \pi z \cdot \cos kx \cos \pi z \big)$$

$$= \frac{\pi k^2}{2(k^2+\pi^2)}\, a(\tau)^2 \sin 2\pi z$$

$$\epsilon^2 \;:\; Pr\Delta^2\psi_2 - PrR_c\theta_{2x} = 0$$

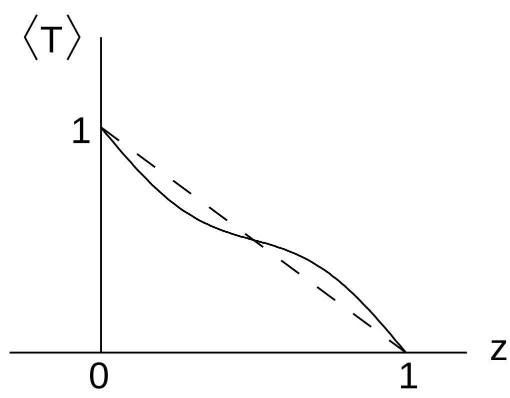
$$\Delta\theta_2 - \psi_{2x} = \frac{\pi k^2}{2(k^2+\pi^2)}\, a(\tau)^2 \sin 2\pi z$$

take $\psi_2 = 0 \quad , \quad \theta_2 = -\dfrac{k^2}{8\pi(k^2+\pi^2)}\, a(\tau)^2 \sin 2\pi z$

recall : $T = T_h + \theta = 1 - z + \epsilon\theta_1 + \epsilon^2\theta_2 + \cdots$ (nondimensional , $T_0 = 1$)

$$\Rightarrow\; T = 1 - z - \frac{\epsilon\, k}{k^2+\pi^2}\, a(\tau) \cos kx \sin \pi z - \frac{\epsilon^2 k^2}{8\pi(k^2+\pi^2)}\, a(\tau)^2 \sin 2\pi z + \cdots$$

$$\langle T \rangle = \frac{k}{2\pi}\int_0^{2\pi/k} T(x,z,t)\, dx = 1 - z - \frac{\epsilon^2 k^2}{8\pi(k^2+\pi^2)}\, a(\tau)^2 \sin 2\pi z$$



1. Convection changes the horizontally averaged temperature profile; it drops more rapidly near the edges of the layer ($z = 0\,,1$) and becomes more uniform near the middle of the layer ($z = 1/2$).

2. The amplitude $a(\tau)$ is still undetermined.

$$\epsilon^3 \; : \; Pr\Delta^2\psi_3 \,-\, PrR_c\theta_{3x} \,=\, \psi_{2z}\Delta\psi_{1x} \,-\, \psi_{2x}\Delta\psi_{1z} \,+\, \psi_{1z}\Delta\psi_{2x} \,-\, \psi_{1x}\Delta\psi_{2z}$$

$$+\,\Delta\psi_{1\tau} \,+\, Pr\,c\,\theta_{1x}$$

$$\Delta\theta_3 \,-\, \psi_{3x} \,=\, \psi_{2z}\theta_{1x} \,-\, \psi_{2x}\theta_{1z} \,+\, \psi_{1z}\theta_{2x} \,-\, \psi_{1x}\theta_{2z} \,+\, \theta_{1\tau}$$

$$Pr\Delta^2\psi_3 \,-\, PrR_c\theta_{3x} \,=\, \Big(-\,(k^2+\pi^2)\,a'(\tau) \,+\, \frac{Pr\,c\,k^2}{k^2+\pi^2}\,a(\tau)\Big)\sin kx \sin \pi z$$

$$\Delta\theta_3 \,-\, \psi_{3x} \,=\, \Big(\frac{k^3}{4(k^2+\pi^2)}\,a^3(\tau)\cos 2\pi z \,-\, \frac{k}{k^2+\pi^2}\,a'(\tau)\Big)\cos kx \sin \pi z$$

<u>note</u> : $Lu \,=\, f \quad , \quad u \,=\, \begin{pmatrix} \psi_3 \\ \theta_3 \end{pmatrix} \quad , \quad L \,=\, \begin{pmatrix} Pr\Delta^2 & -PrR_c\partial_x \\ -\partial_x & \Delta \end{pmatrix}$

$$D(L) \,=\, \big\{ u = (u_1\,,u_2)^T \;:\; u_1 = u_{1zz} = u_2 = 0 \;\text{ on }\; z = 0\,,1 \big\}$$

$$<u\,,v> \,=\, \int_0^1 \int_0^{2\pi/k} \big(u_1v_1 + u_2v_2\big)\,dx\,dz$$

$$L^* \,=\, \begin{pmatrix} Pr\Delta^2 & \partial_x \\ PrR_c\partial_x & \Delta \end{pmatrix} \quad , \quad D(L^*) \,=\, D(L)$$

$$L^*u \,=\, 0 \quad \Leftrightarrow \quad u \,=\, \begin{pmatrix} \sin kx \sin \pi z \\ \dfrac{Pr(k^2+\pi^2)^2}{k}\;\cos kx \sin \pi z \end{pmatrix}$$

check . . .

solvability condition for $\psi_3\,,\theta_3 \;:\; <f, u> = 0$

$$\int_0^1 \int_0^{2\pi/k} \Bigg(\Big(-\,(k^2+\pi^2)\,a'(\tau) \,+\, \frac{Pr\,c\,k^2}{k^2+\pi^2}\,a(\tau)\Big)\sin^2 kx\,\sin^2 \pi z$$

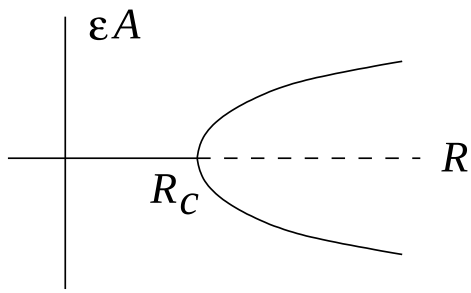
$$+\Big(\frac{k^3\,a^3(\tau)}{4(k^2+\pi^2)}\cos 2\pi z \,-\, \frac{k\,a'(\tau)}{k^2+\pi^2}\Big)\frac{Pr(k^2+\pi^2)^2}{k}\;\cos^2 kx\,\sin^2 \pi z\Bigg)dx\,dz \,=\, 0$$

$$\int_0^1 \cos 2\pi z\,\sin^2 \pi z\,dz \,=\, \int_0^1 \cos 2\pi z \cdot \frac{1}{2}\big(1 - \cos 2\pi z\big)\,dz \,=\, -\,\frac{1}{4}$$

$$\Rightarrow \quad \left(\frac{1+Pr}{Pr}\right) a'(\tau) \;=\; \frac{c\,k^2}{(k^2+\pi^2)^2}\,a(\tau) \;-\; \frac{k^2}{8}\,a^3(\tau) \quad : \quad \text{Landau equation}$$

$\Rightarrow$ there exist stable steady rolls of amplitude A, where $A^2 \;=\; \dfrac{8c}{\epsilon^2(k^2+\pi^2)^2}$

$$R - R_c \;=\; c\,\epsilon^2 \quad \Rightarrow \quad R \;=\; R_c \;+\; \frac{(k^2+\pi^2)^2}{8}\,\epsilon^2 A^2 \quad : \quad \text{defines } A \;=\; A(R,k,\epsilon)$$



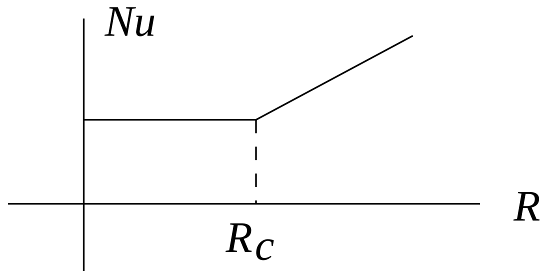
$F \;=\; \kappa\,\dfrac{[T]}{d}\displaystyle\int_0^{\pi/k} T_z(x\,,0)\,dx$: <u>total heat flux through cell bottom</u>

$$T \;=\; T_0 - z \;-\; \frac{\epsilon\,k}{k^2+\pi^2}\,a(\tau)\cos kx \sin \pi z \;-\; \frac{\epsilon^2 k^2}{8\pi(k^2+\pi^2)}\,a(\tau)^2 \sin 2\pi z \;+\; \cdots$$

$$F \;=\; \kappa\,\frac{[T]}{d}\left(\frac{\pi}{k} \;+\; \frac{\epsilon^2 k^2}{8\pi(k^2+\pi^2)}\,A^2\cdot 2\pi\cdot\frac{\pi}{k} \;+\; \cdots\right)$$

$$\;=\; \kappa\,\frac{[T]}{d}\,\frac{\pi}{k_c}\left(1 \;+\; 2\,\frac{R-R_c}{R_c} \;+\; \cdots\right) \;=\; \frac{\kappa\,[T]\,\pi}{d\,k_c}\,Nu$$

: <u>Nusselt number</u>



1. The onset of convection for $R > R_c$ increases the heat flux through the top and bottom of the cell.

2. Experiments and theory show that $Nu \sim R^{\alpha}$ for $R >> 1$ where $0 < \alpha < 1$.

double-diffusive convection (article of John Neu)

$$\rho \;=\; \rho_0(1 \;-\; \alpha_t(T-T_0) \;+\; \alpha_s(S-S_0)) \quad , \quad 0 < \alpha_s << 1$$

$$S \;:\; \text{solute concentration (e.g. salt)} \quad , \quad S = \begin{cases} S_0 - [S] & , \; z = d \\ S_0 & , \; z = 0 \end{cases} \quad , \quad [S] > 0$$

$$\text{decreasing } z \quad \Rightarrow \quad \begin{cases} \text{increasing } T \;\; \Rightarrow \;\; \text{decreasing } \rho \; : \; \text{destabilizing} \\ \text{increasing } S \;\; \Rightarrow \;\; \text{increasing } \rho \; : \; \text{stabilizing} \end{cases}$$

Boussinesq eqs

$$\partial_t \vec{u} + (\vec{u}\cdot\nabla)\vec{u} \;=\; -\,\frac{\nabla p}{\rho_0} - \frac{\rho}{\rho_0}\, g\,\vec{e}_z + \nu\Delta\vec{u} \quad , \quad \nabla\cdot\vec{u} \;=\; 0$$

$$\partial_t T + (\vec{u}\cdot\nabla)T \;=\; \kappa_t \Delta T$$

$$\partial_t S + (\vec{u}\cdot\nabla)S \;=\; \kappa_s \Delta S$$

hydrostatic equilibrium

$$\vec{u} \;=\; 0 \quad , \quad T \;=\; T_0 - \frac{[T]}{d}\, z \quad , \quad S \;=\; S_0 - \frac{[S]}{d}\, z \quad , \quad p \;=\; \ldots$$

perturbations , nondimensional : $\vec{u}\;,\;p\;,\;T\;,\;S \;\rightarrow\; \psi\;,\;\theta\;,\;\sigma$

$$\big(\partial_t - Pr\Delta\big)\Delta\psi \;+\; PrR\,\theta_x \;-\; PrS\,\sigma_x \;=\; -\psi_z\Delta\psi_x \;+\; \psi_x\Delta\psi_z$$

$$\big(\partial_t - \Delta\big)\theta \;+\; \psi_x \;=\; -\psi_z\theta_x \;+\; \psi_x\theta_z$$

$$\big(\partial_t - \tau\Delta\big)\sigma \;+\; \psi_x \;=\; -\psi_z\sigma_x \;+\; \psi_x\sigma_z$$

$$Pr \;=\; \frac{\nu}{\kappa_t} \quad , \quad R \;=\; \frac{\alpha_t\, g\, d^3\, [T]}{\nu\,\kappa_t} \quad , \quad S \;=\; \frac{\alpha_s\, g\, d^3\, [S]}{\nu\,\kappa_t} \quad , \quad \tau \;=\; \frac{\kappa_s}{\kappa_t}$$

bc : $\psi \;=\; \psi_{zz} \;=\; \theta \;=\; \sigma \;=\; 0$ on $z \;=\; 0\,,1$

linear stability (about $\psi = \theta = \sigma = 0$)

$$\psi \;=\; a_\psi \sin kx \sin \pi z \quad , \quad \theta \;=\; a_\theta \cos kx \sin \pi z \quad , \quad \sigma \;=\; a_\sigma \cos kx \sin \pi z$$

marginal stability curve : $R = R_0(k) + \dfrac{S}{\tau}$, $R_0(k) = \dfrac{(k^2+\pi^2)^3}{k^2}$ (hw)

$\Rightarrow$ the solute increases the critical value of the thermal Rayleigh number, i.e. a larger temperature gradient is required to produce convection

qualitative analysis (Chandrasekhar , Stuart)

goal : determine a_ψ , a_θ , a_σ (finite-amplitude steady convection cells)

claim

Let $\overline{\theta} = \theta - \langle\theta\rangle$, $\overline{\sigma} = \sigma - \langle\sigma\rangle$, where $\langle \cdot \rangle$ denotes the horizontal average over a cell, $0 \le x \le \pi/k$. Then steady solutions satisfy the following integral relations.

$$\int_0^1 \langle\psi\Delta^2\psi\rangle\, dz + R\int_0^1 \langle\psi_x\overline{\theta}\,\rangle\, dz - S\int_0^1 \langle\psi_x\overline{\sigma}\rangle\, dz = 0$$

$$\int_0^1 \langle\,\overline{\theta}\Delta\overline{\theta}\,\rangle\, dz - \int_0^1 \langle\psi_x\overline{\theta}\,\rangle\, dz = \int_0^1 \langle\psi_x\overline{\theta}\,\rangle^2 dz - \Big(\int_0^1 \langle\psi_x\overline{\theta}\,\rangle\, dz\Big)^2$$

$$\tau\int_0^1 \langle\overline{\sigma}\Delta\overline{\sigma}\rangle\, dz - \int_0^1 \langle\psi_x\overline{\sigma}\rangle\, dz = \frac{1}{\tau}\Big(\int_0^1 \langle\psi_x\overline{\sigma}\rangle^2 dz - \Big(\int_0^1 \langle\psi_x\overline{\sigma}\rangle\, dz\Big)^2\Big)$$

pf 2

step 1 : $\big(\partial_t - \Delta\big)\theta + \psi_x = -\psi_z\theta_x + \psi_x\theta_z$

steady : $\Delta\theta - \psi_x = \psi_z\theta_x - \psi_x\theta_z = \big(\psi_z\theta\big)_x - \big(\psi_x\theta\big)_z$

take $\langle \cdot \rangle$: $\Delta\langle\theta\rangle - \langle\psi_x\rangle = \langle\big(\psi_z\theta\big)_x\rangle - \langle\big(\psi_x\theta\big)_z\rangle$

$$\langle\theta\rangle_{zz} = -\langle\psi_x\theta\rangle_z \quad\Rightarrow\quad \langle\theta\rangle = -\int_0^z \langle\psi_x\theta\rangle\, dz + c_1 z + c_2$$

$$z = 0 \quad\Rightarrow\quad \langle\theta\rangle\big|_{z=0} = c_2 = 0$$

$$z = 1 \quad\Rightarrow\quad \langle\theta\rangle\big|_{z=1} = -\int_0^1 \langle\psi_x\theta\rangle\, dz + c_1 = 0$$

$$\langle\theta\rangle_z \;=\; -\langle\psi_x\theta\rangle \;+\; \int_0^1 \langle\psi_x\theta\rangle\,dz$$

$$\theta \;=\; \overline{\theta} + \langle\theta\rangle \quad\Rightarrow\quad \langle\theta\rangle_z \;=\; -\langle\psi_x\overline{\theta}\,\rangle \;+\; \int_0^1 \langle\psi_x\overline{\theta}\,\rangle\,dz$$

step 2 : $\Delta\theta \;-\; \psi_x \;=\; \big(\psi_z\theta\big)_x \;-\; \big(\psi_x\theta\big)_z$

$$\Delta\overline{\theta} \;+\; \langle\theta\rangle_{zz} - \;\psi_x \;=\; \big(\psi_z\overline{\theta}\,\big)_x \;-\; \big(\psi_x\overline{\theta}\,\big)_z \;+\; \underbrace{\big(\psi_z\langle\theta\rangle\big)_x \;-\; \big(\psi_x\langle\theta\rangle\big)_z}_{\psi_z\langle\theta\rangle_x \,+\, \psi_{xz}\langle\theta\rangle \,-\, \psi_x\langle\theta\rangle_z \,-\, \psi_{xz}\langle\theta\rangle}$$

multiply by $\overline{\theta}$: $\overline{\theta}\Delta\overline{\theta} \;+\; \overline{\theta}\langle\theta\rangle_{zz} \;-\; \overline{\theta}\psi_x \;=\; \underbrace{\overline{\theta}\big(\psi_z\overline{\theta}\,\big)_x \;-\; \overline{\theta}\big(\psi_x\overline{\theta}\,\big)_z} \;-\; \overline{\theta}\psi_x\langle\theta\rangle_z$

$$\overline{\theta}\psi_z\overline{\theta}_x \;+\; \overline{\theta}\psi_{xz}\overline{\theta} \;-\; \overline{\theta}\psi_x\overline{\theta}_z \;-\; \overline{\theta}\psi_{xz}\overline{\theta} \;=\; \psi_z\Big(\tfrac{1}{2}\overline{\theta}^2\Big)_x \;-\; \psi_x\Big(\tfrac{1}{2}\overline{\theta}^2\Big)_z$$

$$=\; \Big(\psi_z\cdot\tfrac{1}{2}\overline{\theta}^2\Big)_x \;-\; \Big(\psi_x\cdot\tfrac{1}{2}\overline{\theta}^2\Big)_z$$

$$\overline{\theta}\Delta\overline{\theta} \;+\; \overline{\theta}\langle\theta\rangle_{zz} - \;\overline{\theta}\psi_x \;=\; \big(\psi_z\tfrac{1}{2}\overline{\theta}^2\,\big)_x \;-\; \big(\psi_x\tfrac{1}{2}\overline{\theta}^2\,\big)_z \;-\; \overline{\theta}\psi_x\langle\theta\rangle_z$$

take $\langle\,\cdot\,\rangle$: $\langle\,\overline{\theta}\Delta\overline{\theta}\,\rangle \;-\; \langle\,\overline{\theta}\psi_x\rangle \;=\; -\langle\psi_x\tfrac{1}{2}\overline{\theta}^2\rangle_z - \;\langle\,\overline{\theta}\psi_x\rangle\langle\theta\rangle_z$

integrate over z : $\displaystyle\int_0^1 \langle\,\overline{\theta}\Delta\overline{\theta}\,\rangle\,dz \;-\; \int_0^1 \langle\,\overline{\theta}\psi_x\rangle\,dz \;=\; -\int_0^1 \langle\,\overline{\theta}\psi_x\rangle\langle\theta\rangle_z dz$ <u>ok</u>

<u>heuristic</u> : In the slightly supercritical regime, the convection amplitude is small and a nonlinear solution ψ, θ, σ can be approximated by the linear eigenfunctions. Substituting these into the integral relations yields a system of algebraic equations for $a_\psi, a_\theta, a_\sigma$.

