Computing Vortex Sheet Motion

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1. Introduction

Coherent vortex structures occur in many types of fluid flow including mixing layers, jets and wakes. A vortex sheet is a mathematical model for such structures, in which the shear layer is approximated by a surface across which the tangential fluid velocity has a jump discontinuity. Vortex sheet motion belongs to the field of vortex dynamics, one of the main approaches to understanding fluid turbulence.

Careful numerical experiments have helped advance the mathematical study of vortex sheets. Difficulties arise in computing vortex sheet motion due to short wavelength instability, singularity formation, and spiral roll-up. This paper reviews the problem of computing vortex sheet motion and presents several applications. See [2] for a sample of other vortex models and numerical methods.

2. Analytic Evolution and Singularity Formation

A vortex sheet is defined by a curve $z(\Gamma, t)$ in the complex plane, where Γ is the circulation parameter and t is time. The evolution equation is [4,32],

$$\overline{\frac{\partial z}{\partial t}}(\Gamma, t) = \int_{-\infty}^{\infty} K(z(\Gamma, t) - z(\tilde{\Gamma}, t))d\tilde{\Gamma} \quad , \quad K(z) = \frac{1}{2\pi i z} \; . \tag{1}$$

The Cauchy principal value of the integral is taken. Equation (1) says that a point on the vortex sheet moves with the average of the two limiting velocities, as the curve is approached from either side.

A flat vortex sheet of constant strength $z(\Gamma, t) = \Gamma$ is an equilibrium solution of (1). Linear stability analysis shows that short wavelength perturbations can grow arbitrarily fast (Kelvin-Helmholtz instability). This means that the linearized initial value problem is ill-posed in the sense of Hadamard. However, Sulem et al. [35] have proven that if the initial perturbation is an analytic function of Γ, then the solution of (1) remains analytic for a positive time interval.

Birkhoff conjectured that instability and nonlinearity would cause a singularity to form during the vortex sheet's evolution [4, 5]. An asymptotic analysis by Moore [24, 26] supports this conjecture, indicating that with initial perturbation amplitude ε, a $\Gamma^{3/2}$ branch point forms in the vortex sheet at a finite critical time $t = t_c(\varepsilon)$. Meiron et al. [23] analyzed the Taylor series coefficients of $z(\Gamma, t)$

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with respect to the time variable and obtained results consistent with Moore's. The validity of Moore's approximation for $t < t_c$ has been proven [6] and special solutions have been studied [7, 14], but proving that a singularity forms for general initial data is an open problem.

3. The Point Vortex Approximation

Rosenhead performed the first vortex sheet computation in 1931 [31], using the periodic Cauchy kernel in (1). The sheet was discretized by a finite number of point vortices per period $z_j(t) \sim z(\Gamma_j, t), j = 1, \dots, N$, leading to the ordinary differential equations,

$$\frac{\overline{dz_j}}{dt} = \sum_{k \neq j} K(z_j - z_k) N^{-1}, \quad K(z) = \frac{1}{2i} \cot \pi z . \tag{2}$$

The sum omits the singular term $k = j$, but if the vortex sheet has a bounded 2nd Γ-derivative, then the discretization error is $O(N^{-1})$ [25]. If the vortex sheet is analytic, then infinite order accuracy may be obtained by applying one step of Richardson extrapolation [34, 16].

Rosenhead used $N \sim 10$ points and the 1st order Euler method with time step $\Delta t \sim 0.05$ to integrate in time. He drew a smooth interpolating curve through the point vortices, suggesting that a perturbed vortex sheet rolls up into a smooth spiral. In the 1950's, Birkhoff performed computations using a larger number of point vortices and more accurate time integration [4, 5]. In contrast to Rosenhead's results, the points' computed motion was irregular, leading Birkhoff to question whether the vortex sheet rolls up into a spiral. Later workers sought to obtain convergent numerical results by using higher order accurate quadrature rules for the principal value integral, e.g. [15, 38]. Another approach was to stabilize the problem by adding surface tension [28]. In spite of much effort, the computations failed to converge as the number of points increased.

