MATH 671 Particle Methods in Scientific Computing Winter 2022

hw#2 due: Thursday, February 17

Typewriter font indicates a Matlab command, e.g. `fft`. It is not necessary to submit your code unless specifically requested.

1. The logistic map, a model for population dynamics, is defined by $x_{n+1} = ax_n(1 - x_n)$, where a is a parameter and x_0 is given. Let $N = 128$ and compute the iterates of the logistic map x_n for $n = 0 : N-1$, starting from the value $x_0 = \frac{1}{2}$, and let $x = (x_0, \ldots, x_{N-1})^T$ be the resulting time series. Compute the DFT of the time series, $\widehat{x} = F_N x$, using `fft`. Do this for six parameter values, $a = 2.75, 3.25, 3.5, 3.555, 3.5665, 3.57$. For each value of a, plot the time series x_n using `plot`, and also plot the spectral amplitudes $|\widehat{x}_n|$ using `semilogy`, both as functions of n. In the case of the time series, just plot a symbol for each x_n (don't connect the symbols by lines). Present the plots on one side of a single sheet (hint: present the six time series in one plot and the six spectral amplitudes in another plot; each of these plots can be a 3×2 matrix created using `subplot(3,2,i)`, where $i = 1 : 6$ corresponds to the values of a). Print the value of a in the title of each subplot. Discuss the results.

2. Write a DFT code in Matlab that computes $\widehat{v} = F_N v$ by direct summation. To be consistent with Matlab, omit the prefactor $1/\sqrt{N}$ that we used in class. Apply your code and Matlab's `fft` to the vector $v = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{N})^T$, for $N = 2^q, q = 1 : 12$. Plot the run time versus N using `loglog` for your DFT and Matlab's `fft`. Use `tic, toc` to compute the run time. Also make a table with the following format; column 1: q, column 2: max-norm difference between your DFT and Matlab's FFT (use `format shorte`). Discuss the results; what is expected or unexpected? Can you offer a plausible reason for anything unexpected?

3. Consider the BVP from class, $-\phi'' + \sigma^2\phi = f$, on $0 \le x \le 1$ with periodic boundary conditions, $\phi(0) = \phi(1)$, $\phi'(0) = \phi'(1)$. We showed that the eigenvalues of the finite-difference matrix are $\lambda_n = \sigma^2 + \frac{4\sin^2(\pi n h)}{h^2}$ for $n = 0 : N-1$ with $h = 1/N$. We also showed that the eigenvalues of a general $N \times N$ circulant matrix are given by $\lambda_n = \sqrt{N}\,\widehat{c}_n$, for some vector c. Since the finite-difference matrix is circulant, show explicitly that the two expressions for λ_n are equivalent.

4. Consider the BVP from class, $-\phi''+\sigma^2\phi = f$ on $0 \le x \le 1$ with periodic boundary conditions, $\phi(0) = \phi(1)$, $\phi'(0) = \phi'(1)$. Let $\sigma = 2$ and consider two cases, $f_1(x) = x, f_2(x) = \sin \pi x$. In each case find the exact solution $\phi(x)$ (make sure your result satisfies the differential equation and boundary conditions). Then solve the problem numerically by the three methods discussed in class: (1) finite-difference/FFT, (2) pseudospectral, (3) Green's function/Riemann sum. All three methods have the form $u = F_N^* D F_N f$, where D is a diagonal matrix; use Matlab's `fft` and `ifft`, but note that in case (3) the relation $D = \text{diag}(\sqrt{N}\,\widehat{c}_n)$, which was derived in class, should be modified to be consistent with Matlab's `fft`. Take $N = 2^q$ for $q = 2 : 7$. Plot the exact solution and numerical solution using `subplot` in a 3×2 matrix corresponding to the 6 values of q. Include both boundary points in the plots. For each method present a table with the following format; column 1: h, column 2: $\max|\phi_n - u_n|$, column 3: $\max|\phi_n - u_n|/h$, column 4: $\max|\phi_n - u_n|/h^2$. Discuss the results; what is expected or unexpected? Can you offer a plausible reason for anything unexpected?

5. Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} a & c & b \\ b & a & c \\ c & b & a \end{pmatrix}$.