$$(k^2+\pi^2)^2\,a_\psi^2 \;+\; Rk\,a_\theta\,a_\psi \;-\; Sk\,a_\sigma\,a_\psi \;=\; 0$$

$$(k^2+\pi^2)\,a_\theta^2 \;+\; k\,a_\theta\,a_\psi \;=\; -\,\frac{1}{8}\,k^2\,a_\theta^2\,a_\psi^2$$

$$\tau(k^2+\pi^2)\,a_\sigma^2 \;+\; k\,a_\sigma\,a_\psi \;=\; -\,\frac{1}{8\tau}\,k^2\,a_\sigma^2\,a_\psi^2$$

<u>pf</u> 2

$$\psi = a_\psi \sin kx \sin \pi z \quad , \quad \theta = a_\theta \cos kx \sin \pi z \quad , \quad \sigma = a_\sigma \cos kx \sin \pi z$$

$$\theta = \overline{\theta} - \langle \theta \rangle \quad , \quad \langle \theta \rangle = \frac{k}{\pi} \int_0^{\pi/k} \theta(x,z,t)\, dx = 0 \quad \Rightarrow \quad \theta = \overline{\theta} \quad , \quad \sigma = \overline{\sigma}$$

$$\int_0^1 \langle\, \overline{\theta} \Delta \overline{\theta}\, \rangle \, dz - \int_0^1 \langle \psi_x \overline{\theta}\, \rangle \, dz = \int_0^1 \langle \psi_x \overline{\theta}\, \rangle^2 \, dz - \left(\int_0^1 \langle \psi_x \overline{\theta}\, \rangle \, dz \right)^2$$

$$\langle\, \overline{\theta} \Delta \overline{\theta}\, \rangle = a_\theta^2 \cdot -(k^2 + \pi^2) \frac{1}{2} \sin^2 \pi z \quad , \quad \langle \psi_x \overline{\theta}\, \rangle = a_\psi a_\theta \, k \frac{1}{2} \sin^2 \pi z$$

$$\int_0^1 \sin^4 \pi z \, dz = \frac{3}{8}$$

$$a_\theta^2 \cdot -(k^2 + \pi^2) \frac{1}{2} \cdot \frac{1}{2} - a_\psi a_\theta \, k \frac{1}{2} \cdot \frac{1}{2} = a_\psi^2 a_\theta^2 \, k^2 \frac{1}{4} \cdot \frac{3}{8} - a_\psi^2 a_\theta^2 \, k^2 \frac{1}{16} \qquad \underline{\text{ok}}$$

look for a nonzero solution , eliminate a_θ , a_σ , set $k^2 a_\psi^2 = 8\tau^2 (k^2 + \pi^2) a^2$

$$(k^2 + \pi^2)\, a_\theta + k\, a_\psi = -\frac{1}{8} \, k^2 \, a_\theta \, a_\psi^2$$

$$\Rightarrow \quad a_\theta = \frac{-k\, a_\psi}{(k^2 + \pi^2) + \frac{1}{8} k^2 a_\psi^2} = \frac{-k\, a_\psi}{(k^2 + \pi^2)(1 + \tau^2 a^2)}$$

similarly , $a_\sigma = \dfrac{-k\, a_\psi}{\tau (k^2 + \pi^2)(1 + a^2)}$

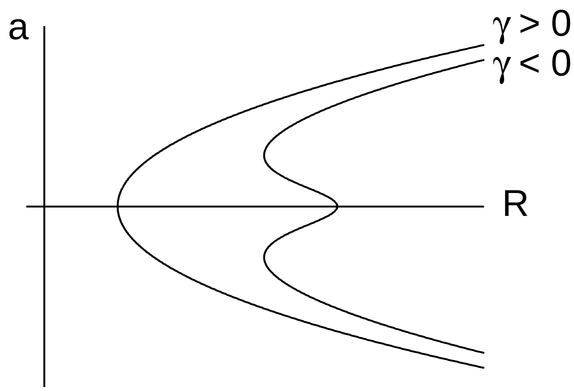
$$(k^2 + \pi^2)^2 \, a_\psi^2 + \frac{Rk\, a_\psi \cdot -k\, a_\psi}{(k^2 + \pi^2)(1 + \tau^2 a^2)} - \frac{Sk\, a_\psi \cdot -k\, a_\psi}{\tau (k^2 + \pi^2)(1 + a^2)} = 0$$

$$\Rightarrow \quad R = R_0(k) \big(1 + \tau^2 a^2 \big) + \frac{S}{\tau} \, \frac{1 + \tau^2 a^2}{1 + a^2} \quad : \ \text{defines } a = a(R\,, S\,, k\,, \tau)$$

$$= R_0(k) + \frac{S}{\tau} + \gamma a^2 + \cdots \quad , \quad \gamma = R_0(k) \tau^2 + S \left(\tau - \frac{1}{\tau} \right) \quad , \quad \tau = \frac{\kappa_s}{\kappa_t}$$

<u>note</u> : $\gamma > 0 \Rightarrow$ supercritical , $\gamma < 0 \Rightarrow$ subcritical

bifurcation diagram of steady states



quantitative analysis

We will use the results of the qualitative analysis to suggest the form of a perturbation expansion for a nonsteady solution ψ, θ, σ under the following assumptions.

weak solute gradient : $S = \epsilon$

slightly subcritical regime : $\gamma < 0 \quad , \quad |\gamma| << 1$

consequences

1) $\gamma \sim \tau^2 + \epsilon\tau - \dfrac{\epsilon}{\tau} \quad \Rightarrow \quad \tau^2 \sim \dfrac{\epsilon}{\tau} \quad \Rightarrow \quad \tau \sim \epsilon^{1/3}$

2) $\tau\Delta\sigma \sim \psi_z\sigma_x - \psi_x\sigma_z \quad \Rightarrow \quad \psi \sim \tau \quad \Rightarrow \quad \psi \sim \epsilon^{1/3}$

3) $a_\theta \sim a_\psi \quad \Rightarrow \quad \theta \sim \psi \quad \Rightarrow \quad \theta \sim \epsilon^{1/3}$

4) $a_\sigma \sim \dfrac{a_\psi}{\tau} \quad \Rightarrow \quad \sigma \sim 1$

5) $R \sim R_c + \dfrac{S}{\tau} + \cdots \quad \Rightarrow \quad R - R_c \sim \epsilon^{2/3}$

6) recall the linear dispersion relation for pure thermal convection

$$(s + Pr(k^2+\pi^2))(k^2+\pi^2)(s+k^2+\pi^2) - Pr\,R\,k^2 = 0 \quad , \quad s \,:\, \text{growth rate}$$

$$\Rightarrow \quad \frac{(k^2+\pi^2)}{k^2\,Pr}\,s^2 + \frac{(k^2+\pi^2)^2}{k^2}\,\frac{(1+Pr)}{Pr}\,s + R_c - R = 0$$

$$\Rightarrow \quad s \sim R - R_c \quad \Rightarrow \quad s \sim \epsilon^{2/3}$$

<u>scaling</u>

$$S = \epsilon \quad , \quad R = R_c + c\,\epsilon^{2/3} \quad , \quad t = \epsilon^{-2/3} T \quad , \quad \tau = \epsilon^{1/3}\tau_0$$

$$\psi = \epsilon^{1/3}\psi_0 + \epsilon^{2/3}\psi_1 + \epsilon\,\psi_2 + \cdots$$

$$\theta = \epsilon^{1/3}\theta_0 + \epsilon^{2/3}\theta_1 + \epsilon\,\theta_2 + \cdots$$

$$\sigma = \sigma_0 + \epsilon^{1/3}\sigma_1 + \epsilon^{2/3}\sigma_2 + \cdots$$

<u>solution</u>

$$\big(\partial_T \epsilon^{2/3} - Pr\Delta\big)\Delta\psi + Pr\big(R_c + c\,\epsilon^{2/3}\big)\,\theta_x - Pr\,\epsilon\,\sigma_x = -\psi_z\Delta\psi_x + \psi_x\Delta\psi_z$$

$$\big(\partial_T \epsilon^{2/3} - \Delta\big)\theta + \psi_x = -\psi_z\theta_x + \psi_x\theta_z$$

$$\big(\partial_T \epsilon^{2/3} - \epsilon^{1/3}\tau_0\Delta\big)\sigma + \psi_x = -\psi_z\sigma_x + \psi_x\sigma_z$$

$$Pr\Delta^2\psi - PrR_c\theta_x = \psi_z\Delta\psi_x - \psi_x\Delta\psi_z + \epsilon^{2/3}\big(\Delta\psi_T + Pr\,c\,\theta_x\big) + \epsilon\,Pr\,\sigma_x$$

$$\Delta\theta - \psi_x = \psi_z\theta_x - \psi_x\theta_z + \epsilon^{2/3}\theta_T$$

$$\epsilon^{1/3}\tau_0\Delta\sigma - \psi_x = \psi_z\sigma_x - \psi_x\sigma_z + \epsilon^{2/3}\sigma_T$$

$$\underline{\epsilon^{1/3}} \; : \; \left.\begin{aligned} \Delta^2\psi_0 - R_c\theta_{0x} &= 0 \\ \Delta\theta_0 - \psi_{0x} &= 0 \end{aligned}\right\} + \text{ bc } \Rightarrow \left\{\begin{aligned} \psi_0 &= \alpha(T)\sin kx \sin \pi z \\ \theta_0 &= -\frac{k}{k^2+\pi^2}\,\alpha(T)\cos kx \sin \pi z \end{aligned}\right.$$

$$\cdots \quad \Rightarrow \quad \frac{1+Pr}{Pr}\,(k^2+\pi^2)\alpha' = \frac{c\,k^2}{k^2+\pi^2}\,\alpha - \frac{1}{8}\,k^2(k^2+\pi^2)\alpha^3 + <\sigma_0\,,\phi_x>$$

$$\phi = \sin kx \sin \pi z \quad , \quad <\sigma_0\,,\phi_x> = \frac{4k}{\pi}\int_0^1\int_0^{\pi/k}\sigma_0\,\phi_x\,dx\,dz$$

$$\tau_0\Delta\sigma_0 - \psi_{0x} = \psi_{0z}\sigma_{0x} - \psi_{0x}\sigma_{0z} \quad \Rightarrow \quad \frac{\tau_0}{\alpha}\Delta\sigma_0 - \phi_z\sigma_{0x} + \phi_x\sigma_{0z} = \phi_x$$

problem

Given a differential operator L and a function f, compute $<u\,,f>$ where u is the solution of $Lu = f$.

variational formulation (J. B. Keller)

Given L, f as above, define $g(x,y) = <f,y> + <x,f> - <Lx,y>$ for arbitrary functions x, y. Let u, v be the solutions of $Lu = f, L^{*}v = f$. Then $g(u+x, v+y)$ has a critical point at $x = y = 0$ and the critical value is $g(u,v) = <u,f>$, the required inner product.

pf

$$g(u+x, v+y) = <f, v+y> + <u+x, f> - <L(u+x), v+y>$$

$$= <f,v> + <f,y> + <u,f> + <x,f>$$

$$- <Lu,v> - <Lu,y> - <Lx,v> - <Lx,y>$$

$$= <u,f> - <Lx,y> \qquad \underline{\text{ok}}$$

application

$$u = \sigma_0 \quad , \quad Lu = \frac{\tau_0}{\alpha}\Delta u - \phi_z u_x + \phi_x u_z \quad , \quad f = \phi_x$$

$$g(u,v) = <\phi_x, v> + <u, \phi_x> - <\frac{\tau_0}{\alpha}\Delta u - \phi_z u_x + \phi_x u_z, v>$$

$$L^{*}v = \frac{\tau_0}{\alpha}\Delta v + \partial_x(\phi_z v) - \partial_z(\phi_x v) = \frac{\tau_0}{\alpha}\Delta + \phi_z v_x - \phi_x v_z$$

recall : $\phi = \sin kx \sin \pi z \quad \Rightarrow \quad \phi(x,z) = \phi(x, 1-z)$

$$1. \;\; z \to 1-z \quad \Rightarrow \quad \begin{cases} \phi(x,z) = \phi(x,1-z) \\ L^{*}v = f \;\to\; Lu = f \end{cases} \quad \Rightarrow \quad \begin{cases} v(x,1-z) = u(x,z) \\ v(x,z) = u(x,1-z) \end{cases}$$

$$\Rightarrow \quad g(u,v) = g(u(x,z), u(x,1-z)) = \overline{g}(u)$$

$$2. \;\; Lu = \phi_x \quad , \quad \text{bc} \; : \; u = 0 \text{ on } z = 0\,,1 \quad , \quad u_x = 0 \text{ on } x = 0\,,\pi/k$$

$$\Rightarrow \; u = c_1 \cos kx \sin \pi z + c_2 \sin 2\pi z + c_3 \cos kx \sin 3\pi z + \cdots$$

3. $\overline{g}_c \;=\; -\sqrt{8}\,k\,(k^2+\pi^2)^{1/2}\dfrac{A}{(1+2A^2)^2}\left(1+3A^2+2\left(\dfrac{k^2+5\pi^2}{k^2+9\pi^2}\right)A^4\right)$

$$A \;=\; \frac{k}{(3(k^2+\pi^2))^{1/2}}\,\frac{\alpha(T)}{\tau_0}$$

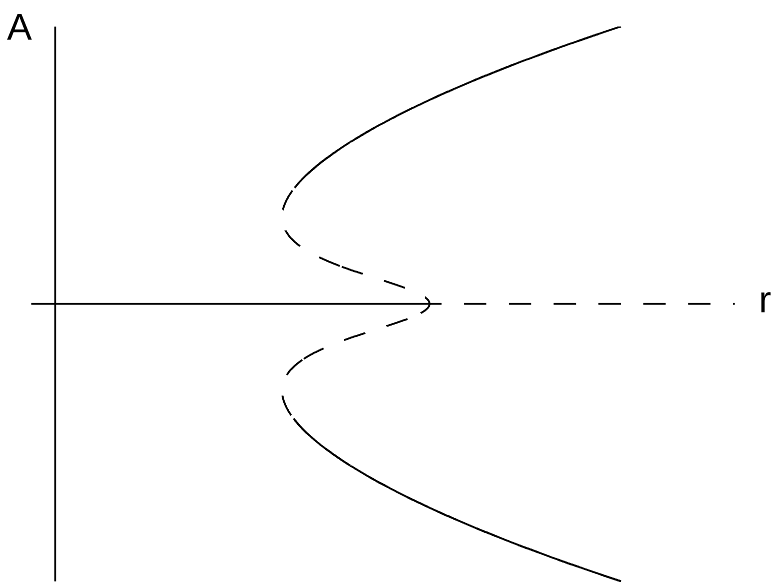
$< \sigma_0\,,\phi_x > \;\rightarrow\; \overline{g}_c$

Landau equation

$$\lambda A' \;=\; rA \;-\; A^3 \;-\; \mu\,\frac{A}{(1+2A^2)^2}\left(1+3A^2+2\left(\frac{k^2+5\pi^2}{k^2+9\pi^2}\right)A^4\right)$$

$$\lambda \;=\; \frac{1+Pr}{Pr\,\tau_0^2(k^2+\pi^2)} \quad , \quad r \;=\; \frac{c}{R_c\tau_0^2} \quad , \quad \mu \;=\; \frac{1}{R_c\tau_0^3}$$

$\mu > 1 \quad \Rightarrow \quad$ subcritical bifurcation



note

1. Small-amplitude subcritical convection cells are unstable; they decay to zero as $t \rightarrow \infty$.

2. Large-amplitude subcritical convection cells are stable; fluid convection causes the solute profile to become nearly uniform and the stabilizing effect of the solute gradient is neutralized, thereby enabling the convection cells to persist.

next goal : 3D equations , linearized , pattern formation

recall : Boussinesq eqs II (for perturbations about hydrostatic equilibrium)

$$\nabla \cdot \vec{u} = 0$$

$$\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = -\frac{\nabla \pi}{\rho_0} + \alpha g \, \theta \, \vec{e}_z + \nu \Delta \vec{u}$$

$$\theta_t + (\vec{u} \cdot \nabla)\theta = \frac{[T]}{d} w + \kappa \Delta \theta$$

note : $\nabla = (\partial_x, \partial_y, \partial_z)$, $\vec{u} = (u, v, w)$

nondimensionalize , linearize

$$\nabla \cdot \vec{u} = 0$$

$$\vec{u}_t = -\nabla \pi + Pr \, R \, \theta \, \vec{e}_z + Pr \Delta \vec{u}$$

$$\theta_t = w + \Delta \theta$$

take curl of momentum equation , set $\vec{\omega} = \nabla \times \vec{u}$

$$\Rightarrow \quad \vec{\omega}_t = Pr \, R \, \nabla \theta \times \vec{e}_z + Pr \Delta \vec{\omega}$$

take curl again

$$\nabla \times \vec{\omega} = \nabla \times (\nabla \times \vec{u}) = \nabla(\nabla \cdot \vec{u}) - \Delta \vec{u} = -\Delta \vec{u}$$

$$\nabla \times (\nabla \theta \times \vec{e}_z) = \nabla \times (\nabla \times (\theta \vec{e}_z)) = \nabla(\nabla \cdot (\theta \vec{e}_z)) - \Delta(\theta \vec{e}_z) = \nabla \theta_z - (\Delta \theta)\vec{e}_z$$

$$\nabla \times (\Delta \vec{\omega}) = \Delta(\nabla \times \vec{\omega}) = \Delta(-\Delta \vec{u}) = -\Delta^2 \vec{u}$$

$$\Rightarrow \quad -\Delta \vec{u}_t = Pr \, R \, (\nabla \theta_z - (\Delta \theta)\vec{e}_z) - Pr \Delta^2 \vec{u}$$

take z-component , $\partial_x^2 + \partial_y^2 = \Delta_1$

$$\Rightarrow \quad \left.\begin{aligned} \Delta w_t &= Pr \, R \, \Delta_1 \theta + Pr \Delta^2 w \\ \theta_t &= w + \Delta \theta \end{aligned}\right\} : \ \text{coupled PDEs in } (x, y, z, t)$$

<u>boundary conditions</u>

$$\text{vertical} \; : \; z = 0\,,1 \; \begin{cases} \text{rigid} \;\;,\;\; \text{no-slip} \; : \; w = w_z = \theta = 0 \\ \text{free} \;\;,\;\; \text{zero stress} \; : \; w = w_{zz} = \theta = 0 \end{cases}$$

horizontal : $(x, y) \in \partial\Omega$, more later

<u>normal modes</u>

$$w \,=\, e^{st} f(x,y) W(z) \quad , \quad \theta \,=\, e^{st} f(x,y) T(z) \quad , \quad D = \frac{d}{dz}$$

$$\theta_t \,=\, w \,+\, \Delta\theta \quad \Rightarrow \quad sfT \,=\, fW \,+\, (\Delta_1 + D^2) fT \,=\, fW \,+\, T\Delta_1 f \,+\, fD^2 T$$

$$\Rightarrow \quad \frac{(D^2 - s)T + W}{T} \,=\, -\,\frac{\Delta_1 f}{f} \,=\, a^2$$

$$\Rightarrow \quad (D^2 - a^2 - s)T \,=\, -W \quad , \quad \Delta_1 f \,+\, a^2 f \,=\, 0 \; : \; \underline{\text{Helmholtz equation}}$$

<u>note</u> : a has units of L^{-1} , horizontal wavenumber

$$\Delta w_t \,=\, Pr\, R\, \Delta_1 \theta \,+\, Pr \Delta^2 w$$

$$\Rightarrow \quad s(\Delta_1 + D^2) fW \,=\, Pr\, R\, \Delta_1 fT \,+\, Pr(\Delta_1 + D^2)^2 fW$$

$$\Rightarrow \quad sf(D^2 - a^2)W \,=\, -Pr\, R\, a^2 fT \,+\, Prf(D^2 - a^2)^2 W$$

<u>summary</u>

$$\Delta_1 f \,+\, a^2 f \,=\, 0 \; : \; \text{PDE in } (x, y)$$

$$\left.\begin{aligned} &(D^2 - a^2 - s)T \,=\, -W \\ &\Big(D^2 - a^2\Big)\Big(D^2 - a^2 - \frac{s}{Pr}\Big)W \,=\, a^2 R\, T \end{aligned}\right\} \; : \; \text{coupled ODEs in } z$$

rigid bc : $W \,=\, DW \,=\, T \,=\, 0$, free bc : $W \,=\, D^2 W \,=\, T \,=\, 0$

this defines two eigenvalue problems : $a = a(\Omega)$, $s = s(a, R, Pr)$

def : $s = \sigma + i\omega$, a system is called marginally stable if $\sigma = 0$

note

1. In a marginally stable system we may have $\omega = 0$ (e.g. turning point, transcritical/pitchfork bifurcation) or $\omega \neq 0$ (e.g. Hopf bifurcation). If $\omega = 0$ at a point of marginal stability, then we say that the principle of exchange of stability is satisfied.