The key to obtaining convergent numerical results for $t < t_c$ lies in Fourier analysis of the computed solution [19]. Sulem et al. [36] showed that the singularity structure of nonlinear evolution equations could be obtained from spectral computations, by analyzing the rate of decay of the discrete Fourier coefficients. For vortex sheet computations, discrete Fourier coefficients of the perturbation quantities $p_j(t) = z_j(j) - \Gamma_j$ can be similarly analyzed.

Figure 1 shows computations with $N = 50$ in single and double precision arithmetic. Irregular small scale motion develops in single precision, but the double precision results are smooth. The corresponding spectral amplitudes are plotted in Fig. 2. The initial spectrum has a spike at wavenumber $k = 1$ (an explicit perturbation of amplitude $\varepsilon = 0.01$), as well as broad band noise in the higher modes. In Fig. 2a, the noise is amplified by the system's instability, leading to the irregular motion in Fig. 1a for $t \geq 0.3$. In Fig. 2b, the spectrum spreads smoothly to higher wavenumbers, due to genuine nonlinear effects [19].

A stable physical process is modeled by a well-posed initial value problem, and if the difference scheme is consistent and stable, then the solution converges as the mesh is refined [30]. Shear flows however are physically unstable and this appears as ill-posedness in the vortex sheet initial value problem. The point vortex approximation for an analytic vortex sheet defines a consistent but unstable

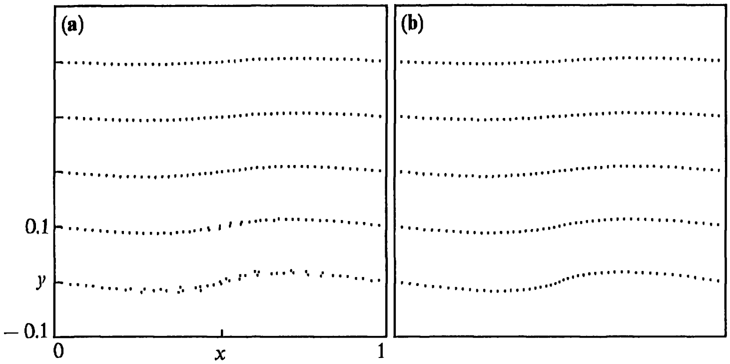

Fig. 1. Point vortex computations at times $t = 0, 0.1, 0.2, 0.3, 0.4$. **(a)** single precision. **(b)** double precision

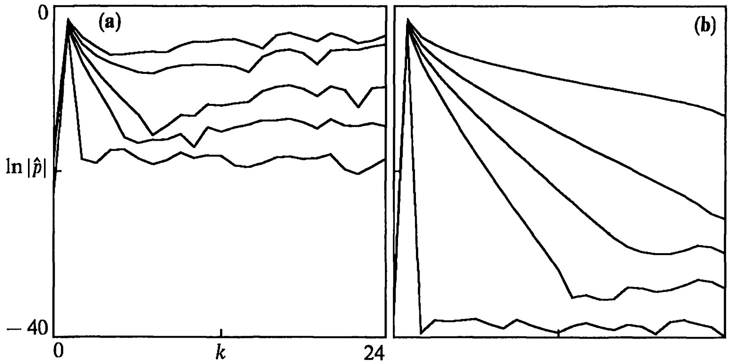

Fig. 2. Discrete Fourier coefficient amplitudes corresponding to Fig. 1. **(a)** single precision. **(b)** double precision

difference scheme. Fritz John has observed [18], "Instability of a difference scheme under small perturbations does not exclude the possibility that in special cases the scheme converges towards the correct function, if no errors are permitted in the data or the computation." This refers to roundoff error, due to the computer's finite precision arithmetic, as opposed to discretization error, due to replacing a continuous operator by a discrete approximation. Using higher precision arithmetic is one way to see convergence as the mesh is refined, but for vortex sheet computations, a more practical remedy is to filter out the spurious roundoff error perturbations [19]. Computations and theory [8] now show that the point vortex approximation converges as $N \to \infty$ for $t < t_c$. A consistent picture of singularity formation in a vortex sheet has been obtained: infinite curvature forms at an isolated point, but the vortex sheet remains continuously differentiable at $t = t_c$, showing no sign of roll-up [19, 23, 24, 26, 34].

