MATH 671 Particle Methods in Scientific Computing Winter 2022

hw#3 due: Thursday, March 17

1. Let $\rho(x)$ be a charge density on $0 \le x \le 1$ satisfying charge neutrality, $\int_0^1 \rho(x)dx = 0$. In class we derived an integral expression for the associated potential function in the case of periodic boundary conditions, $\phi(x) = \int_0^1 (g(x,y) - xy)\rho(y)dy$, where $g(x,y) = -\frac{1}{2}|x-y|$ is the free-space Green's function in 1d. Verify that the integral expression for ϕ satisfies the boundary conditions, $\phi(0) = \phi(1), \phi_x(0) = \phi_x(1)$, and the Poisson equation, $-\phi_{xx} = \rho$; show this directly, i.e. don't use $-g_{xx} = \delta$.

2. Show that $\displaystyle\int_0^1 (-g_x(x,y) + y)dy = x$, where $g(x,y) = -\frac{1}{2}\text{sign}(x-y)$. (page 28 of the notes)

3. The <u>convolution</u> of two functions f, g is defined by $(f * g)(x) = \int_{-\infty}^{\infty} f(x-s)g(s)ds$.

a) Let W_0 be the nearest-mesh-point weight function, and let W_1 be the cloud-in-cell weight function (defined on pages 25-26 of the notes). Show that $W_1 = W_0 * W_0$; take $\Delta x = 1$ for simplicity.

b) Find $W_2 = W_1 * W_0$.

c) Show that $\int_{-\infty}^{\infty} W_i(x)dx = 1$ for $i = 0, 1, 2$.

d) Plot W_0, W_1, W_2 in the same plot.

4. Download the file `m671b_Vlasov-Poisson_PIC_hw3.m` from the course site. The file implements the PIC method for the 1d Vlasov-Poisson equations with periodic boundary conditions for initial condition corresponding to the two-stream instability (two cold electron beams moving in opposite directions with a perturbation in charge density). The code however is missing the Poisson solver; your task is to fill in those lines (use either the FFT solver or the elimination solver discussed in class); run the code and present plots of the particles in phase space at time $t = 0, 0.25, 0.5, 0.75, 1$ (all on one page using `subplot` is fine). You may experiment with different numerical parameters, perturbations, and weight functions, but the results you submit should use the settings in the downloaded file. Also fill in the blanks in the following text.

> At early times the beams propagate ____________, but the perturbation causes some particles to ____________ and others to ____________. Eventually vortices form in phase space in which some particles change ____________.

5. The stream function of a vortex-blob is defined by $\psi_\delta(x,y) = -\frac{1}{2\pi}\ln(x^2 + y^2 + \delta^2)^{1/2}$.

a) Find the associated vorticity $\omega_\delta(x,y) = -\nabla^2\psi_\delta(x,y)$ and show that $\int_{\mathbb{R}^2} \omega_\delta(x,y)\,dx\,dy = 1$ for all $\delta > 0$. Plot $w_\delta(x,0)$ for $-4 \le x \le 4$ with $\delta = 0.2,\ 0.1,\ 0.05$ all on the same plot. What can you say about $\omega_\delta(x,0)$ in the limit $\delta \to 0$?

b) Solve the vortex-blob equations with initial data $x_j(0) = \cos\theta_j$, $y_j(0) = 0$, $\theta_j = (-1 + \frac{j}{N+1})\pi$, and strength $\Gamma_j = \frac{\pi}{N+1}\cos\theta_j$ for $j = 1 : N$. See page 33 of the notes for the equations. Take $N = 200$, $\delta = 2 \cdot 10^{-1}$. Plot the location of the vortex-blobs at $t = 0, 1, 2, 4, 8, 16$ using square axes. Use direct summation to evaluate the right hand side of the differential equations. Use any time-stepping scheme you like, but make sure the results are correct to plotting accuracy. Discuss the results. (Note: this is a 2d model for an aircraft wake; accounting for 3d effects is an important research problem).

6. Show that $g(x,y,z) = \frac{1}{4\pi}(x^2 + y^2 + z^2)^{-1/2}$ is the Green's function for the Laplace operator in 3d, i.e. $-\nabla^2 g = \delta$. Follow the steps for the 2d case on pages 31-32 of the notes.