2. If $R > 0$, then $\omega = 0$ (and hence the PES is satisfied). pf : hw

3. Assume $R > 0$ and a is given. Since the PES is satisfied, we may set $s = 0$ to determine the curve of marginal stability.

$$(D^2 - a^2)T = -W \quad , \quad (D^2 - a^2)^2 W = a^2 R T \quad \Rightarrow \quad (D^2 - a^2)^3 W = -a^2 R W$$

assume free bc on $z = 0, 1$: $W = D^2 W = T = 0 \quad \Rightarrow \quad D^4 W = 0$

eigenfunctions : $W(z) = \sin j\pi z \quad , \quad j = 1, 2, \ldots$

eigenvalues : $(-j^2\pi^2 - a^2)^3 = -a^2 R \quad \Rightarrow \quad R = \dfrac{a^2 + j^2\pi^2}{a^2}$

$$j = 1 \quad \Rightarrow \quad R_0(a) = \frac{a^2 + \pi^2}{a^2} \quad , \quad W(z) = \sin \pi z \quad \text{as before}$$

4. To complete the description of the normal modes we must determine $f(x, y)$, where $\Delta_1 f + a^2 f = 0$.

def

A cell is a region of space with vertical boundary st no fluid crosses the boundary, i.e. $\vec{n} \cdot (u, v) = 0$ on the cell boundary (of course in general $\vec{\tau} \cdot (u, v) \neq 0$, $w \neq 0$ on the cell boundary).

claim

$$\Delta_1 u = -w_{xz} - \omega_{3y} \quad , \quad \Delta_1 v = -w_{yz} - \omega_{3x} \quad , \quad \omega_{3t} = Pr \Delta \omega_3$$

where $\omega_3 = v_x - u_y$ is the z-component of vorticity pf : hw

note

If $\omega_3 = 0$ on the cell boundary, then $\omega_3 \to 0$ as $t \to \infty$ throughout the cell, so we may ignore the ω_3 terms above.

ex : cylindrical rolls (as before)

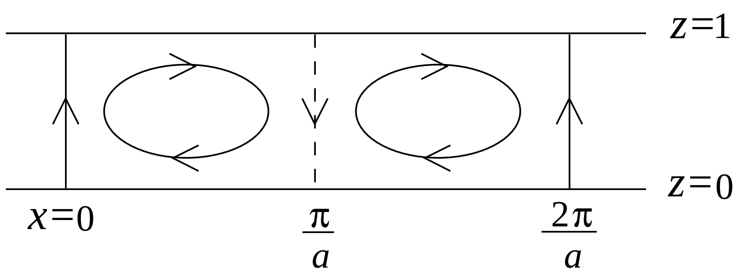
$f(x,y) = \cos ax$

$w = e^{st} \cos ax\, W(z)$

$\Rightarrow \quad \Delta_1 u = -w_{xz} = a e^{st} \sin ax\, DW(z) \quad , \quad \Delta_1 v = -w_{yz} = 0$

$\Rightarrow \quad u = -\frac{1}{a}\, e^{st} \sin ax\, DW(z) \quad , \quad v = 0$

cell boundary : $u = 0 \quad \Leftrightarrow \quad x = \frac{2\pi n}{a}$



ex : rectangular cells

$f(x,y) = \cos a_1 x \cos a_2 y \quad , \quad a_1^2 + a_2^2 = a^2$

$w = e^{st} \cos a_1 x \cos a_2 y\, W(z)$

$$\Rightarrow \quad u = -\frac{a_1}{a^2}\, e^{st} \sin a_1 x \cos a_2 y\, DW(z)$$

$$v = -\frac{a_2}{a^2}\, e^{st} \cos a_1 x \sin a_2 y\, DW(z)$$

projection of streamlines onto horizontal plane : $\frac{dy}{dx} = \frac{v}{u} = \frac{a_2 \tan a_2 y}{a_1 \tan a_1 x}$

picture : $a_1 = \frac{\sqrt{3}}{2}\, a \quad , \quad a_2 = \frac{1}{2}\, a$

$(x,y) \sim (0,0) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{a_2^2 y}{a_1^2 x} = \frac{y}{3x} \quad \Rightarrow \quad y = c x^{1/3}$

<u>ex</u> : hexagonal cells

$$f(x,y) \,=\, \cos\frac{a}{2}\big(\sqrt{3}\,x+y\big) \,+\, \cos\frac{a}{2}\big(\sqrt{3}\,x-y\big) \,+\, \cos ay$$

$$\Delta_1 f = -\,\frac{3a^2}{4}\cos\frac{a}{2}\big(\sqrt{3}\,x+y\big) \,-\, \frac{3a^2}{4}\cos\frac{a}{2}\big(\sqrt{3}\,x-y\big)$$

$$-\,\frac{a^2}{4}\cos\frac{a}{2}\big(\sqrt{3}\,x+y\big) \,-\, \frac{a^2}{4}\cos\frac{a}{2}\big(\sqrt{3}\,x-y\big) \,-\, a^2\cos ay \,=\, -a^2 f \qquad \underline{\text{ok}}$$

<u>properties</u>

1. $f(-x,-y) \,=\, f(x,y)$

2. $f\Big(x+\dfrac{4\pi m}{\sqrt{3}\,a}\,,\,y+\dfrac{4\pi n}{a}\Big) \,=\, f(x,y)$: doubly-periodic

3. $f(x,y)$ is invariant under rotation by $\pi/3$ radians

<u>pf</u> :

$$x \,=\, r\cos\theta \quad , \quad y \,=\, r\sin\theta$$

$$\frac{1}{2}\big(\sqrt{3}\,x \,\pm\, y\big) \,=\, \frac{1}{2}\big(\sqrt{3}\,r\cos\theta \,\pm\, r\sin\theta\big)$$

$$= r\Big(\sin\frac{\pi}{3}\cos\theta \,\pm\, \cos\frac{\pi}{3}\sin\theta\Big) \,=\, r\sin\Big(\frac{\pi}{3} \,\pm\, \theta\Big)$$

$$f(r,\theta) \,=\, \cos\Big(a\,r\sin\Big(\frac{\pi}{3}+\theta\Big)\Big) \,+\, \cos\Big(a\,r\sin\Big(\frac{\pi}{3}-\theta\Big)\Big) \,+\, \cos\big(a\,r\sin\theta\big)$$

$$f\Big(r,\theta+\frac{\pi}{3}\Big) \,=\, \cos\Big(ar\sin\Big(\underbrace{\frac{2\pi}{3}+\theta}_{\substack{\downarrow \\ -\left(\frac{2\pi}{3}+\theta\right) \\ \downarrow \\ \frac{\pi}{3}-\theta}}\Big)\Big) + \cos\big(ar\sin\big(\underbrace{-\,\theta}_{\substack{\downarrow \\ \theta}}\big)\big) \,+\, \cos\Big(ar\sin\Big(\theta+\frac{\pi}{3}\Big)\Big)$$

<u>ok</u>

<u>note</u>

$$w \;=\; e^{st} f(x,y)\, W(z)$$

$$\Delta_1 u \;=\; -w_{xz} \;=\; -\, e^{st} f_x \, DW \qquad \Rightarrow \qquad u \;=\; \frac{1}{a^2}\, e^{st} f_x \, DW$$

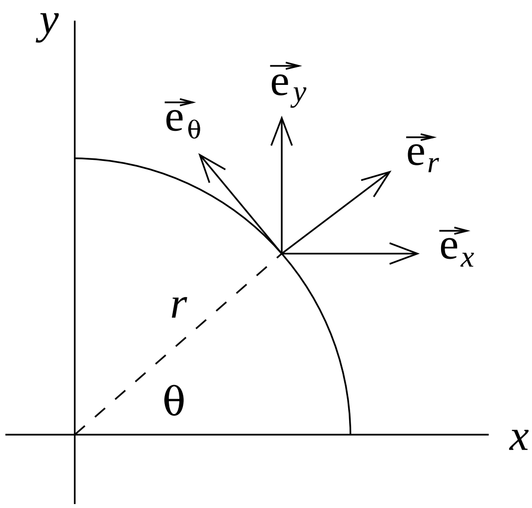
$$u \;=\; \frac{1}{a^2}\, e^{st} \left(-\,\frac{\sqrt{3}\,a}{2} \right) \Big(\sin \frac{a}{2} \big(\sqrt{3}\, x + y \big) \;+\; \sin \frac{a}{2} \big(\sqrt{3}\, x - y \big) \Big)\, DW$$

$$x \;=\; \frac{2\pi}{\sqrt{3}\, a} \qquad \Rightarrow \qquad u \;=\; e^{st} \left(-\,\frac{\sqrt{3}}{2\,a} \right) \Big(\sin \Big(\pi + \frac{a}{2} y \Big) + \sin \Big(\pi - \frac{a}{2} y \Big) \Big) DW \;=\; 0$$

7. centrifugal instability

cylindrical coordinates

$$(x, y, z) \quad \rightarrow \quad (r, \theta, z) \quad , \quad x = r\cos\theta \quad , \quad y = r\sin\theta$$



$$\vec{e}_r = \cos\theta\,\vec{e}_x + \sin\theta\,\vec{e}_y$$

$$\vec{e}_\theta = -\sin\theta\,\vec{e}_x + \cos\theta\,\vec{e}_y$$

$$\Rightarrow \quad \begin{pmatrix} \vec{e}_r \\ \vec{e}_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \vec{e}_x \\ \vec{e}_y \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} \vec{e}_x \\ \vec{e}_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \vec{e}_r \\ \vec{e}_\theta \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} \vec{e}_x &= \cos\theta\,\vec{e}_r - \sin\theta\,\vec{e}_\theta \\ \vec{e}_y &= \sin\theta\,\vec{e}_r + \cos\theta\,\vec{e}_\theta \end{aligned}$$

<u>gradient</u>

$$\nabla f \;=\; \frac{\partial f}{\partial x}\,\vec{e}_x \;+\; \frac{\partial f}{\partial y}\,\vec{e}_y \;=\; f_x\,\vec{e}_x \;+\; f_y\,\vec{e}_y \quad \big(\text{note : } f_x \text{ is temporary notation}\big)$$

$$\begin{aligned} f_x \;&=\; f_r\,r_x \;+\; f_\theta\,\theta_x \\ f_y \;&=\; f_r\,r_y \;+\; f_\theta\,\theta_y \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} r_x & \theta_x \\ r_y & \theta_y \end{pmatrix} \begin{pmatrix} f_r \\ f_\theta \end{pmatrix}$$

$$x \;=\; r\cos\theta \quad \Rightarrow \quad 1 \;=\; r_x\cos\theta \;+\; r\cdot -\sin\theta\,\theta_x \quad , \quad 0 \;=\; r_y\cos\theta \;+\; r\cdot -\sin\theta\,\theta_y$$

$$y \;=\; r\sin\theta \quad \Rightarrow \quad 0 \;=\; r_x\sin\theta \;+\; r\cdot\cos\theta\,\theta_x \quad , \quad 1 \;=\; r_y\sin\theta \;+\; r\cdot\cos\theta\,\theta_y$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_x & \theta_x \\ r_y & \theta_y \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$

$$\begin{pmatrix} r_x & \theta_x \\ r_y & \theta_y \end{pmatrix} = r^{-1} \begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} f_x \\ f_y \end{pmatrix} = r^{-1} \begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} f_r \\ f_\theta \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} f_x \;&=\; \cos\theta f_r \;-\; r^{-1}\sin\theta f_\theta \\ f_y \;&=\; \sin\theta f_r \;+\; r^{-1}\cos\theta f_\theta \end{aligned}$$

$$\begin{aligned} \nabla f \;=\; f_x\,\vec{e}_x \;+\; f_y\,\vec{e}_y \;&=\; \big(\cos\theta f_r \;-\; r^{-1}\sin\theta f_\theta\big)\big(\cos\theta\,\vec{e}_r \;-\; \sin\theta\,\vec{e}_\theta\big) \\ &\quad+\; \big(\sin\theta f_r \;+\; r^{-1}\cos\theta f_\theta\big)\big(\sin\theta\,\vec{e}_r \;+\; \cos\theta\,\vec{e}_\theta\big) \\ &=\; f_r\,\vec{e}_r \;+\; f_\theta\,\frac{\vec{e}_\theta}{r} \;=\; \left(\vec{e}_r\,\frac{\partial}{\partial r} \;+\; \frac{\vec{e}_\theta}{r}\,\frac{\partial}{\partial\theta}\right) f \end{aligned}$$

$$\Rightarrow \quad \nabla \;=\; \vec{e}_r\,\frac{\partial}{\partial r} \;+\; \frac{\vec{e}_\theta}{r}\,\frac{\partial}{\partial\theta}$$

<u>divergence</u>

$\vec{u} \,=\, u_r\,\vec{e}_r \,+\, u_\theta\,\vec{e}_\theta \qquad \big(\text{note}: \; u_r \text{ is the } r\text{-component of } \vec{u}\,\big)$

$$\nabla\cdot\vec{u} \,=\, \left(\vec{e}_r\,\frac{\partial}{\partial r} \,+\, \frac{\vec{e}_\theta}{r}\,\frac{\partial}{\partial\theta}\right)\cdot\Big(u_r\,\vec{e}_r \,+\, u_\theta\,\vec{e}_\theta\Big)$$

$$= \vec{e}_r\cdot\left(\frac{\partial u_r}{\partial r}\,\vec{e}_r \,+\, u_r\,\frac{\partial\vec{e}_r}{\partial r} \,+\, \frac{\partial u_\theta}{\partial r}\,\vec{e}_\theta \,+\, u_\theta\,\frac{\partial\vec{e}_\theta}{\partial r}\right)$$

$$+\,\frac{\vec{e}_\theta}{r}\cdot\left(\frac{\partial u_r}{\partial\theta}\,\vec{e}_r \,+\, u_r\,\frac{\partial\vec{e}_r}{\partial\theta} \,+\, \frac{\partial u_\theta}{\partial\theta}\,\vec{e}_\theta \,+\, u_\theta\,\frac{\partial\vec{e}_\theta}{\partial\theta}\right)$$

$$\vec{e}_r \,=\, \cos\theta\,\vec{e}_x \,+\, \sin\theta\,\vec{e}_y \quad , \quad \vec{e}_\theta \,=\, -\sin\theta\,\vec{e}_x \,+\, \cos\theta\,\vec{e}_y$$

$$\frac{\partial\vec{e}_r}{\partial r} \,=\, 0 \quad , \quad \frac{\partial\vec{e}_\theta}{\partial r} \,=\, 0$$

$$\frac{\partial\vec{e}_r}{\partial\theta} \,=\, -\sin\theta\,\vec{e}_x \,+\, \cos\theta\,\vec{e}_y \,=\, \vec{e}_\theta \quad , \quad \frac{\partial\vec{e}_\theta}{\partial\theta} \,=\, -\cos\theta\,\vec{e}_x \,-\, \sin\theta\,\vec{e}_y \,=\, -\vec{e}_r$$

$$\Rightarrow \quad \nabla\cdot\vec{u} \,=\, \frac{\partial u_r}{\partial r} \,+\, \frac{u_r}{r} \,+\, \frac{1}{r}\,\frac{\partial u_\theta}{\partial\theta}$$

<u>convection term</u>

$$\vec{u}\cdot\nabla \,=\, \Big(u_r\,\vec{e}_r \,+\, u_\theta\,\vec{e}_\theta\Big)\cdot\left(\vec{e}_r\,\frac{\partial}{\partial r} \,+\, \frac{\vec{e}_\theta}{r}\,\frac{\partial}{\partial\theta}\right) \,=\, u_r\,\frac{\partial}{\partial r} \,+\, \frac{u_\theta}{r}\,\frac{\partial}{\partial\theta}$$

$$(\vec{u}\cdot\nabla)\vec{u} \,=\, \left(u_r\,\frac{\partial}{\partial r} \,+\, \frac{u_\theta}{r}\,\frac{\partial}{\partial\theta}\right)\Big(u_r\,\vec{e}_r \,+\, u_\theta\,\vec{e}_\theta\Big)$$

$$= u_r\left(\frac{\partial u_r}{\partial r}\,\vec{e}_r \,+\, u_r\,\frac{\partial\vec{e}_r}{\partial r} \,+\, \frac{\partial u_\theta}{\partial r}\,\vec{e}_\theta \,+\, u_\theta\,\frac{\partial\vec{e}_\theta}{\partial r}\right)$$

$$+\,\frac{u_\theta}{r}\left(\frac{\partial u_r}{\partial\theta}\,\vec{e}_r \,+\, u_r\,\frac{\partial\vec{e}_r}{\partial\theta} \,+\, \frac{\partial u_\theta}{\partial\theta}\,\vec{e}_\theta \,+\, u_\theta\,\frac{\partial\vec{e}_\theta}{\partial\theta}\right)$$

$$= \left(u_r\,\frac{\partial u_r}{\partial r} \,+\, \frac{u_\theta}{r}\,\frac{\partial u_r}{\partial\theta} \,-\, \frac{u_\theta^2}{r}\right)\vec{e}_r \,+\, \left(u_r\,\frac{\partial u_\theta}{\partial r} \,+\, \frac{u_r u_\theta}{r} \,+\, \frac{u_\theta}{r}\,\frac{\partial u_\theta}{\partial\theta}\right)\vec{e}_\theta$$

$$\Rightarrow \quad (\vec{u}\cdot\nabla)\vec{u} \,=\, \left((\vec{u}\cdot\nabla)u_r \,-\, \frac{u_\theta^2}{r}\right)\vec{e}_r \,+\, \left((\vec{u}\cdot\nabla)u_\theta \,+\, \frac{u_r u_\theta}{r}\right)\vec{e}_\theta$$