An obvious question is whether the vortex sheet continues to exist past the critical time. Note that in other problems, a physically valid weak solution can be defined past a critical time, e.g. shock formation in a nonlinear hyperbolic equation. Computations show that the point vortex approximation does not converge for $t > t_c$ as $N \to \infty$ [19]. A different type of small scale motion occurs in the point vortex system (2) for $t > t_c$, but it is not relevant to vortex sheet evolution. Based on work with self-similar vortex sheets [27], Pullin conjectured that a periodically perturbed sheet rolls up into a spiral for $t > t_c$, the spiral vanishes in size as $t \to t_c{}^+$, and for any $t > t_c$ it has an infinite number of turns [29]. As described in the next section, numerical experiments using Chorin's vortex blob method support this conjecture [1, 9, 10, 11, 20, 21, 22].

4. Vortex Sheet Roll-Up

Let $\delta > 0$ be a smoothing parameter and consider a regularized approximation to (1),

$$\overline{\frac{\partial z}{\partial t}}(\Gamma, t) = \int_{-\infty}^{\infty} K_\delta(z(\Gamma, t) - z(\tilde{\Gamma}, t))d\tilde{\Gamma}\,, \quad K_\delta(z) = K(z)\frac{|z|^2}{|z|^2 + \delta^2}\,. \tag{3}$$

When (3) is discretized, the computational elements are called "vortex blobs". For fixed $\delta > 0$, short wavelength perturbations no longer have unbounded growth rates and computed solutions converge as the number of blobs $N \to \infty$, even for $t > t_c$ [20]. Figure 3a shows the evolution for $0 \leq t \leq 1$, with the smoothing parameter value $\delta = 0.03$, in a case for which the vortex sheet's critical time is $t_c \sim 0.375$. Figure 3b shows the solution at time $t = 1$ with decreasing amounts of smoothing $0.05 \leq \delta \leq 0.2$. Figure 4 shows that the smoothed solutions at $t = 1$ converge to a spiral as $\delta \to 0$. The limit spiral is a candidate extension for the vortex sheet past the critical time.

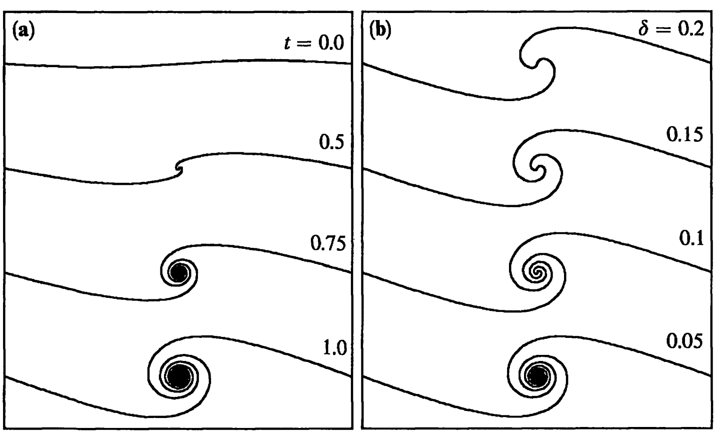

Fig. 3. Regularized vortex sheet roll-up past the critical time $t_c \sim 0.375$. **(a)** $\delta = 0.03$, increasing time. **(b)** $t = 1$, decreasing δ