<u>Laplacian</u>

$$\Delta = \nabla \cdot \nabla = \left(\vec{e}_r \frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r} \frac{\partial}{\partial \theta}\right) \cdot \left(\vec{e}_r \frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r} \frac{\partial}{\partial \theta}\right)$$

$$= \vec{e}_r \cdot \left(\frac{\partial \vec{e}_r}{\partial r} \frac{\partial}{\partial r} + \vec{e}_r \frac{\partial^2}{\partial r^2} - \frac{\vec{e}_\theta}{r^2} \frac{\partial}{\partial \theta} + \frac{1}{r}\left(\frac{\partial \vec{e}_\theta}{\partial r} \frac{\partial}{\partial \theta} + \vec{e}_\theta \frac{\partial^2}{\partial r \partial \theta}\right)\right)$$

$$+ \frac{\vec{e}_\theta}{r} \cdot \left(\frac{\partial \vec{e}_r}{\partial \theta} \frac{\partial}{\partial r} + \vec{e}_r \frac{\partial^2}{\partial r \partial \theta} + \frac{1}{r}\left(\frac{\partial \vec{e}_\theta}{\partial \theta} \frac{\partial}{\partial \theta} + \vec{e}_\theta \frac{\partial^2}{\partial \theta^2}\right)\right)$$

$$\Rightarrow \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

$$\Delta \vec{u} = \Delta\big(u_r \vec{e}_r + u_\theta \vec{e}_\theta\big)$$

$$\Delta\big(u_r \vec{e}_r\big) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}\right)\big(u_r \vec{e}_r\big)$$

$$= \frac{\partial^2 u_r}{\partial r^2} \vec{e}_r + \frac{1}{r} \frac{\partial u_r}{\partial r} \vec{e}_r + \frac{1}{r^2}\left(\frac{\partial^2 u_r}{\partial \theta^2} \vec{e}_r + 2 \frac{\partial u_r}{\partial \theta} \frac{\partial \vec{e}_r}{\partial \theta} + u_r \frac{\partial^2 \vec{e}_r}{\partial \theta^2}\right)$$

$$= \big(\Delta u_r\big)\vec{e}_r + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \vec{e}_\theta - \frac{u_r}{r^2} \vec{e}_r$$

$$\Delta\big(u_\theta \vec{e}_\theta\big) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}\right)\big(u_\theta \vec{e}_\theta\big)$$

$$= \frac{\partial^2 u_\theta}{\partial r^2} \vec{e}_\theta + \frac{1}{r} \frac{\partial u_\theta}{\partial r} \vec{e}_\theta + \frac{1}{r^2}\left(\frac{\partial^2 u_\theta}{\partial \theta^2} \vec{e}_\theta + 2 \frac{\partial u_\theta}{\partial \theta} \frac{\partial \vec{e}_\theta}{\partial \theta} + u_\theta \frac{\partial^2 \vec{e}_\theta}{\partial \theta^2}\right)$$

$$= \big(\Delta u_\theta\big)\vec{e}_\theta - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \vec{e}_r - \frac{u_\theta}{r^2} \vec{e}_\theta$$

$$\Rightarrow \quad \Delta \vec{u} = \left(\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta}\right)\vec{e}_r + \left(\Delta u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta}\right)\vec{e}_\theta$$

7.1 swirl flow

$\vec{u} \,=\, u_r\,\vec{e}_r \,+\, u_\theta\,\vec{e}_\theta \,+\, u_z\,\vec{e}_z$

inviscid flow

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p \qquad , \qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u}\cdot\nabla = \frac{\partial}{\partial t} + u_r\,\frac{\partial}{\partial r} + \frac{u_\theta}{r}\,\frac{\partial}{\partial \theta} + u_z\,\frac{\partial}{\partial z}$$

$$\frac{Du_r}{Dt} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\,\frac{\partial p}{\partial r} \qquad , \qquad \frac{u_\theta^2}{r} \;:\; \underline{\text{centrifugal acceleration}}$$

$$\frac{Du_\theta}{Dt} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho}\,\frac{1}{r}\,\frac{\partial p}{\partial \theta}$$

$$\frac{Du_z}{Dt} = -\frac{1}{\rho}\,\frac{\partial p}{\partial z}$$

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\,\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

We will investigate the stability of axisymmetric swirl flow wrt various types of perturbations (e.g. inviscid, viscous, 2D, 3D, axisymmetric).

axisymmetric inviscid flow

$$\frac{\partial}{\partial \theta} = 0 \quad \Rightarrow \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\,\frac{\partial}{\partial r} + u_z\,\frac{\partial}{\partial z}$$

$$\frac{Du_r}{Dt} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\,\frac{\partial p}{\partial r}$$

$$\frac{Du_\theta}{Dt} + \frac{u_r u_\theta}{r} = 0$$

$$\frac{Du_z}{Dt} = -\frac{1}{\rho}\,\frac{\partial p}{\partial z}$$

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0$$

claim

$$\frac{D(ru_\theta)}{Dt} = 0 \quad , \quad ru_\theta \ : \ \underline{\text{angular momentum}} \text{ about } z\text{-axis}$$

pf

$$\frac{D(ru_\theta)}{Dt} = \frac{\partial(ru_\theta)}{\partial t} + u_r \frac{\partial(ru_\theta)}{\partial r} + u_z \frac{\partial(ru_\theta)}{\partial z}$$

$$= r\left(\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_r u_\theta}{r} + u_z \frac{\partial u_\theta}{\partial z}\right) = r\left(\frac{Du_\theta}{Dt} + \frac{u_r u_\theta}{r}\right) = r \cdot 0 = 0 \quad \underline{\text{ok}}$$

basic flow

$u_r = u_z = 0$, $u_\theta = V(r)$, $p = P(r)$: axisymmetric swirl flow

$$-\frac{V^2}{r} = -\frac{1}{\rho}\frac{dP}{dr} \quad \Rightarrow \quad P = \rho \int \frac{V^2}{r}\, dr \quad , \quad 0 \le r \le R \quad \text{or} \quad R_1 \le r \le R_2$$

heuristic stability argument (Rayleigh)

$$H = rV \quad \Rightarrow \quad \frac{DH}{DT} = 0 \ , \quad \int_{C_r} V dr = 2\pi r V = 2\pi H \ : \ \text{circulation around } C_r$$

$$F = \frac{V^2}{r} = \frac{H^2}{r^3} \ : \ \text{centrifugal force} \quad , \quad E = V^2 = \frac{H^2}{r^2} \ : \ \text{kinetic energy}$$

$$F \sim \frac{\partial E}{\partial r} \quad \Rightarrow \quad E \sim \text{potential energy}$$

consider 2 rings of fluid : $r = r_1 , r_2$, $z = z_1 , z_2$ where $r_1 < r_2$

$$E_i = \frac{H_1^2}{r_1^2} + \frac{H_2^2}{r_2^2} \quad , \quad \text{interchange ring locations} \ : \ E_f = \frac{H_2^2}{r_1^2} + \frac{H_1^2}{r_2^2}$$

$$E_i - E_f = \left(H_1^2 - H_2^2\right)\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$$

then the basic flow minimizes the energy $\Leftrightarrow$ $E_i < E_f$ $\Leftrightarrow$ $H_1 < H_2$

Rayleigh's criterion : If the circulation $2\pi r V(r)$ is an increasing function of r, then the basic flow is stable wrt axisymmetric perturbations.

inviscid linear stability : 3D perturbations

$$u_r = u_r' \;,\; u_z = u_z' \;,\; u_\theta = V(r) + u_\theta' \;,\; p = P(r) + p'$$

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_z \frac{\partial u_r}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} + u_z \frac{\partial u_\theta}{\partial z} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta}$$

$$\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} - u_z \frac{\partial u_z}{\partial z} = +\frac{1}{\rho}\frac{\partial p}{\partial z}$$

$$\frac{\partial u_r'}{\partial t} + \frac{V}{r}\frac{\partial u_r'}{\partial \theta} - 2\frac{V}{r}u_\theta' = -\frac{1}{\rho}\frac{\partial p'}{\partial r}$$

$$\frac{\partial u_\theta'}{\partial t} + u_r'(DV) + \frac{V}{r}\frac{\partial u_\theta'}{\partial \theta} + u_r'\frac{V}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p'}{\partial \theta} \quad , \quad DV = \frac{dV}{dr}$$

$$\frac{\partial u_z'}{\partial t} + \frac{V}{r}\frac{\partial u_z'}{\partial \theta} = -\frac{1}{\rho}\frac{\partial p'}{\partial z}$$

define $\Omega(r) = \dfrac{V(r)}{r}$: <u>angular velocity</u>

$$\frac{\partial u_r'}{\partial t} + \Omega\frac{\partial u_r'}{\partial \theta} - 2\,\Omega\,u_\theta' = -\frac{1}{\rho}\frac{\partial p'}{\partial r}$$

$$\frac{\partial u_\theta'}{\partial t} + \Omega\frac{\partial u_\theta'}{\partial \theta} + (D_*V)u_r' = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p'}{\partial \theta} \quad , \quad D_*V = DV + \frac{V}{r}$$

$$\frac{\partial u_z'}{\partial t} + \Omega\frac{\partial u_z'}{\partial \theta} = -\frac{1}{\rho}\frac{\partial p'}{\partial z}$$

$$\frac{\partial u_r'}{\partial r} + \frac{u_r'}{r} + \frac{1}{r}\frac{\partial u_\theta'}{\partial \theta} + \frac{\partial u_z'}{\partial z} = 0$$

<u>normal modes</u>

$$\left(u_r' \,,\, u_\theta' \,,\, u_z' \,,\, \frac{p'}{\rho}\right) = \underbrace{(u \,,\, v \,,\, w \,,\, p)}_{\text{functions of } r} e^{st + i(n\theta + kz)}$$

n : azimuthal wavenumber , k : axial wavenumber

define $\gamma = s + in\Omega$

$$\gamma u - 2\Omega\, v = -Dp$$

$$\gamma v + (D_* V)u = -\frac{in}{r}\, p$$

$$\gamma w = -ikp$$

$$D_* u + \frac{in}{r}\, v + ikw = 0$$

<u>claim</u>

$$\gamma^2 D\left(\frac{r^2(D_* u)}{n^2 + k^2 r^2}\right) - \left(\gamma^2 + \frac{k^2 r^2 \Phi}{n^2 + k^2 r^2} + in\gamma r D\left(\frac{D_* V}{n^2 + k^2 r^2}\right)\right) u = 0$$

where $\Phi = \dfrac{1}{r^3}\, D\Big(\big(rV\big)^2\Big)$: <u>Rayleigh discriminant</u>

<u>note</u>

1. This is a 2nd order ODE for $u(r)$. With appropriate bc, it defines an eigenvalue problem for the eigenvalue s and corresponding eigenfunction $u(r)$.

2. Rayleigh's criterion says that if $\Phi > 0$, then the basic flow is stable wrt axisymmetric perturbations. We will verify this below. It is known that $\Phi > 0$ does not guarantee stability for non-axisymmetric perturbations.

3. $\Phi = \dfrac{1}{r^3}\, 2rV D(rV) = \dfrac{2V}{r^2}\,(rDV + V) = 2\Omega(D_* V)$

<u>pf</u>

$$p \;=\; \frac{\gamma w}{-ik} \;=\; \frac{i\gamma w}{k} \;=\; \frac{i\gamma}{k}\cdot\frac{-1}{ik}\left(D_* u \,+\, \frac{inv}{r}\right) \;=\; \frac{-\gamma}{k^2}\left(D_* u \,+\, \frac{inv}{r}\right)$$

$$\gamma v \,+\, (D_* V)u \;=\; \frac{-in}{r}\cdot\frac{-\gamma}{k^2}\left(D_* u \,+\, \frac{inv}{r}\right) \;=\; \frac{in\gamma(D_* u)}{k^2 r} \,-\, \frac{n^2\gamma v}{k^2 r^2}$$

$$\gamma\left(1 + \frac{n^2}{k^2 r^2}\right) v \;=\; \frac{in\gamma(D_* u)}{k^2 r} \,-\, (D_* V)u \quad\Rightarrow\quad v \;=\; \frac{inr(D_* u)}{n^2 + k^2 r^2} \,-\, \frac{k^2 r^2 (D_* V)u}{\gamma(n^2 + k^2 r^2)}$$

$$\gamma u \,-\, 2\Omega\, v \;=\; -D\left(\frac{-\gamma}{k^2}\left(D_* u \,+\, \frac{inv}{r}\right)\right) \;=\; \frac{1}{k^2}\, D\left(\gamma\left(D_* u \,+\, \frac{inv}{r}\right)\right)$$

$$= \frac{1}{k^2}\left(\gamma D\left(D_* u \,+\, \frac{inv}{r}\right) \,+\, in\,(D\Omega)\left(D_* u \,+\, \frac{inv}{r}\right)\right)$$

$$D_* u + \frac{inv}{r} \;=\; D_* u \,+\, \frac{in}{r}\left(\frac{inr(D_* u)}{n^2 + k^2 r^2} \,-\, \frac{k^2 r^2 (D_* V)u}{\gamma(n^2 + k^2 r^2)}\right)$$

$$= \left(1 - \frac{n^2}{n^2 + k^2 r^2}\right) D_* u \,-\, \frac{ink^2 r (D_* V)u}{\gamma(n^2 + k^2 r^2)} \;=\; \frac{k^2 r^2 (D_* u)}{n^2 + k^2 r^2} \,-\, \frac{ink^2 r(D_* V)u}{\gamma(n^2 + k^2 r^2)}$$

$$\gamma u \,-\, 2\Omega\left(\frac{inr(D_* u)}{n^2 + k^2 r^2} \,-\, \frac{k^2 r^2 (D_* V)u}{\gamma(n^2 + k^2 r^2)}\right)$$

$$= \gamma D\left(\frac{r^2 (D_* u)}{n^2 + k^2 r^2} \,-\, \frac{inr(D_* V)u}{\gamma(n^2 + k^2 r^2)}\right) + in\,(D\Omega)\left(\frac{r^2(D_* u)}{n^2 + k^2 r^2} \,-\, \frac{inr(D_* V)u}{\gamma(n^2 + k^2 r^2)}\right)$$

$$\gamma u \,-\, \frac{2\Omega inr(D_* u)}{n^2 + k^2 r^2} \,+\, \frac{k^2 r^2 \Phi u}{\gamma(n^2 + k^2 r^2)}$$

$$= \gamma D\left(\frac{r^2(D_* u)}{n^2 + k^2 r^2}\right) \,-\, in\gamma\left(\frac{1}{\gamma} D\left(\frac{r(D_* V)u}{n^2 + k^2 r^2}\right) \,+\, \frac{-in(D\Omega)}{\gamma^2}\cdot\frac{r(D_* V)u}{n^2 + k^2 r^2}\right)$$

$$+\, \frac{inr^2(D\Omega)(D_* u)}{n^2 + k^2 r^2} \,+\, \frac{n^2 r(D\Omega)(D_* V)u}{\gamma(n^2 + k^2 r^2)}$$

$$= \gamma D\left(\frac{r^2(D_* u)}{n^2 + k^2 r^2}\right) - in\left(D\left(\frac{D_* V}{n^2 + k^2 r^2}\right) ru + \frac{(D_* V)D(ru)}{n^2 + k^2 r^2}\right) + \frac{inr^2(D\Omega)(D_* u)}{n^2 + k^2 r^2}$$

$$D(ru) \,=\, rDu \,+\, u \,=\, r(D_*u)$$

$$\gamma D\left(\frac{r^2(D_*u)}{n^2+k^2r^2}\right) \,-\, inD\left(\frac{D_*V}{n^2+k^2r^2}\right)ru \,-\, \gamma u - \frac{k^2r^2\Phi u}{\gamma(n^2+k^2r^2)}$$

$$= -\,\frac{2\Omega inr(D_*u)}{n^2+k^2r^2} \,+\, \frac{in(D_*V)(rD_*u)}{n^2+k^2r^2} \,-\, \frac{inr^2(D\Omega)(D_*u)}{n^2+k^2r^2}$$

$$= \frac{inr(D_*u)}{n^2+k^2r^2}\underbrace{\big(-2\Omega \,+\, D_*V \,-\, r(D\Omega)\big)}$$

$$= -2\,\frac{V}{r} \,+\, DV \,+\, \frac{V}{r} \,-\, rD\left(\frac{V}{r}\right)$$

$$= -2\,\frac{V}{r} \,+\, DV \,+\, \frac{V}{r} \,-\, r\left(-\,\frac{V}{r^2} \,+\, \frac{DV}{r}\right) \,=\, 0 \qquad \underline{\text{ok}}$$

<u>inviscid linear stability</u> : axisymmetric perturbations

recall : $\gamma \,=\, s \,+\, in\Omega$

$$\gamma^2 D\left(\frac{r^2(D_*u)}{n^2+k^2r^2}\right) - \left(\gamma^2 \,+\, \frac{k^2r^2\Phi}{n^2+k^2r^2} \,+\, in\gamma rD\left(\frac{D_*V}{n^2+k^2r^2}\right)\right)u \,=\, 0$$

$$n \,=\, 0 \quad \Rightarrow \quad \gamma \,=\, s \quad , \quad \big(DD_* - k^2\big)u \,-\, \frac{k^2}{s^2}\Phi\, u \,=\, 0$$

$$Lu \,=\, \lambda u \quad , \quad L \,=\, \Phi^{-1}\big(DD_* - k^2\big) \quad , \quad \lambda \,=\, \frac{k^2}{s^2}$$

$$D(L) \,=\, \big\{u\,, u' \in L^2[R_1, R_2]\;,\; u(R_1) = u(R_2) = 0\big\} \quad , \quad <u\,, v> \,=\, \int_{R_1}^{R_2} uv\, r\Phi\, dr$$

1. If $\Phi > 0$, this is a regular <u>Sturm-Liouville problem</u>, i.e. L is self-adjoint (hw) and there is a countable sequence of eigenvalues $\lambda_j < 0$ st $\lambda_j \to -\infty$ as $j \to \infty$. Hence s_j is imaginary and the flow is marginally stable, so Rayleigh's stability criterion for axisymmetric perturbations is verified.