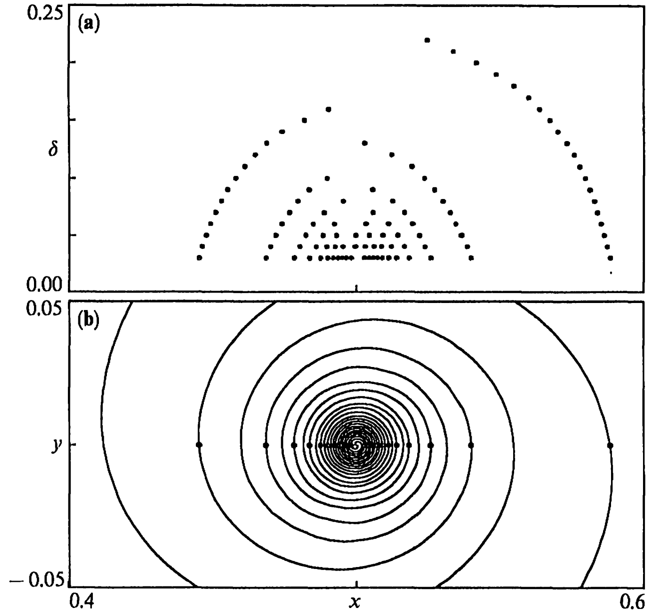

Fig. 4. Convergence as $\delta \to 0$ for $t = 1 > t_c \sim 0.375$. **(a)** x-axis intercepts of one spiral branch plotted against δ. **(b)** closeup of the solution for $\delta = 0.03$

The numerical experiments suggest that the vortex blob method provides a convergent discretization of vortex sheet motion for $t > 0$. This has been proven for $t < t_c$ [8], but proving convergence for the physically important roll-up regime $t \geq t_c$ is an outstanding problem. Other interesting issues concern uniqueness of the limit for different regularizations [3, 37], existence of a weak solution to the incompressible Euler equations with general vortex sheet initial data and the possible presence of concentrations in the limit $\delta \to 0$ [12, 13].

5. Applications

The vortex blob method has advantageous mathematical and numerical properties, but the smoothing parameter δ has no precise physical meaning. One would like to know whether computations performed with a value $\delta > 0$ approximate real fluid motion. Some applications presented below demonstrate the vortex blob method's potential for simulating shear layer dynamics.

Aircraft Trailing Vortices. On takeoff and landing, an aircraft sheds vortices at the wing's trailing edge. Figures 5 and 6 show a free-space vortex sheet simulation of this process, including the effects of the wing tips and deployed flaps [21]. The computation illustrates different types of vortex interactions: rotation of like-sign vortex pairs, translation of opposite-sign vortex pairs, core deformation due to collision, and vortex sheet folding.

Separation at a Sharp Edge. Vortices are shed from the edges of a flat plate that is moving in a viscous fluid. As the viscosity is reduced, an ideal flow emerges

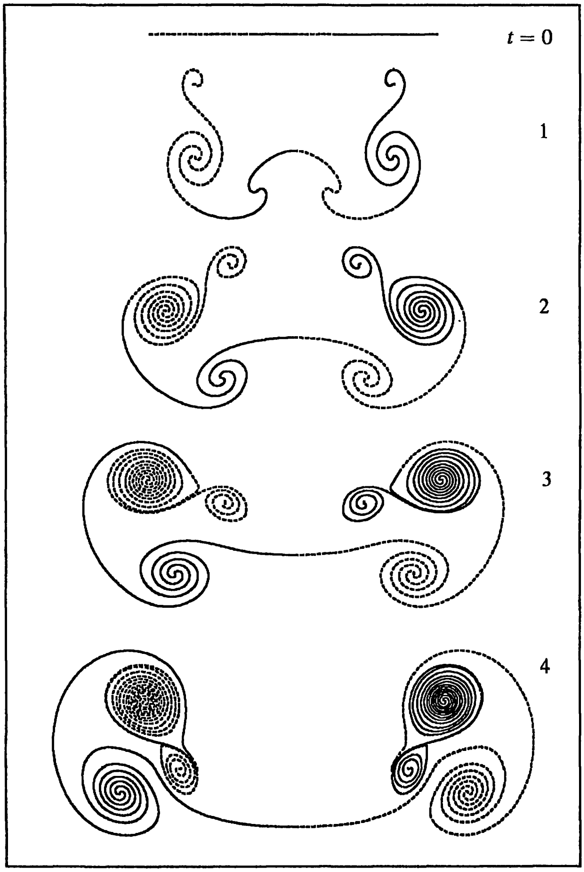

Fig. 5. Roll-up of an aircraft trailing vortex sheet, including tip and flap vortices. The solid and dotted lines indicate opposite senses of rotation

having embedded vortex sheets that emanate from the edges. This problem is more difficult to compute than the periodic and free space problems considered above. New issues arise, in satisfying the flow tangency condition on the plate, and shedding the correct amount of circulation at the edges. Previous numerical studies did not obtain smooth spiral roll-up, e.g. [17, 33].

A new implementation of the vortex blob method has been developed. Figure 7a is a computation of the vortex sheets that separate from an impulsively started flat plate. The velocity field plotted in Fig. 7b shows that the sheets form a recirculating region behind the plate.

To validate the algorithm, a comparison with Pullin's computation of self-similar vortex sheet roll-up [27] has been performed. The similarity assumption circumvents the difficulty of solving the initial value problem. Figure 8 compares a time dependent vortex blob computation with Pullin's self-similar result. The two plots may be superimposed to verify that the spiral shapes are in good agreement. Further details are given in [22] and a more complete validation is in preparation.

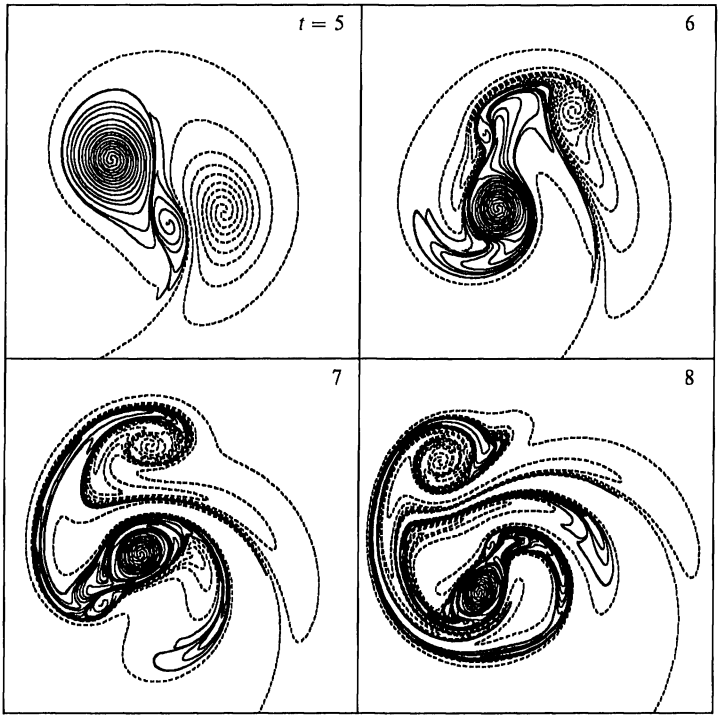

Fig. 6. Continuation of Fig. 5, showing details of core deformation and vortex sheet folding

Instability of a Jet. Figure 9 shows the evolution of a jet being expelled from a box. The jet is driven by two point sources in the lower corners of the box, which are turned on at time $t = 0$. A starting vortex forms and propagates away from the outlet, leaving behind a thin straight jet. Waves form along the jet, rolling up into a small vortex which propagates through the large starting vortex.

6. Final Remarks

Vortex sheet motion poses interesting mathematical problems concerning singular integrals, weak limits, and nonlinear dynamics. Vortex blob computations may provide a useful tool for clarifying the role of coherent vortex structures in shear flow. Future computational work will focus on improved treatment of boundary conditions, the effects of parametric forcing, and three dimensionality.