2. If $\Phi < 0$ or Φ changes sign, then $\lambda_j > 0$ for some j. Hence s_j may be positive or negative, and the flow is unstable.

special case : $V(r) = r\Omega$, $\Omega =$ constant (Kelvin)

Since $\Phi = 2\Omega(D_*V) = 4\Omega^2 > 0$, we know by Raleigh's criterion that the flow is stable wrt axisymmetric perturbations, so we will consider 3D perturbations. We already derived the equation for u, but the equation for p is simpler in this case.

$$\gamma u - 2\Omega\, v = -Dp$$

$$\gamma v + (D_*V)u = -\frac{in}{r}\,p$$

$$\gamma w = -ikp$$

$$D_*u + \frac{in}{r}\,v + ikw = 0$$

$$V = r\Omega \quad , \quad D_*V = DV + \frac{V}{r} = \Omega + \Omega = 2\Omega$$

$$\begin{pmatrix} \gamma & -2\Omega \\ 2\Omega & \gamma \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -Dp \\ -\frac{in}{r}\,p \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\gamma^2 + 4\Omega^2} \begin{pmatrix} \gamma & 2\Omega \\ -2\Omega & \gamma \end{pmatrix} \begin{pmatrix} -Dp \\ -\frac{inp}{r} \end{pmatrix} = \frac{1}{\gamma^2 + 4\Omega^2} \begin{pmatrix} -\gamma Dp - \frac{2in\Omega p}{r} \\ 2\Omega Dp - \frac{in\gamma p}{r} \end{pmatrix}$$

$$D_*\left(-\gamma Dp - \frac{2in\Omega p}{r}\right) + \frac{in}{r}\left(2\Omega Dp - \frac{in\gamma p}{r}\right) + \left(\gamma^2 + 4\Omega^2\right)ik\left(\frac{-ikp}{\gamma}\right) = 0$$

$$-\gamma D_*Dp - 2in\Omega D_*\left(\frac{p}{r}\right) + \frac{2in\Omega Dp}{r} + \frac{n^2\gamma p}{r^2} + \left(\gamma^2 + 4\Omega^2\right)\frac{k^2 p}{\gamma} = 0$$

$$D_*\left(\frac{p}{r}\right) = D\left(\frac{p}{r}\right) + \frac{p}{r^2} = \frac{Dp}{r} - \frac{p}{r^2} + \frac{p}{r^2} = \frac{Dp}{r}$$

$$D_*Dp - \frac{n^2}{r^2}\,p = k^2\left(1 + \frac{4\Omega^2}{\gamma^2}\right)p$$

bc : $u = 0$ on $r = R_1$, R_2 $\quad\Rightarrow\quad$ $\gamma Dp + \dfrac{2in\Omega}{r}\,p = 0$ on $r = R_1$, R_2

<u>solution</u>

$$D_* D = D^2 + \frac{D}{r} \quad : \text{ radial Laplacian}$$

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = -\lambda f$$

look for $f(r,\theta) = p(r)e^{in\theta}$

$$D^2 p + \frac{1}{r} Dp + \frac{1}{r^2}(in)^2 p = D_* Dp - \frac{n^2}{r^2} p = -\lambda p \quad : \text{ above}$$

$$r^2 D^2 p + rDp + \big(\lambda r^2 - n^2\big)p = 0$$

$$x = \sqrt{\lambda}\, r \quad , \quad p(r) = g(x) \quad \Rightarrow \quad Dp = \frac{dg}{dx}\sqrt{\lambda}$$

$$\left(\frac{x}{\sqrt{\lambda}}\right)^2 \frac{d^2 g}{dx^2}\lambda + \frac{x}{\sqrt{\lambda}}\frac{dg}{dx}\sqrt{\lambda} + \left(\lambda\left(\frac{x}{\sqrt{\lambda}}\right)^2 - n^2\right)g = 0$$

$$\Rightarrow \quad x^2 \frac{d^2 g}{dx^2} + x\frac{dg}{dx} + \big(x^2 - n^2\big) = 0 \quad : \quad \underline{\text{Bessel equation}} \quad , \quad J_n(x)\ ,\ J_{-n}(x)$$

<u>note</u>

The eigenvalues are purely imaginary and hence the perturbed flow undergoes marginally stable oscillations. (hw)

<u>inviscid linear stability</u> : 2D perturbations

recall : $\gamma = s + in\Omega$

$$\gamma^2 D\left(\frac{r^2(D_* u)}{n^2 + k^2 r^2}\right) - \left(\gamma^2 + \frac{k^2 r^2 \Phi}{n^2 + k^2 r^2} + in\gamma r D\left(\frac{D_* V}{n^2 + k^2 r^2}\right)\right) u = 0$$

$$k = 0 \quad \Rightarrow \quad \gamma^2 D\left(\frac{r^2(D_* u)}{n^2}\right) - \left(\gamma^2 + in\gamma r D\left(\frac{D_* V}{n^2}\right)\right) u = 0$$

$$D(ru) = r(D_* u)$$

$$D(r^2(D_* u)) = D(r \cdot r(D_* u)) = D(r \cdot D(ru)) = r D_* D(ru)$$

$$\phi = ru \quad \Rightarrow \quad \gamma^2 r D_* D\phi - \gamma^2 n^2 \frac{\phi}{r} - in\gamma(DD_* V)\phi = 0$$

$$\gamma\left(D_* D - \frac{n^2}{r^2}\right)\phi - \frac{in(DD_* V)}{r}\phi = 0$$

<u>note</u>

1. z-component of vorticity in polar coordinates : $\omega_z = \dfrac{\partial u_\theta}{\partial r} + \dfrac{u_\theta}{\partial r} - \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}$

Hence for the basic flow we have $\omega_z = DV + \dfrac{V}{r} = D_* V$.

2. $\phi(R_1) = \phi(R_2) = 0 \quad \Rightarrow \quad \text{real}(s) \cdot n \displaystyle\int_{R_1}^{R_2} \frac{(DD_* V)|\phi|^2}{|s + in\Omega|^2}\, dr = 0$ (hw)

<u>thm</u> (Rayleigh)

A necessary condition for inviscid instability of an axisymmetric swirl flow wrt 2D perturbations is that $DD_* V$ should change sign in the interval $R_1 < r < R_2$, i.e. the basic vorticity $D_* V$ should have a local max or local min in the interval $R_1 < r < R_2$.

<u>note</u>

This is an analogue of a well-known result for planar shear flow. (more later) viscous flow

$$\frac{Du_r}{Dt} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)$$

$$\frac{Du_\theta}{Dt} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + \nu\left(\Delta u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right)$$

$$\frac{Du_z}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\Delta u_z$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z} \quad , \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

basic flow

$$u_r = u_z = 0 \quad , \quad u_\theta = V(r) \quad , \quad p = P(r)$$

$$-\frac{V^2}{r} = -\frac{1}{\rho}\frac{dP}{dr} \quad \Rightarrow \quad P = \rho\int\frac{V^2}{r}\,dr \ : \ \text{as before}$$

$$\Delta u_\theta - \frac{u_\theta}{r^2} = 0 \quad \Rightarrow \quad D^2V + \frac{DV}{r} - \frac{V}{r^2} = 0$$

$$V(r) = r^\alpha \quad \Rightarrow \quad \alpha(\alpha-1) + \alpha - 1 = 0 \quad \Rightarrow \quad \alpha = \pm 1$$

$$V(r) = Ar + \frac{B}{r} \quad \Rightarrow \quad \Omega(r) = A + \frac{B}{r^2}$$

Couette flow : viscous flow between 2 cylinders , $R_1 \le r \le R_2$

$$\Omega(R_1) = \Omega_1 \quad \Rightarrow \quad A + \frac{B}{R_1^2} = \Omega_1 \quad \Rightarrow \quad AR_1^2 + B = \Omega_1 R_1^2$$

$$\Omega(R_2) = \Omega_2 \quad \Rightarrow \quad A + \frac{B}{R_2^2} = \Omega_2 \quad \Rightarrow \quad AR_2^2 + B = \Omega_2 R_2^2$$

$$\begin{pmatrix} R_1^2 & 1 \\ R_2^2 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \Omega_1 R_1^2 \\ \Omega_2 R_2^2 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{R_1^2 - R_2^2}\begin{pmatrix} 1 & -1 \\ -R_2^2 & R_1^2 \end{pmatrix}\begin{pmatrix} \Omega_1 R_1^2 \\ \Omega_2 R_2^2 \end{pmatrix}$$

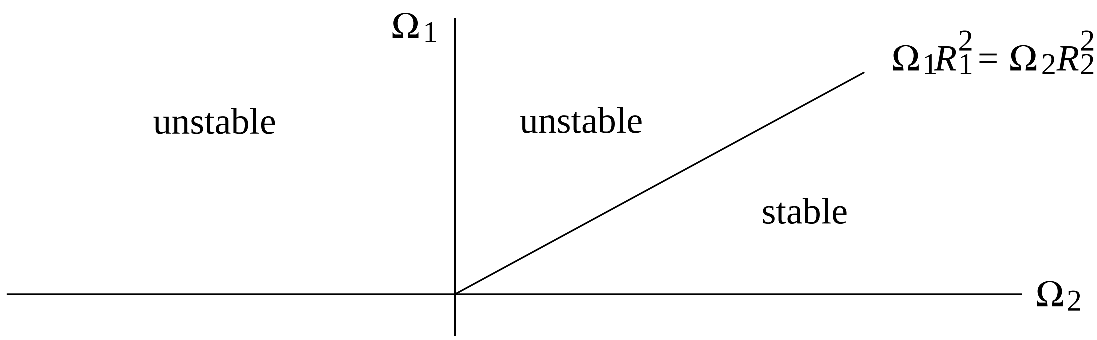
$$A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} \quad , \quad B = \frac{R_1^2 R_2^2 (\Omega_1 - \Omega_2)}{R_2^2 - R_1^2}$$

<u>axisymmetric inviscid perturbations</u> : Rayleigh's criterion

$$\Phi = \frac{1}{r^3} D\Big(\big(r^2\Omega\big)^2\Big) = \frac{2r^2\Omega}{r^3} D(Ar^2 + B) = 4A\Omega = 4\,\frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}\,\Omega(r)$$

<u>case 1</u> : Ω_1 , Ω_2 have the same sign

WLOG assume $\Omega_1 > 0, \Omega_2 > 0$. Then $\Omega(r) > 0$ for all r st $R_1 \le r \le R_2$. This follows from $\Omega'(r) = -2B/r^3$; if $B \ne 0$, then $\Omega(r)$ has it's max and min at the endpoints, while if $B = 0$, then $\Omega_1 = \Omega_2 = \Omega(r)$. Then the flow is stable $\Leftrightarrow \ \Phi > 0 \ \Leftrightarrow \ \Omega_2 R_2^2 > \Omega_1 R_1^2$.



1. For given Ω_1, R_1, R_2, the flow is stable $\Leftrightarrow$ $\Omega_2 > \Omega_1 R_1^2 / R_2^2$.

2. For given Ω_1, Ω_2, R_1, the flow is stable $\Leftrightarrow$ $R_2^2 > \Omega_1 R_1^2 / \Omega_2$.

3. Kelvin's example : $\Omega_1 = \Omega_2 \quad \Rightarrow \quad$ stable

<u>case 2</u> : Ω_1 , Ω_2 have opposite sign

Then there is an r_0 st $\Omega(r_0) = 0, R_1 < r_0 < R_2$. WLOG assume $\Omega_2 > 0 > \Omega_1$. Then the flow is stable $\Leftrightarrow$ $\Phi > 0$ $\Leftrightarrow$ $\Omega(r) > 0$ $\Leftrightarrow$ $r_0 < r \le R_2$. Hence the flow is unstable for $R_1 \le r \le r_0$, i.e. in a layer adjacent to the inner cylinder.

<u>2D inviscid perturbations</u> : Rayleigh's thm

$$V = Ar + \frac{B}{r} \;\Rightarrow\; D_*V = A - \frac{B}{r^2} + A + \frac{B}{r^2} = 2A \;\Rightarrow\; D_*DV = 0$$

$\Rightarrow$ Rayleigh's thm doesn't apply

$$(s + in\Omega)\left(D_*D - \frac{n^2}{r^2}\right)\phi = 0$$

<u>case 1</u> : $s + in\Omega(r) \neq 0$ for $R_1 \leq r \leq R_2$

$$\left(D_*D - \frac{n^2}{r^2}\right)\phi = D^2\phi + \frac{D\phi}{r} - \frac{n^2}{r^2}\phi = 0$$

$$\phi = r^\alpha \;\Rightarrow\; \alpha(\alpha-1) + \alpha - n^2 = 0 \;\Rightarrow\; \alpha = \pm n \;,\; \phi = c_1 r^n + c_2 r^{-n}$$

$$\phi(R_1) = \phi(R_2) = 0 \;\Rightarrow\; \cdots \;\Rightarrow\; c_1 = c_2 = 0$$

$\Rightarrow$ There are no eigenvalues for Couette flow, i.e. there is no <u>discrete spectrum</u>.

<u>case 2</u> : $s + in\Omega(r_0) = 0$ for some r_0 st $R_1 \leq r_0 \leq R_2$

There is a <u>continuous spectrum</u> of stable singular eigenfunctions associated with the <u>critical layer</u> at $r = r_0$. (more later)

<u>note</u>

1. $xf(x) = 0 \;\Rightarrow\; f(x) = \delta(x)$ <u>pf</u> : $\displaystyle\int_{-\infty}^{\infty} x\delta(x)\phi(x)\,dx = x\phi(x)\Big|_0 = 0$ <u>ok</u>

2. $\phi'' = \lambda\phi \;\Rightarrow\; \phi = e^{ikx} \;,\; \lambda = -k^2$

a) periodic bc on $0 \leq x \leq 2\pi \;\Rightarrow\; k = 0, \pm 1, \pm 2, \ldots$: discrete spectrum

Fourier series , $\displaystyle f(x) = \sum_{k=-\infty}^{\infty} \widehat{f}_k\, e^{ikx}$

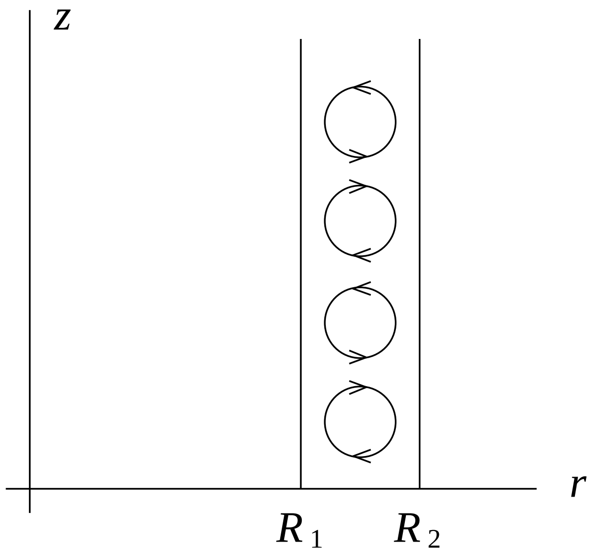
b) free-space bc on $-\infty < x < \infty \;\Rightarrow\; -\infty < k < \infty$: continuous spectrum

Fourier transform , $\displaystyle f(x) = \int_{-\infty}^{\infty} \widehat{f}(k)\, e^{ikx} dk$

Taylor's experiment (1921) : $\Omega_1 > 0$, $\Omega_2 = 0$: unstable by inviscid theory

small Ω_1 : Couette flow

larger Ω_1 : periodic array of counter-rotating axisymmetric vortex rings



: Taylor vortices

questions : Since inviscid theory predicts that this particular Couette flow is unstable, why is it seen in the experiment? What determines the wavelength and amplitude of the Taylor vortices? Taylor (1923) performed a theoretical stability analysis to answer these questions.

linearized equations : viscous axisymmetric perturbations

$$\frac{\partial u_r'}{\partial t} + \Omega \frac{\partial u_r'}{\partial \theta} - 2\,\Omega\, u_\theta' = -\frac{1}{\rho}\frac{\partial p'}{\partial r} + \nu\left(\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)$$

$$\frac{\partial u_\theta'}{\partial t} + \Omega \frac{\partial u_\theta'}{\partial \theta} + (D_* V) u_r' = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p'}{\partial \theta} + \nu\left(\Delta u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right)$$

$$\frac{\partial u_z'}{\partial t} + \Omega \frac{\partial u_z'}{\partial \theta} = -\frac{1}{\rho}\frac{\partial p'}{\partial z}$$

$$\frac{\partial u_r'}{\partial r} + \frac{u_r'}{r} + \frac{1}{r}\frac{\partial u_\theta'}{\partial \theta} + \frac{\partial u_z'}{\partial z} = 0$$

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

<u>note</u>

$$\frac{\partial}{\partial r}\Delta = \frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) = \frac{\partial^3}{\partial r^3} + \frac{1}{r}\frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\frac{\partial}{\partial r} + \frac{\partial^3}{\partial r \partial z^2}$$

$$= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}\right)\frac{\partial}{\partial r} = \Delta_*\frac{\partial}{\partial r}$$

$$\Delta_* = \Delta - \frac{1}{r^2}$$

$$\left(\frac{\partial}{\partial t} - \nu\Delta_*\right)u_r' - 2\,\Omega\,u_\theta' = -\frac{1}{\rho}\frac{\partial p'}{\partial r}$$

$$\left(\frac{\partial}{\partial t} - \nu\Delta_*\right)\frac{\partial^2 u_r'}{\partial z^2} - 2\,\Omega\,\frac{\partial^2 u_\theta'}{\partial z^2} = -\frac{1}{\rho}\frac{\partial^3 p'}{\partial r \partial z^2} = \frac{\partial^2}{\partial r \partial z}\left(-\frac{1}{\rho}\frac{\partial p'}{\partial z}\right)$$

$$= \frac{\partial^2}{\partial r \partial z}\left(\frac{\partial}{\partial t} - \nu\Delta\right)u_z' = \frac{\partial}{\partial r}\left(\frac{\partial}{\partial t} - \nu\Delta\right)\frac{\partial u_z'}{\partial z}$$

$$= -\frac{\partial}{\partial r}\left(\frac{\partial}{\partial t} - \nu\Delta\right)\left(\frac{\partial u_r'}{\partial r} + \frac{u_r'}{r}\right) = -\left(\frac{\partial}{\partial t} - \nu\Delta_*\right)\frac{\partial}{\partial r}\left(\frac{\partial u_r'}{\partial r} + \frac{u_r'}{r}\right)$$

$$\left(\frac{\partial}{\partial t} - \nu\Delta_*\right)\left(\frac{\partial^2 u_r'}{\partial r^2} + \frac{1}{r}\frac{\partial u_r'}{\partial r} - \frac{u_r'}{r^2} + \frac{\partial u_r'}{\partial z^2}\right) = 2\,\Omega\,\frac{\partial^2 u_\theta'}{\partial z^2}$$

$$\left.\begin{aligned} &\left(\frac{\partial}{\partial t} - \nu\Delta_*\right)\Delta_* u_r' = 2\,\Omega\,\frac{\partial^2 u_\theta'}{\partial z^2} \\ &\left(\frac{\partial}{\partial t} - \nu\Delta_*\right)u_\theta' = -(D_*V)u_r' \end{aligned}\right\} : \text{ coupled PDEs for } u_r',\ u_\theta'$$