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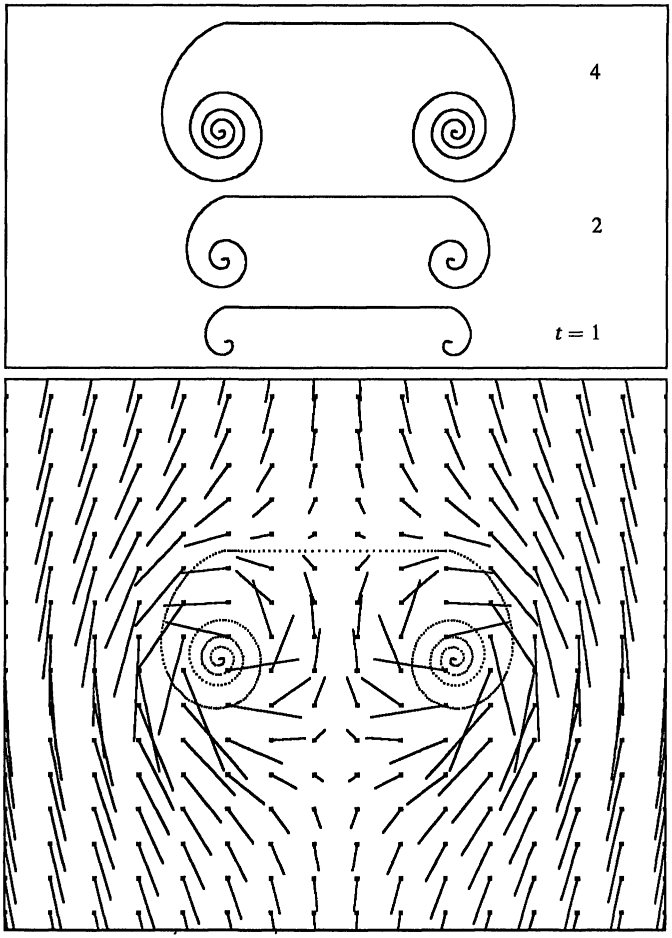

Fig. 7. (a) Vortex sheet roll-up due to the impulsively started upward motion of a flat plate. **(b)** Velocity field at time $t = 4$

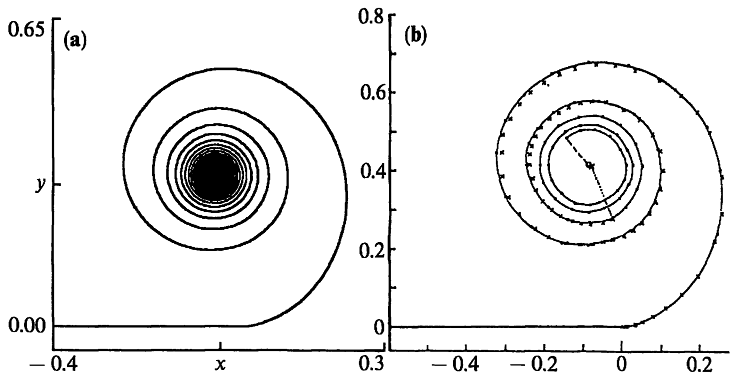

Fig. 8. (a) Time dependent vortex blob computation, $\delta = 0.025, t = 1$. **(b)** Self-similar vortex sheet roll-up past a semi-infinite flat plate, reproduced from [27]

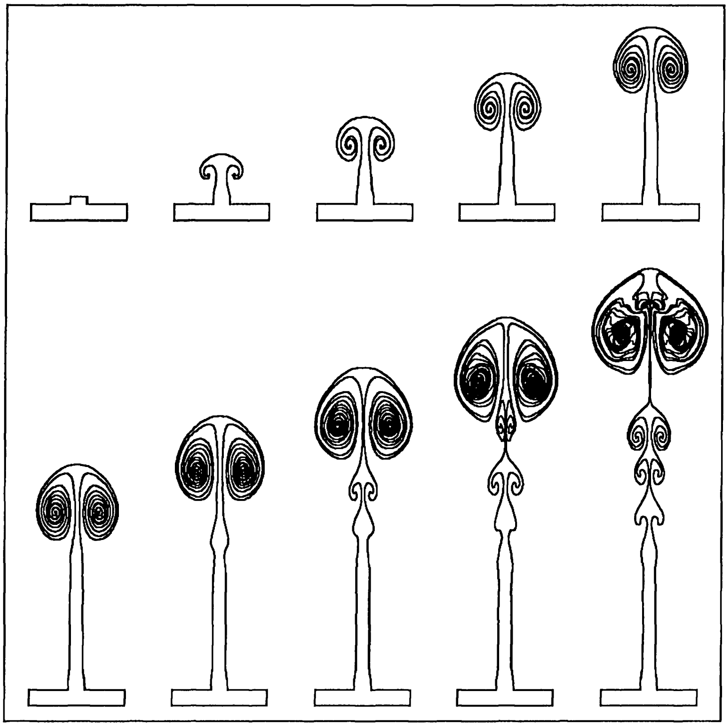

Fig. 9. Computation of a jet being expelled from a box

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