<u>normal modes</u>

$$\big(u_r', u_\theta'\big) = \big(u(r), v(r)\big)e^{st+ikz}$$

note : $D^2u + \dfrac{Du}{r} - \dfrac{u}{r^2} = D\left(Du + \dfrac{u}{r}\right) = DD_*u$

$$\big(s - \nu\big(DD_* - k^2\big)\big)\big(DD_* - k^2\big)u = 2\,\Omega \cdot - k^2\,v$$

$$\big(s - \nu\big(DD_* - k^2\big)\big)v = -(D_* V)u$$

$$\left.\begin{aligned}&\big(\nu\big(DD_* - k^2\big) - s\big)\big(DD_* - k^2\big)u = 2\,k^2\,\Omega\,v\\&\big(\nu\big(DD_* - k^2\big) - s\big)v = (D_* V)u\end{aligned}\right\} \ : \ \text{coupled ODEs for } u\ ,\ v$$

<u>boundary conditions</u>

$u = v = 0$ on $r = R_1\ ,\ R_2$

$$\frac{\partial u_r'}{\partial r} + \frac{u_r'}{r} + \frac{\partial u_z'}{\partial z} = 0 \quad \Rightarrow \quad Du = 0 \ \text{ on } \ r = R_1\ ,\ R_2$$

<u>note</u>

1. $\nu = 0 \quad \Rightarrow \quad -s\big(DD_* - k^2\big)u = 2\,k^2\,\Omega\,v = 2\,k^2\,\Omega \cdot \dfrac{(D_* V)u}{-s}$

$$\Rightarrow \quad \big(DD_* - k^2\big)u - \frac{k^2}{s^2}\Phi\,u = 0 \ : \ \text{as before}$$

2. $V = Ar + \dfrac{B}{r} \quad \Rightarrow \quad D_* V = A - \dfrac{B}{r^2} + A + \dfrac{B}{r^2} = 2A < 0$ in Taylor's experiment

<u>thin gap approximation</u>

$d = R_2 - R_1 << R_1 \quad , \quad r = R_1 + \xi d \quad , \quad 0 \le \xi \le 1$

$$A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} = \frac{\Omega_2 (R_1 + d)^2 - \Omega_1 R_1^2}{d(2R_1 + d)} \sim \frac{\Omega_2 (R_1^2 + 2R_1 d) - \Omega_1 R_1^2}{2R_1 d(1 + d/2R_1)}$$

$$\sim \frac{(\Omega_2 - \Omega_1)R_1^2 + 2\Omega_2 R_1 d}{2R_1 d}\big(1 - d/2R_1\big)$$

$$= \frac{(\Omega_2 - \Omega_1)R_1^2 + 2\Omega_2 R_1 d - \frac{1}{2}(\Omega_2 - \Omega_1)R_1 d}{2R_1 d}$$

$$= \frac{(\Omega_2 - \Omega_1)R_1}{2d} + \Omega_2 - \frac{1}{4}(\Omega_2 - \Omega_1)$$

$$B \;=\; \frac{R_1^2 R_2^2(\Omega_1-\Omega_2)}{R_2^2-R_1^2} \;=\; \frac{R_1^2(R_1+d)^2(\Omega_1-\Omega_2)}{2R_1 d}\left(1-d/2R_1\right)$$

$$\sim\; \frac{(\Omega_1-\Omega_2)R_1(R_1^2+2R_1 d-\frac{1}{2}R_1 d)}{2d} \;=\; \frac{(\Omega_1-\Omega_2)R_1^3}{2d} \;+\; \frac{3}{4}(\Omega_1-\Omega_2)R_1^2$$

$$\frac{1}{r^2} \;=\; \frac{1}{(R_1+\xi d)^2} \;=\; \frac{1}{R_1^2(1+\xi d/R_1)^2} \;\sim\; \frac{1}{R_1^2}\left(1 \,-\, \frac{2\xi d}{R_1}\right)$$

$$\frac{B}{r^2} \;\sim\; \left(\frac{(\Omega_1-\Omega_2)R_1^3}{2d} \;+\; \frac{3}{4}(\Omega_1-\Omega_2)R_1^2\right)\frac{1}{R_1^2}\left(1 \,-\, \frac{2\xi d}{R_1}\right)$$

$$=\; \frac{(\Omega_1-\Omega_2)R_1}{2d} \;+\; \frac{3}{4}(\Omega_1-\Omega_2) \,-\, (\Omega_1-\Omega_2)\xi$$

$$\Omega(r) \;=\; A \,+\, \frac{B}{r^2} \;\sim\; \frac{(\Omega_2-\Omega_1)R_1}{2d} \,+\, \Omega_2 \,-\, \frac{1}{4}(\Omega_2-\Omega_1)$$

$$+\; \frac{(\Omega_1-\Omega_2)R_1}{2d} \;+\; \frac{3}{4}(\Omega_1-\Omega_2) \,-\, (\Omega_1-\Omega_2)\xi$$

$$=\; \Omega_2 \,-\, (\Omega_2-\Omega_1) \,-\, (\Omega_1-\Omega_2)\xi \;=\; \Omega_1 \,+\, (\Omega_2-\Omega_1)\xi$$

$$\Omega(r) \;\sim\; \Omega_1(1 \,+\, \alpha\xi) \quad , \quad \alpha \;=\; \frac{\Omega_2-\Omega_1}{\Omega_1}$$

$$D \;=\; \frac{d}{dr} \;=\; \frac{d}{d\xi}\,\frac{d\xi}{dr} \;=\; \frac{\widetilde{D}}{d} \quad , \quad \widetilde{D} \;=\; \frac{d}{d\xi} \quad , \quad D_* \;=\; D+\frac{1}{r} \;=\; \frac{\widetilde{D}}{d}+\frac{1}{R_1+\xi d} \;\sim\; \frac{\widetilde{D}}{d}$$

$$\left(\nu\left(\frac{\widetilde{D}^2}{d^2} \,-\, k^2\right) \,-\, s\right)\left(\frac{\widetilde{D}^2}{d^2} \,-\, k^2\right)u \;=\; 2\,k^2\,\Omega_1(1 \,+\, \alpha\xi)\,v$$

$$\left(\nu\left(\frac{\widetilde{D}^2}{d^2} \,-\, k^2\right) \,-\, s\right)v \;=\; 2Au$$

$$\left(\left(\widetilde{D}^2 \,-\, k^2 d^2\right) \,-\, \frac{sd^2}{\nu}\right)\left(\widetilde{D}^2 \,-\, k^2 d^2\right)u \;=\; \frac{2k^2 d^4\Omega_1}{\nu}\,(1 \,+\, \alpha\xi)\,v$$

$$\left(\left(\widetilde{D}^2 \,-\, k^2 d^2\right) \,-\, \frac{sd^2}{\nu}\right)v \;=\; \frac{2Ad^2}{\nu}\,u$$

$kd = a$: nondimensional axial wavenumber

$\dfrac{sd^2}{\nu} = \sigma$: ...... " ...... growth rate

$$u = \frac{2k^2d^4\Omega_1}{\nu}\,\widetilde{u} = \frac{2d^2\Omega_1}{\nu}\,a^2\,\widetilde{u}$$

$$\big(\widetilde{D}^2 - a^2 - \sigma\big)\big(\widetilde{D}^2 - a^2\big)\frac{2d^2\Omega_1}{\nu}\,a^2\,\widetilde{u} = \frac{2d^2\Omega_1}{\nu}\,a^2\,(1 + \alpha\xi)\,v$$

$$\big(\widetilde{D}^2 - a^2 - \sigma\big)v = \frac{2Ad^2}{\nu}\cdot\frac{2d^2\Omega_1}{\nu}\,a^2\,\widetilde{u}$$

drop tildes

$$\big(D^2 - a^2 - \sigma\big)\big(D^2 - a^2\big)u = (1 + \alpha\xi)\,v$$

$$\big(D^2 - a^2 - \sigma\big)v = -T\,a^2\,u$$

$$T = \frac{-4Ad^4\Omega_1}{\nu^2} \quad : \quad \underline{\text{Taylor number}} \quad , \quad A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}$$

<u>recall</u> : thermal convection

$$\Big(D^2 - a^2 - \frac{s}{Pr}\Big)\Big(D^2 - a^2\Big)W = a^2 R\,T$$

$$\big(D^2 - a^2 - s\big)T = -W$$

$$D = \frac{d}{dz} \quad , \quad a \ : \ \text{horizontal wavenumber} \quad , \quad T \ : \ \text{temperature}$$

<u>note</u>

1. Rayleigh's criterion for the stability of axisymmetric inviscid perturbations says that $\Omega_2 R_2^2 > \Omega_1 R_1^2$, which is equivalent to $T < 0$. Next we will show that viscosity stabilizes the flow for $0 < T < T_c$.

2. To determine the curve of marginal stability, we will assume that the PES is satisfied and set $\sigma = 0$.

$$\left.\begin{aligned}&\big(D^2 - a^2\big)^2 u = (1 + \alpha\xi)\, v\\&\big(D^2 - a^2\big) v = -T_0\, a^2\, u\\&\text{bc} : u = Du = v = 0 \text{ at } \xi = 0 , 1\end{aligned}\right\} \Rightarrow \quad T_0 = T_0(a)$$

<u>note</u>

One approach is to replace the function $1 + \alpha\xi$ by its average value $1 + \alpha/2$. Instead we will consider a more general method due to Chandrasekhar.

$$v = \sum_{m=1}^{\infty} c_m \sin m\pi\xi$$

$$\Rightarrow \quad \big(D^2 - a^2\big)^2 u = (1 + \alpha\xi) \sum_{m=1}^{\infty} c_m \sin m\pi\xi \quad , \quad u = Du = 0 \text{ at } \xi = 0 , 1$$

$$\Rightarrow \quad u = \sum_{n=1}^{\infty} c_n\, u_n(\xi)$$

$$\Rightarrow \quad \big(D^2 - a^2\big)^2 u_n = (1 + \alpha\xi) \sin n\pi\xi \quad , \quad u_n = Du_n = 0 \text{ at } \xi = 0 , 1$$

$$u_n(\xi) = \sum_{m=1}^{\infty} A_{mn} \sin m\pi\xi \quad , \quad A_{mn} = 2 \int_0^1 u_n(\xi) \sin m\pi\xi \, d\xi$$

$$\big(D^2 - a^2\big) v = -T_0\, a^2\, u$$

$$\Rightarrow \quad -\sum_{m=1}^{\infty} \big(m^2\pi^2 + a^2\big) c_m \sin m\pi\xi = -T_0\, a^2 \sum_{n=1}^{\infty} c_n \sum_{m=1}^{\infty} A_{mn} \sin m\pi\xi$$

$$\Rightarrow \quad \big(m^2\pi^2 + a^2\big) c_m = T_0\, a^2 \sum_{n=1}^{\infty} A_{mn} c_n \quad , \quad m = 1 , 2 , \ldots$$

This is an infinite-dimensional homogeneous system of linear equations for the coefficients $\{c_m\}$. In this system, T_0 plays the role of an eigenvalue and we can obtain an approximation $T_0^{(m)}$ by truncation.

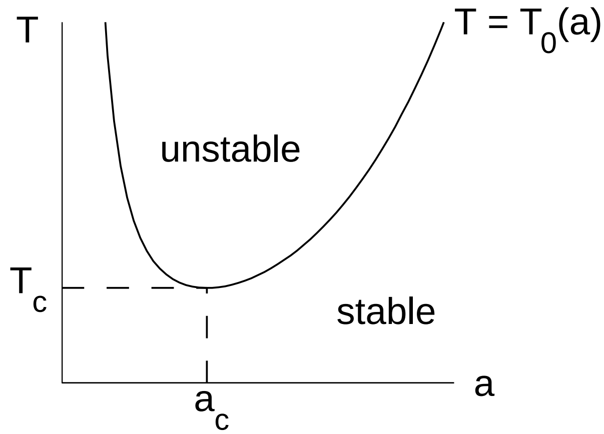
ex : $m = n = 1$, $v = c_1 \sin \pi\xi$, $u = c_1 u_1(\xi)$

$$\left(D^2 - a^2\right)^2 u_1 = (1 + \alpha\xi) \sin \pi\xi \quad , \quad u_1 = Du_1 = 0 \ \text{ at } \ \xi = 0\,,\ 1$$

$$u_1(\xi) = \cdots \quad , \quad A_{11} = 2\int_0^1 u_1(\xi) \sin \pi\xi \, d\xi = \cdots$$

$$\left(\pi^2 + a^2\right) c_1 = T_0^{(1)}\, a^2\, A_{11} c_1 \quad \Rightarrow \quad T_0^{(1)} = \frac{\pi^2 + a^2}{a^2\, A_{11}}$$

$$\Rightarrow \quad T_0^{(1)} = \frac{2}{2+\alpha} \cdot \frac{(\pi^2 + a^2)^3}{a^2(1 - 16\pi^2 a(\cosh^2(a/2))/((\pi^2+a^2)^2(\sinh a + a)))}$$



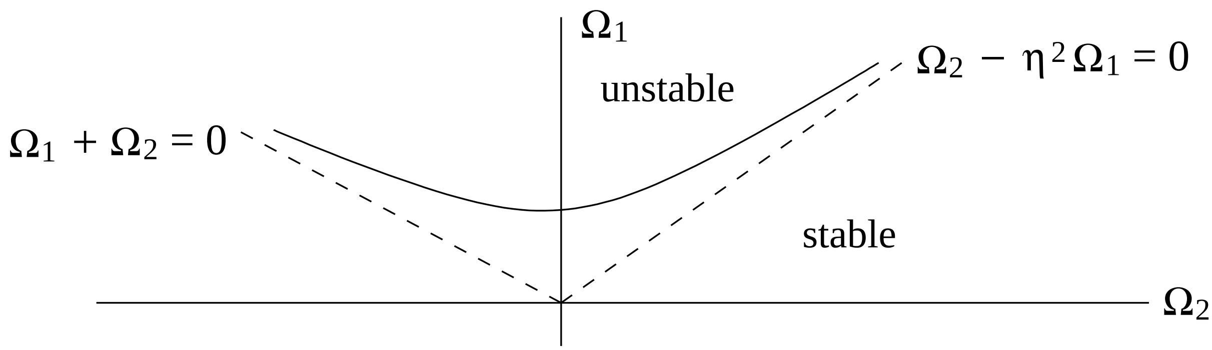
$$T_0 \sim a^{-2} \ \text{ as } \ a \to 0$$

$$T_0 \sim a^4 \ \text{ as } \ a \to \infty$$

$$T_c = \frac{2}{2+\alpha} \quad , \quad a_c = 3.12$$

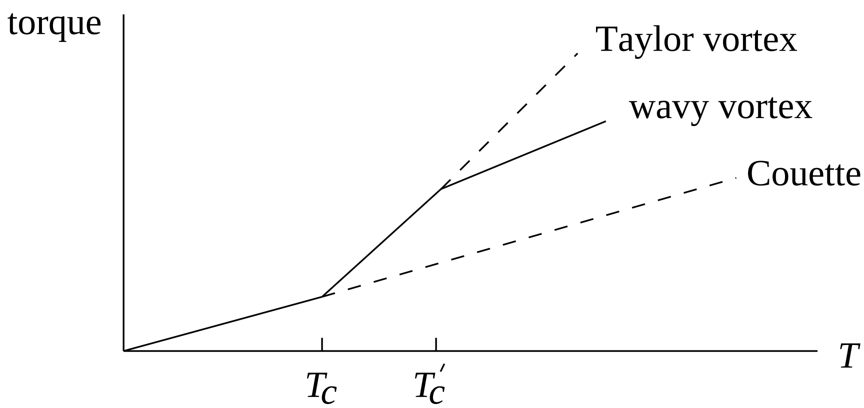
$$T_c = \min_a T_0 \sim \frac{2}{2 + \dfrac{\Omega_2 - \Omega_1}{\Omega_1}} \cdot 1715 = \frac{-4d^4\Omega_1}{\nu^2} \cdot \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} \quad , \quad \eta = \frac{R_1}{R_2}$$

$$\Rightarrow \quad \Omega_2 - \eta^2\Omega_1 = \frac{-\nu^2(1-\eta^2)}{4d^4} \cdot \frac{3430}{\Omega_1 + \Omega_2}$$



<u>further experiments</u> (Coles)

Couette flow : axisymmetric , uniform in z
$\downarrow$
Taylor vortices : periodic in z
$\downarrow$
wavy vortices : periodic in z , θ
$\downarrow$
chaos/turbulence : aperiodic



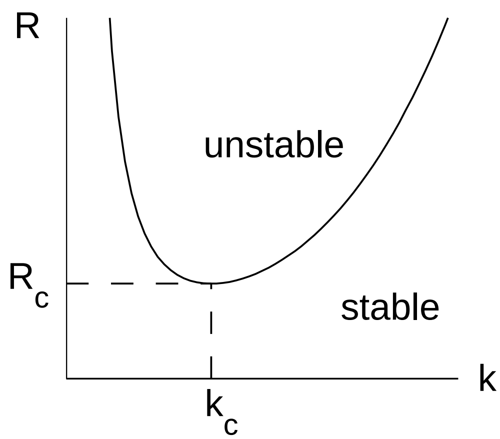
<u>weakly nonlinear analysis of Taylor-Couette flow</u> (Stuart)

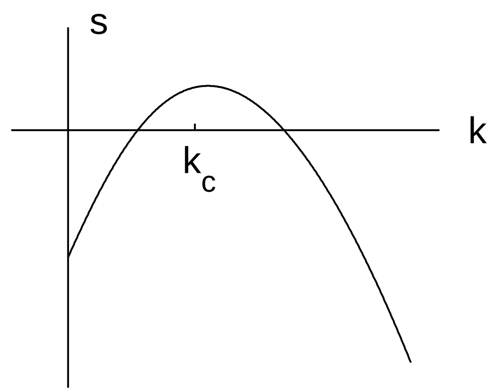
$u_r' = u(r)e^{st+ikz} \to u(r)A(\tau)e^{ik_c z}$, τ : slow time

$$\frac{dA}{d\tau} = sA - \gamma A|A|^3 \ : \ \text{Landau equation}$$

$T > T_c \quad \Rightarrow \quad \gamma > 0$: supercritical pitchfork bifurcation modulation theory (lecture notes of John Neu)

The marginal stability curve for thermal convection (and Taylor-Couette flow) has the following form.





note

1. For $R > R_c$, there is a band of unstable wavenumbers about $k = k_c$.

$$Pr = 1 \quad , \quad R > R_c \quad \Rightarrow \quad s \, = \, -(k^2 + \pi^2) + \frac{\sqrt{R}\,|k|}{(k^2 + \pi^2)^{1/2}}$$

2. Previously we considered the growth or decay of individual normal modes using linear stability theory. In the slightly supercritical regime, we found weakly nonlinear solutions of the form $A(\tau)e^{ik_c x}$, where $\tau = \epsilon^2 t$ is a slow time variable and the amplitude satisfies a Landau equation $A_\tau = sA - \gamma A^3$. A supercritical pitchfork bifurcation occurs at $R = R_c$ and for $R > R_c$ there are stable nonzero solutions of the Landau equation corresponding to convection cells (and Taylor vortices).

3. The normal modes of linear stability theory and the solutions given by the Landau theory are spatially periodic functions of a single wavenumber. However as the value of R increases, more modes with different wavenumbers become unstable and the theory must be extended to account for interactions that occur among these modes.

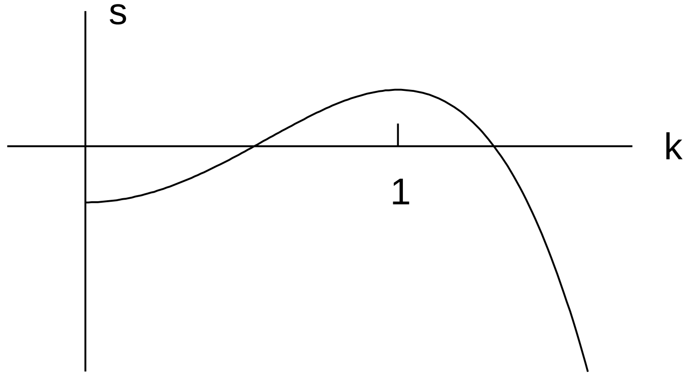
model problem : Swift-Hohenberg equation

$$u_t \,=\, -\,au \,-\, \tfrac{1}{2}\,u_{xx} \,-\, \tfrac{1}{4}\,u_{xxxx} \,-\, \tfrac{4}{3}\,\epsilon^2\,u^3 \quad , \quad a > 0$$

The linearized problem about $u = 0$ has normal modes of the form $u = e^{st+ikx}$, where $s \,=\, -\,a \,-\, \frac{1}{2}(ik)^2 \,-\, \frac{1}{4}(ik)^4 \,=\, -\,a \,+\, \frac{1}{2}k^2 \,-\, \frac{1}{4}k^4$.

$$s(0) \,=\, -a \quad , \quad s'(k) \,=\, k \,-\, k^3 \,=\, 0 \;\;\Rightarrow\;\; k \,=\, 0 \,,\, \pm 1 \quad , \quad s(\pm 1) \,=\, \tfrac{1}{4} \,-\, a$$

$$s''(k) \,=\, 1 - 3k^2 \;\;\Rightarrow\;\; s''(0) \,>\, 0 \;:\; \text{min} \quad , \quad s''(\pm 1) \,<\, 0 \;:\; \text{max}$$



For $0 < a < \frac{1}{4}$, there is a band of unstable wavenumbers about $k = 1$.

perturbation theory

set $\frac{1}{4} \,-\, a \,=\, \kappa\,\epsilon^2 \quad , \quad \kappa > 0 \quad , \quad T \,=\, \epsilon^2 t$: slow time variable

$$s \,=\, 0 \quad \Rightarrow \quad \tfrac{1}{4}k^4 \,-\, \tfrac{1}{2}k^2 \,+\, a \,=\, 0$$

$$k^2 \,=\, \frac{\frac{1}{2} \pm \left(\frac{1}{4} - 4\cdot\frac{1}{4}\cdot a\right)^{1/2}}{\frac{1}{2}} \,=\, 1 \pm (1-4a)^{1/2} \,=\, 1 \pm (4\kappa\,\epsilon^2)^{1/2} \,=\, 1 \pm 2\kappa^{1/2}\epsilon$$

$\Rightarrow \quad k \,=\, 1 \,+\, O(\epsilon) \quad , \quad X \,=\, \epsilon\,x$: long space variable

Look for $u(x,t) \,=\, A(X,T)\cos(x + \psi(X,T)) \,+\, \epsilon^2 u'(x,T) \,+\, \cdots$, i.e. a slowly modulated spatial oscillation with carrier wavenumber $k=1$, amplitude A, and phase shift ψ. To find A and ψ, substitute into the SH eqn and retain terms up to order ϵ^2.

$$u_t \;=\; A_T\,\epsilon^2\cos(x+\psi(X,T)) \;+\; A\cdot -\sin(x+\psi(X,T))\cdot\psi_T\,\epsilon^2 \;+\;\cdots$$

$$\partial_x A \;=\; A_X\,\epsilon \quad , \quad \partial_x^2 A \;=\; A_{XX}\,\epsilon^2 \quad , \quad \ldots$$

$$\partial_x\cos(x+\psi) \;=\; -\,(1+\epsilon\psi_X)\sin(x+\psi)$$

$$\partial_x^2\cos(x+\psi) \;=\; -\,\epsilon^2\psi_{XX}\sin(x+\psi) \;-\; (1+\epsilon\psi_X)^2\cos(x+\psi)$$

$$\partial_x^3\cos(x+\psi) \;=\; -\,\epsilon^2\psi_{XX}(1+\epsilon\psi_X)\cos(x+\psi) \;-\; 2(1+\epsilon\psi_X)\epsilon^2\psi_{XX}\cos(x+\psi)$$
$$+\;(1+\epsilon\psi_X)^3\sin(x+\psi)$$
$$=\;-\,3\epsilon^2\psi_{XX}\cos(x+\psi) \;+\; (1+\epsilon\psi_X)^3\sin(x+\psi)$$

$$\partial_x^4\cos(x+\psi) \;=\; 3\epsilon^2\psi_{XX}\sin(x+\psi) \;+\; 3(1+\epsilon\psi_X)^2\epsilon^2\psi_{XX}\sin(x+\psi)$$
$$+\;(1+\epsilon\psi_X)^4\cos(x+\psi)$$
$$=\;6\epsilon^2\psi_{XX}\sin(x+\psi) \;+\; (1+4\epsilon\psi_X+6\epsilon^2\psi_X^2)\cos(x+\psi)$$

$$\partial_x^2(A\cos(x+\psi)) \;=\; \partial_x^2 A\cdot\cos(x+\psi) \;+\; 2\partial_x A\cdot\partial_x\cos(x+\psi) \;+\; A\cdot\partial_x^2\cos(x+\psi)$$
$$=\;\epsilon^2 A_{XX}\cos(x+\psi) \;+\; 2\epsilon A_X\cdot -\,(1+\epsilon\psi_X)\sin(x+\psi)$$
$$+\,A(-\,\epsilon^2\psi_{XX}\sin(x+\psi) \;-\; (1+\epsilon\psi_X)^2\cos(x+\psi))$$
$$=\;(\epsilon^2 A_{XX} - A(1+\epsilon\psi_X)^2)\cos(x+\psi) \;-\; (2\epsilon A_X(1+\epsilon\psi_X)+\epsilon^2 A\psi_{XX})\sin(x+\psi)$$

$$\partial_x^4(A\cos(x+\psi)) \;=\; 6\,\partial_x^2 A\cdot\partial_x^2\cos(x+\psi) + 4\,\partial_x A\cdot\partial_x^3\cos(x+\psi) + A\cdot\partial_x^4\cos(x+\psi)$$
$$=\;6\epsilon^2\psi_{XX}\cdot -\cos(x+\psi) \;+\; 4\epsilon A_X(1+3\epsilon\psi_X)\sin(x+\psi)$$
$$+\,A(6\epsilon^2\psi_{XX}\sin(x+\psi) \;+\; (1+4\epsilon\psi_X+6\epsilon^2\psi_X^2)\cos(x+\psi))$$
$$=\;(-6\epsilon^2\psi_{XX} + A(1+4\epsilon\psi_X+6\epsilon^2\psi_X^2)\cos(x+\psi)$$
$$+\;(4\epsilon A_X(1+3\epsilon\psi_X)+6\epsilon^2 A\psi_{XX})\sin(x+\psi)$$

$$u_t \,+\, au \,+\, \tfrac{1}{2}\, u_{xx} \,+\, \tfrac{1}{4}\, u_{xxxx} \,+\, \tfrac{4}{3}\,\epsilon^2\, u^3 \,=\, 0$$

$\Rightarrow$

$$(\epsilon^2 A_T + aA + \tfrac{1}{2}(\epsilon^2 A_{XX} - A(1+\epsilon\psi_X)^2) + \tfrac{1}{4}(-6\epsilon^2\psi_{XX} + A(1+4\epsilon\psi_X + 6\epsilon^2\psi_X^2))$$

$$\cdot\ \cos(x+\psi)$$

$$+\ (-\epsilon^2 A\psi_T - \tfrac{1}{2}(2\epsilon A_X(1+\epsilon\psi_X) + \epsilon^2 A\psi_{XX}) + \tfrac{1}{4}(4\epsilon A_X(1+3\epsilon\psi_X) + 6\epsilon^2 A\psi_{XX})$$

$$\cdot\ \sin(x+\psi)$$

$$+\ a\epsilon^2 u' \,+\, \tfrac{1}{2}\epsilon^2 u'_{xx} \,+\, \tfrac{1}{4}\epsilon^2 u'_{xxxx} \,+\, \tfrac{4}{3}\epsilon^2 A^3 \cos^3(x+\psi) \,+\, \cdots \,=\, 0$$

$\Rightarrow$

$$(aA - \tfrac{1}{2}A + \tfrac{1}{4}A + \epsilon(-A\psi_X + A\psi_X)$$

$$+\ \epsilon^2(A_T + \tfrac{1}{2}(A_{XX} - A\psi_X^2) + \tfrac{1}{4}(-6A_{XX} + 6A\psi_X^2))) \cdot \cos(x+\psi)$$

$$+\ (\epsilon(-A_X + A_X) + \epsilon^2(-A\psi_T - A_X\psi_X - \tfrac{1}{2}A\psi_{XX} + 3A_X\psi_X + \tfrac{3}{2}A\psi_{XX})) \cdot \sin(x+\psi)$$

$$+\ \epsilon^2(au' \,+\, \tfrac{1}{2}u'_{xx} \,+\, \tfrac{1}{4}u'_{xxxx} \,+\, \tfrac{4}{3}A^3\cos^3(x+\psi)) \,+\, \cdots \,=\, 0$$

$$a \,=\, \tfrac{1}{4} \,-\, \kappa\,\epsilon^2$$

$$(-\kappa A + A_T - A_{XX} + A\psi_X^2)\cos(x+\psi) \,+\, (-A\psi_T + 2A_X\psi_X + A\psi_{XX})\sin(x+\psi)$$

$$+\ \tfrac{1}{4}u' \,+\, \tfrac{1}{2}u'_{xx} \,+\, \tfrac{1}{4}u'_{xxxx} \,+\, \tfrac{4}{3}A^3\cos^3(x+\psi) \,=\, 0$$

<u>note</u>

1. $\cos^3\theta \,=\, \tfrac{3}{4}\cos\theta \,+\, \tfrac{1}{4}\cos 3\theta$

2. $\cos(x+\psi)\,, \sin(x+\psi)$ are solutions of the homogeneous equation for u'; the solvability condition requires the coefficients of $\cos(x+\psi)\,, \sin(x+\psi)$ to vanish.

$$A_T \,=\, A_{XX} \,-\, A\psi_X^2 \,+\, \kappa A \,-\, A^3$$

$$A\psi_T \,=\, 2A_X\psi_X + A\psi_{XX}$$

set $Z = Ae^{i\psi}$ $\big(\Rightarrow\ u(x,t) = A\cos(x+\psi) = \mathrm{real}(Ae^{i(x+\psi)}) = \mathrm{real}(Ze^{ix})\ \big)$

$$Z_T = A_T e^{i\psi} + Ai\psi_T e^{i\psi} = (A_T + iA\psi_T)e^{i\psi} \quad , \quad Z_X = (A_X + iA\psi_X)e^{i\psi}$$

$$\begin{aligned} Z_{XX} &= (A_{XX} + i(A_X\psi_X + A\psi_{XX})e^{i\psi} + (A_X + iA\psi_X)i\psi_X e^{i\psi} \\ &= (A_{XX} + 2iA_X\psi_X + iA\psi_{XX} - A\psi_X^2)e^{i\psi} \\ &= (A_T - \kappa A + A^3 + iA\psi_T)e^{i\psi} = Z_T - \kappa Z + |Z|^2 Z \end{aligned}$$

$Z_T = \kappa Z - |Z|^2 Z + Z_{XX}$: Ginzburg-Landau equation

note

1. The GL equation arises similarly for a variety of problems; κ is positive for thermal convection and Taylor-Couette flow, but for other problems it can be negative or complex.

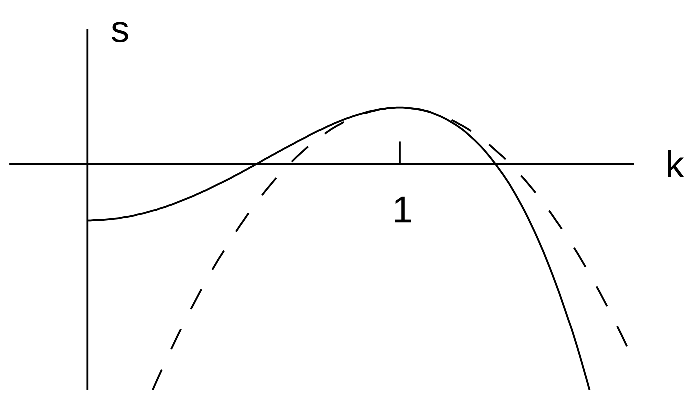
2. linearize about $Z = 0$, $Z = e^{ST+iKX}$ $\Rightarrow$ $S = \kappa - K^2$

return to original variables : $S = \dfrac{s}{\epsilon^2}$, $K = \dfrac{k-1}{\epsilon}$

$$\Rightarrow \quad \frac{s}{\epsilon^2} = \kappa - \left(\frac{k-1}{\epsilon}\right)^2 \quad \Rightarrow \quad s_{GL} = \kappa\epsilon^2 - (k-1)^2 = \tfrac{1}{4} - a - (k-1)^2$$

$$\begin{aligned} s_{SH} &= -a + \tfrac{1}{2}k^2 - \tfrac{1}{4}k^4 = -a + \tfrac{1}{2}(1+k-1)^2 - \tfrac{1}{4}(1+k-1)^4 \\ &= -a + \tfrac{1}{2}(1 + 2(k-1) + (k-1)^2) - \tfrac{1}{4}(1 + 4(k-1) + 6(k-1)^2 + \cdots) \\ &= \tfrac{1}{4} - a - (k-1)^2 + \cdots \end{aligned}$$

$s_{SH} = s_{GL} + O\big((k-1)^3\big)$ $\Rightarrow$ GL is a quadratic approximation near $k = 1$



<u>special solutions</u>

$$Z(X,T) \;=\; A(T)e^{iKX}$$

$$A_T e^{iKX} \;=\; \kappa A e^{iKX} \;-\; A^3 e^{iKX} \;-\; K^2 A e^{iKX}$$

$$A_T \;=\; (\kappa - K^2)A \;-\; A^3 \quad : \quad \text{Landau equation}$$

equilibrium solutions : $A_0^2 \;=\; \kappa \;-\; K^2$ for $|K| \leq \kappa$

Hence $u(x,t) = A_0 \cos((1+\epsilon K)x)$ is a steady solution of the GL equation and a steady approximate solution of SH; it defines a pattern, e.g. convection cells, Taylor vortices.

<u>linear stability</u>

We know that $A(T) = A_0$ is a stable equilibrium of the Landau equation; next we determine the linear stability of $Z(X,T) = A_0 e^{iKX}$ as an equilibrium of GL.

$$Z(X,T) \;=\; A_0 e^{iKX} \;+\; z(X,T)$$

$$Z_T \;=\; \big(\kappa \;-\; |Z|^2\big)Z \;+\; Z_{XX}$$

$$z_T \;=\; \big(\kappa - |A_0 e^{iKX} + z|^2\big)\big(A_0 e^{iKX} + z\big) \;-\; K^2 A_0 e^{iKX} \;+\; z_{XX}$$

$$\kappa \;-\; |A_0 e^{iKX} + z|^2 \;=\; \kappa \;-\; \big(A_0 e^{iKX} + z\big)\big(A_0 e^{-iKX} + \overline{z}\big)$$

$$= \kappa \;-\; A_0^2 \;-\; A_0\big(e^{iKX}\overline{z} + e^{-iKX} z\big) \;+\; |z|^2$$

$$z_T \;=\; \Big(K^2 - A_0\big(e^{iKX}\overline{z} + e^{-iKX} z\big)\Big)\big(A_0 e^{iKX} + z\big) \;-\; K^2 A_0 e^{iKX} \;+\; z_{XX}$$

$$= K^2 A_0 e^{iKX} \;+\; K^2 z \;-\; A_0^2\big(e^{2iKX}\overline{z} + z\big) \;-\; K^2 A_0 e^{iKX} \;+\; z_{XX}$$

$$= \big(K^2 - A_0^2\big)z \;-\; A_0^2 e^{2iKX}\overline{z} \;+\; z_{XX}$$

$$z_T \;=\; (2K^2 - \kappa)z \;-\; (\kappa - K^2)e^{2iKX}\overline{z} \;+\; z_{XX} \quad , \quad \text{look for } z \;=\; e^{\lambda T}\zeta(X)$$

$$\lambda\zeta \;=\; (2K^2 - \kappa)\zeta \;-\; (\kappa - K^2)e^{2iKX}\overline{\zeta} \;+\; \zeta_{XX}$$

$$(\lambda + \kappa - 2K^2)\zeta \;+\; (\kappa - K^2)e^{2iKX}\overline{\zeta} \;-\; \zeta_{XX} \;=\; 0$$

look for $\zeta \,=\, \alpha e^{i(K+q)X} \,+\, \beta e^{i(K-q)X}$ where $\alpha\,,\beta$ are real

$$e^{2iKX}\overline{\zeta} \,=\, e^{2iKX}(\alpha e^{-i(K+q)X} \,+\, \beta e^{-i(K-q)X}) \,=\, \alpha e^{i(K-q)X} \,+\, \beta e^{i(K+q)X}$$

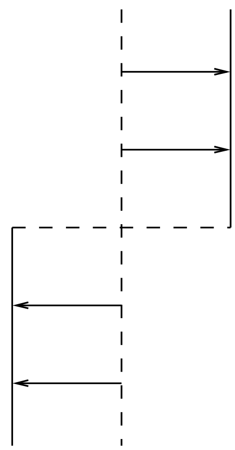
$$\Rightarrow \quad \big((\lambda+\kappa-2K^2)\alpha+(\kappa-K^2)\beta+(K+q)^2\alpha\big)e^{i(K+q)X}$$

$$+\,\big((\lambda+\kappa-2K^2)\beta+(\kappa-K^2)\alpha+(K-q)^2\beta\big)e^{i(K-q)X} \,=\, 0$$

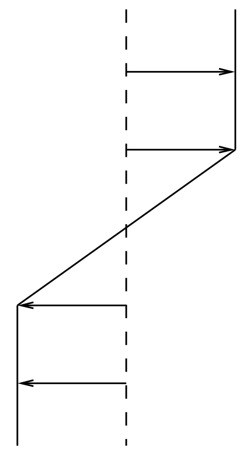
$$\Rightarrow \quad \begin{pmatrix} \lambda+\kappa-K^2+q^2+2Kq & \kappa-K^2 \\ \kappa-K^2 & \lambda+\kappa-K^2+q^2-2Kq \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \lambda \,=\, -\,(\kappa-K^2)\,-\,q^2\,\pm\,\sqrt{4K^2q^2+(\kappa-K^2)^2}$$

piecewise linear profiles



vortex sheet



shear layer

Assume that $U(z)$ and/or $U'(z)$ are piecewise linear and discontinuous at $z = z_0$.

Rayleigh's equation : $(U - c)(D^2 - k^2)\phi - U''\phi = 0$

Assume also that $U(z_0) - c \neq 0$. Then for $z \neq z_0$ we have $(D^2 - k^2)\phi = 0$ and at $z = z_0$, ϕ must satisfy a jump condition.

$$D\big((U - c)D\phi - U'\phi\big) - k^2(U - c)\phi = 0$$

$$\Rightarrow \quad \int_{z_0-\epsilon}^{z_0+\epsilon} \Big(D\big((U - c)D\phi - U'\phi\big) - k^2(U - c)\phi\Big)\, dz = 0$$

$$\Rightarrow \quad \big((U - c)D\phi - U'\phi\big)\Big|_{z_0-\epsilon}^{z_0+\epsilon} - k^2 \int_{z_0-\epsilon}^{z_0+\epsilon} (U - c)\phi\, dz = 0$$

$$\Rightarrow \quad \lim_{\epsilon \to 0} \big((U - c)D\phi - U'\phi\big)\Big|_{z_0-\epsilon}^{z_0+\epsilon} = \big[(U - c)D\phi - U'\phi\big] = 0$$

recall : $p = U'\phi - (U - c)D\phi$

so the jump condition says that the pressure is continuous across the interface

$$\frac{-p}{(U - c)^2} = \frac{(U - c)D\phi - U'\phi}{(U - c)^2} = D\left(\frac{\phi}{U - c}\right)$$

$$\Rightarrow \quad \int_{z_0-\epsilon}^{z_0+\epsilon} \frac{-p}{(U-c)^2}\, dz \; = \; \int_{z_0-\epsilon}^{z_0+\epsilon} D\left(\frac{\phi}{U-c}\right) dz \; = \; \frac{\phi}{U-c}\Bigg|_{z_0-\epsilon}^{z_0+\epsilon}$$

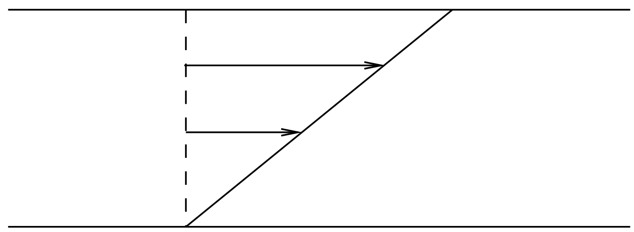
$$\Rightarrow \quad \lim_{\epsilon \to 0} \frac{\phi}{U-c}\Bigg|_{z_0-\epsilon}^{z_0+\epsilon} \; = \; \left[\frac{\phi}{U-c}\right] \; = \; 0$$

recall : $\eta' = Ae^{ik(x-ct)} \quad , \quad \psi' = \phi(z)e^{ik(x-ct)}$

the interface moves with the fluid velocity $\quad \Leftrightarrow \quad \dfrac{\partial \eta'}{\partial t} + U\dfrac{\partial \eta'}{\partial x} \; = \; w' \; = \; -\dfrac{\partial \psi'}{\partial x}$

$$\Leftrightarrow \quad A \cdot -ikc + UA \cdot -ik \; = \; -ik\phi \quad \Leftrightarrow \quad \frac{\phi}{U-c} \; = \; -A \quad \Leftrightarrow \quad \left[\frac{\phi}{U-c}\right] \; = \; 0$$

plane Couette flow



$$U(z) \,=\, z \quad , \quad 0 \le z \le 1$$

$$(U-c)(D^2-k^2)\phi \,-\, U''\phi \,=\, 0 \quad \Rightarrow \quad (z-c)(D^2-k^2)\phi \,=\, 0$$

naive approach

$$(D^2-k^2)\phi \,=\, 0$$

$$\phi(z) \,=\, a\sinh kz \,+\, b\sinh k(1-z)$$

$$\left.\begin{array}{l} \phi(0) \,=\, 0 \quad \Rightarrow \quad b\sinh k \,=\, 0 \quad \Rightarrow \quad b \,=\, 0 \\ \phi(1) \,=\, 0 \quad \Rightarrow \quad a\sinh k \,=\, 0 \quad \Rightarrow \quad a \,=\, 0 \end{array}\right\} \quad \Rightarrow \quad \text{there are no normal modes}$$

Case (1960) clarified the problem by solving the linearized IVP using the Laplace transform in time.

2D Euler equations : $(u,w,p) \,=\, (U+u',w',p')$

$$u_t \,+\, uu_x \,+\, wu_z \,=\, -p_x \quad \Rightarrow \quad u'_t \,+\, Uu'_x \,+\, U'w' \,=\, -p'_x$$

$$w_t \,+\, uw_x \,+\, ww_z \,=\, -p_z \quad \Rightarrow \quad w'_t \,+\, Uw'_x \,=\, -p'_z$$

$$u_x \,+\, w_z \,=\, 0 \quad \Rightarrow \quad u'_x \,+\, w'_z \,=\, 0$$

stream function : $\psi' \quad , \quad u' \,=\, \psi_z \quad , \quad w' \,=\, -\psi_x$

$$\psi'_{tz} \,+\, U\psi'_{xz} \,-\, U'\psi'_x \,=\, -p'_x$$

$$-\psi'_{tx} \,-\, U\psi'_{xx} \,=\, -p'_z$$

$$\psi'_{tzz} \,+\, U\psi'_{xzz} \,+\, U'\psi'_{xz} \,-\, U'\psi'_{xz} \,-\, U''\psi_x \,=\, -\,(\partial_t \,+\, U\partial_x)\psi_{xx}$$

$$\left.\begin{array}{l} (\partial_t + U\partial_x)(\psi'_{xx} + \psi'_{zz}) - U''\psi'_x = 0 \\ \\ \psi'(x,z,0) \;:\; \text{given} \\ \\ \psi'(x,0,t) = \psi'(x,1,t) = 0 \quad , \quad \text{bc in } x \end{array}\right\} \;:\; \text{IVP}$$

<u>viscous theory</u>

<u>Navier-Stokes</u>

$$u_t + uu_x + wu_z = -p_x + \frac{1}{R}\Delta u$$

$$w_t + uw_x + ww_z = -p_z + \frac{1}{R}\Delta w$$

$$u_x + w_z = 0$$

equilibrium : $(u\,,w) = (U(z)\,,0) \quad \Leftrightarrow \quad \frac{1}{R}U'' = p_x = \text{constant}$

Couette flow : $U(z) = z$, $-1 \le z \le 1$, moving walls , $p_x = 0$

Poiseuille flow : $U(z) = 1 - z^2$, $-1 \le z \le 1$, stationary walls , $p_x = -\frac{2}{R}$

linear stability : $(u\,,w\,,p) = (U + u', w', P + p')$

$$u'_t + Uu'_x + U'w' = -p'_x + \frac{1}{R}\Delta u'$$

$$w'_t + Uw'_x = -p'_z + \frac{1}{R}\Delta w'$$

$$u'_x + w'_z = 0$$

bc : $u' = w' = 0$, on $z = z_1$, z_2

eliminate pressure

$$u'_{tz} + Uu'_{xz} + U'u'_x + U'w'_z + U''w' = -p'_{xz} + \frac{1}{R}\Delta u'_z$$

$$w'_{tx} + Uw'_{xx} = -p'_{zx} + \frac{1}{R}\Delta w'_x$$

vorticity : $\omega = w'_x - u'_z \quad \Rightarrow \quad \omega'_t + U\omega'_x - U''w' = \frac{1}{R}\Delta\omega'$

<u>integral relation</u> (Yih, p. 484; see also energy version by Neu)

$$\omega'\omega'_t + \omega' U \omega'_x - \omega' U'' w' = \omega' \frac{1}{R} \Delta\omega'$$

$$\frac{1}{2}\partial_t(\omega')^2 + U\partial_x(\omega')^2 - U''\omega' w' = \frac{1}{R}\omega'\Delta\omega'$$

given $f(x,z,t)$, define $\langle f \rangle = \int_{-\infty}^{\infty} f(x,z,t)\,dx$ $\Big($or $\ldots$ if periodic in $x\Big)$

$$\frac{1}{2}\partial_t\big\langle(\omega')^2\big\rangle = U''\big\langle\omega' w'\big\rangle + \frac{1}{R}\big\langle\omega'\Delta\omega'\big\rangle \quad , \quad \text{assuming } \omega' \to 0 \text{ for } x \to \pm\infty$$

$$\frac{1}{2}\frac{d}{dt}\int_{z_1}^{z_2}\big\langle(\omega')^2\big\rangle\,dz = \int_{z_1}^{z_2} U''\big\langle\omega' w'\big\rangle\,dz + \frac{1}{R}\int_{z_1}^{z_2}\big\langle\omega'\Delta\omega'\big\rangle\,dz$$

(a) $\big\langle\omega' w'\big\rangle = \big\langle(w'_x - u'_z)w'\big\rangle = -\big\langle u'_z w'\big\rangle$, assuming $w' \to 0$ for $x \to \pm\infty$

$$u'_z w' = (u'w')_z - u'w'_z = (u'w')_z - u' \cdot -u'_x$$

$\big\langle\omega' w'\big\rangle = -\big\langle u'w'\big\rangle_z$, assuming $u' \to 0$ for $x \to \pm\infty$

(b) $\omega'\Delta\omega' = \omega'\big(\omega'_{xx} + \omega'_{zz}\big)$

$$\big\langle\omega'\Delta\omega'\big\rangle = -\big\langle(\omega'_x)^2\big\rangle + \big\langle\omega'\omega'_{zz}\big\rangle$$

$$\int_{z_1}^{z_2}\big\langle\omega'\omega'_{zz}\big\rangle\,dz = \left\langle\int_{z_1}^{z_2}\omega'\omega'_{zz}\,dz\right\rangle = \left\langle \omega'\omega'_z\Big|_{z_1}^{z_2} - \int_{z_1}^{z_2}(\omega'_z)^2\,dz\right\rangle$$

$$\int_{z_1}^{z_2}\big\langle\omega'\Delta\omega'\big\rangle\,dz = -\int_{z_1}^{z_2}\big\langle|\nabla\omega'|^2\big\rangle\,dz + \frac{1}{2}\Big\langle\big((\omega')^2\big)_z\Big\rangle\Big|_{z_1}^{z_2}$$

$$\frac{1}{2}\frac{d}{dt}\int_{z_1}^{z_2}\big\langle(\omega')^2\big\rangle\,dz$$

$$= -\int_{z_1}^{z_2} U''\underbrace{\big\langle u'w'\big\rangle_z}_{\uparrow\ \text{Reynolds stress}}\,dz - \frac{1}{R}\int_{z_1}^{z_2}\big\langle|\nabla\omega'|^2\big\rangle\,dz + \frac{1}{2R}\Big\langle\big((\omega')^2\big)_z\Big\rangle\Big|_{z_1}^{z_2}$$

<u>case 1</u> : $U'' = \text{constant}$, e.g. Couette flow, Poiseuille flow

$$\int_{z_1}^{z_2} U'' \big\langle u'w' \big\rangle_z \, dz \;=\; U'' \big\langle u'w' \big\rangle \Big|_{z_1}^{z_2} \;=\; 0 \quad \text{since } w' = 0 \text{ on } z = z_1 \,,\; z_2$$

$$\frac{1}{2}\,\frac{d}{dt}\int_{z_1}^{z_2} \big\langle (\omega')^2 \big\rangle \, dz \;=\; -\,\frac{1}{R}\int_{z_1}^{z_2} \big\langle |\nabla\omega'|^2 \big\rangle \, dz \;+\; \frac{1}{2R}\Big\langle \big((\omega')^2\big)_z \Big\rangle \Big|_{z_1}^{z_2}$$

term 1 : stabilizing , viscous dissipation

term 2 : possibly destabilizing , walls may act as a source of vorticity

<u>case 2</u> : $z_1 = -\infty \,,\; z_2 = \infty$, e.g. free shear flow

$$\frac{1}{2}\,\frac{d}{dt}\int_{-\infty}^{\infty} \big\langle (\omega')^2 \big\rangle \, dz \;=\; -\int_{-\infty}^{\infty} U'' \big\langle u'w' \big\rangle_z \, dz \;-\; \frac{1}{R}\int_{-\infty}^{\infty} \big\langle |\nabla\omega'|^2 \big\rangle \, dz$$

assuming $\omega' \to 0$ for $x \to \pm\infty$

term 1 : possibly destabilizing , energy may be transferred from the base flow to the perturbation

term 2 : stabilizing , as before

<u>conclusion</u> : the effect of viscosity may be stabilizing or destabilizing

<u>stream function</u> : ψ'

$$u' = \psi'_z \quad , \quad w' = -\psi'_x \quad , \quad \omega' = -\Delta\psi'$$

$$\omega'_t + U\omega'_x - U''w' \;=\; \frac{1}{R}\,\Delta\omega'$$

$$\Delta\psi'_t + U\Delta\psi'_x - U''\psi'_x \;=\; \frac{1}{R}\,\Delta^2\psi' \quad , \quad \text{bc} \;:\; \psi' = \psi'_z = 0 \text{ on } z = z_1 \,,\; z_2$$

normal modes : $\psi'(x,z,t) = \phi(z)e^{ik(x-ct)}$

$$-ikc(D^2 - k^2)\phi + Uik(D^2 - k^2)\phi - U''ik\phi \;=\; \frac{1}{R}(D^2 - k^2)^2\phi$$

<u>Orr-Sommerfeld equation</u>

$$\frac{1}{ikR}(D^2 - k^2)^2\phi \,=\, (U - c)(D^2 - k^2)\phi \,-\, U''\phi$$

transient growth (Henningson, Schmidt, Trefethen, ...)

The implicit assumption of normal mode analysis is that an equilibrium flow is stable $\Leftrightarrow$ all of the eigenvalues are stable, i.e. all $c_i \leq 0$. However, this criterion may fail when the linearized solution exhibits transient growth.

ex

$$f(t) = e^{-t} - e^{-2t} \ \Rightarrow \ \lim_{t\to\infty} f(t) = 0 \ , \ \text{but for small } t \text{ we have } f(t) \sim t + O(t^2)$$

We say that $f(t)$ is asymptotically stable, but it displays transient growth. Such behavior can occur in the solution of a hydrodynamic linear stability problem.

def : Consider a matrix A (or a differential operator) acting on a Hilbert space, e.g. $\mathbb{R}^n, \mathbb{C}^n$ (or $L_2(\mathbb{R})$). A is called normal if $A^*A = AA^*$, i.e. if A commutes with its adjoint A^*.

note

1. If A is self-adjoint, then A is normal.

2. A is normal $\Leftrightarrow$ A is unitarily diagonalizable, i.e. $A = UDU^*$, where D is diagonal and U is unitary, i.e. $UU^* = I$. In this case the eigenspaces corresponding to distinct eigenvalues of A are orthogonal and they span the entire Hilbert space.

ex

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \ : \ \text{self-adjoint} \ , \ \text{normal}$$

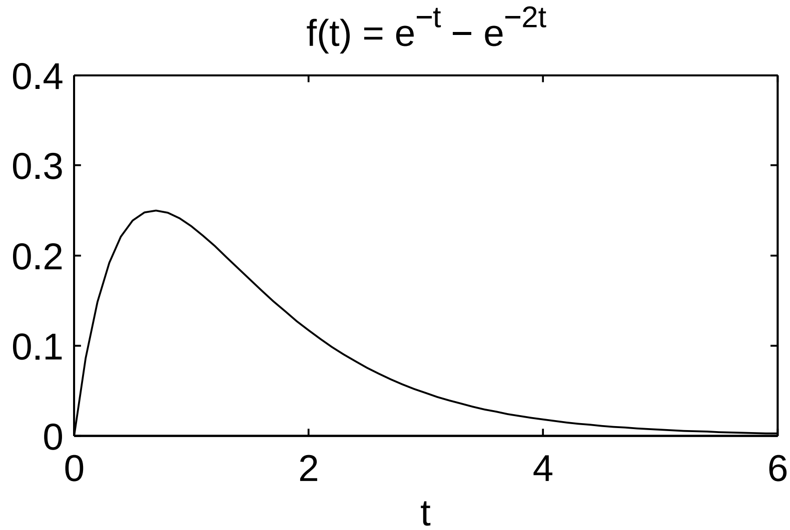
$$\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \ : \ \text{not self-adjoint} \ , \ \text{normal} \ , \ \lambda = -1 \pm i$$

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \ : \ \text{not self-adjoint} \ , \ \text{not normal} \ , \ \text{not diagonalizable}$$

$$\begin{pmatrix} -1 & 5 \\ 0 & -2 \end{pmatrix} \ : \ \text{not self-adjoint} \ , \ \text{not normal} \ , \ \text{diagonalizable}$$

Now consider a linear system of ODEs, $du/dt = Au$, with solution $u(t) = e^{At}u(0)$.

In the examples above, the eigenvalues of A have negative real part. Does it follow that the solution $u(t)$ decays in time? If so, in which case does $u(t)$ decay fastest?

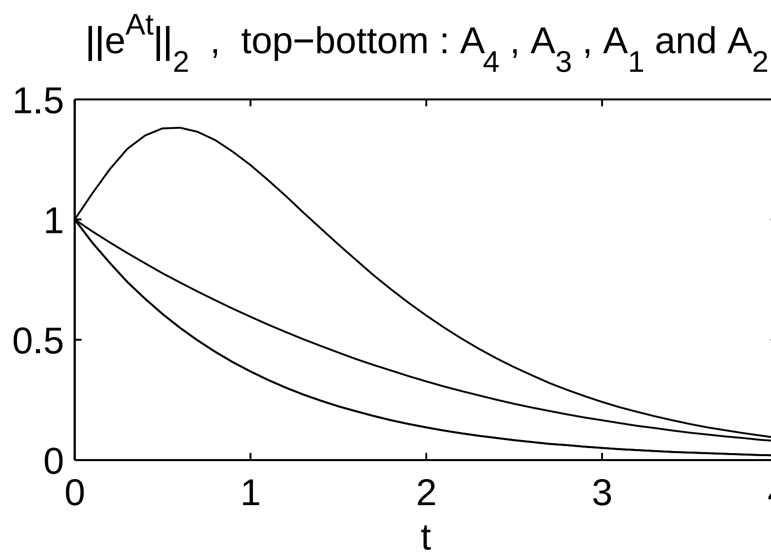


$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} -1 & 5 \\ 0 & -2 \end{pmatrix}$$



For normal matrices whose eigenvalues have negative real part (like A_1, A_2), transient growth cannot occur. For non-normal matrices whose eigenvalues have negative real part (like A_3, A_4), certain solutions can exhibit transient growth - the key factor is that eigenvectors of a non-normal matrix are not orthogonal.