A NUMERICAL METHOD FOR VORTEX SHEET ROLL-UP

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SUMMARY

The problem of computing vortex sheet roll-up from periodic analytic initial data is studied. Previous theoretical and numerical work is reviewed. Computational difficulties arising from ill posedness and singularity formation are discussed. A desingularization method is proposed to diminish these difficulties. Computations indicate that this approach converges past the time at which previous numerical investigations have failed to converge.

INTRODUCTION

The aim of this paper is to discuss some recent progress in computing vortex sheet evolution. The vortex sheet is an asymptotic model for shear layer instability in which the transition region between the two parallel streams is approximated by a surface across which the tangential velocity component has a jump discontinuity. In pursuing these calculations we have been influenced by Corcos & Sherman [7] who advocate the use of deterministic models to study the dynamics of the coherent structures that occur in turbulent mixing layers. The specific problem considered here is that of an infinite plane vortex sheet embedded in two dimensional ideal fluid which at time $t = 0$ is given a periodic analytic perturbation. The numerical work to be discussed aims at providing a convergent discretization of periodic vortex sheet roll-up, with the hope that this will contribute to a better understanding of shear layer dynamics.

GOVERNING EQUATIONS

A vortex sheet in two dimensional ideal flow can be described by a complex curve $x(\Gamma,t) + iy(\Gamma,t)$ where t is time and Γ is a Lagrangian parameter which measures the circulation between a base point $\Gamma = 0$ and an arbitrary point along the sheet [5]. The vortex sheet strength is the jump in the tangential component of velocity across the sheet and is given by $(x_\Gamma^2 + y_\Gamma^2)^{-1/2}$. Consider a vortex sheet for which $x(\Gamma,t) - \Gamma + iy(\Gamma,t) = p(\Gamma,t)$ is a periodic function of Γ with period one. The equations to be studied are,

$$\frac{\partial x}{\partial t} = \frac{-1}{2}\int_0^1 \frac{\sinh 2\pi(y-\tilde{y})}{\cosh 2\pi(y-\tilde{y}) - \cos 2\pi(x-\tilde{x}) + \delta^2}\, d\tilde{\Gamma}\,, \tag{1a}$$

$$\frac{\partial y}{\partial t} = \frac{1}{2}\int_0^1 \frac{\sin 2\pi(x-\tilde{x})}{\cosh 2\pi(y-\tilde{y}) - \cos 2\pi(x-\tilde{x}) + \delta^2}\, d\tilde{\Gamma}\,, \tag{1b}$$

where $x = x(\Gamma,t)$, $\tilde{x} = x(\tilde{\Gamma},t)$, etc. The term δ^2 in the integrands' denominator is a smoothing parameter whose inclusion is motivated by the work of Chorin & Bernard [6] and Anderson [1]. For now we confine ourselves to the case $\delta = 0$ in which the integrals are interpreted as Cauchy principal value integrals. Then equations (1a,b) are the exact vortex sheet evolution equations, expressing the kinematical condition that the vortex sheet move with the fluid. The dynamical requirement that circulation around material curves be preserved is contained in the statement that Γ is a Lagrangian variable.

THEORETICAL ISSUES WHEN $\delta = 0$

A flat vortex sheet of constant strength one (given by $x(\Gamma,t) + iy(\Gamma,t) = \Gamma$) is in equilibrium. Linear stability analysis about this equilibrium yields perturbation solutions proportional to $\exp 2\pi(\omega t + ik\Gamma)$ when the dispersion relation $\omega^2 = \frac{1}{4}k^2$ is satisfied [3]. Since there are short wavelength solutions having arbitrarily large growth rates ("Kelvin-Helmholtz instability") the linearized initial value problem is not well posed in the sense of Hadamard, i.e. the solution does not depend continuously on the initial data. Birkhoff [5] conjectured the short time existence of an analytic solution to the nonlinear vortex sheet problem. This was proven by Sulem et al. [17].

Another conjecture of Birkhoff & Fisher [4] was that an initially analytic vortex sheet can stop being analytic at a finite time t_c. Support for this was given by Moore [12,14] in an asymptotic analysis based on small initial amplitude. Meiron et al. [11] used Taylor series in time and obtained good agreement with Moore's results. Both investigations found that at the critical time t_c, the vortex sheet strength has a cusp and the curvature has an infinite jump discontinuity although the sheet's slope remains bounded and its tangent vector is continuous. Moore and Meiron et al. also express the opinion that the singularity formation implies a restriction on the validity of the vortex sheet model. This view however may be too pessimistic since weak solutions to other model systems (e.g. nonlinear hyperbolic equations) can be theoretically justified and remain physically relevant beyond the time of singularity formation. In fact, D. Pullin has conjectured (private communication 1983) that past the vortex sheet's critical time, the sheet is a double branched spiral with an infinite number of turns, and that as t_c is approached from above, the spiral vanishes in size. This idea is motivated by the study of self-similar spiral formation for initially flat vortex sheets which have a singular strength distribution [15].

The work of Moore, Meiron et al. and Sulem et al. has led to a better understanding of vortex sheet evolution from analytic initial data. This contrasts with the failure of standard numerical methods to solve equations (1a,b) when $\delta = 0$. We now briefly discuss some of the issues pertaining to these calculations.

NUMERICAL ISSUES WHEN $\delta = 0$

The classical method, introduced by Rosenhead [16], is to approximate the curve by a finite number of points per wavelength corresponding to a uniform Γ-mesh. Thus $x_j(t) + iy_j(t)$ approximates $x(\Gamma_j,t) + iy(\Gamma_j,t)$ for $\Gamma_j = (j-1)\Delta\Gamma$, $j = 1,\ldots N$ and $N = 1/\Delta\Gamma$. The Cauchy principal value integrals are approximated using a simple quadrature rule which omits the integrands' singularity at $\tilde{\Gamma} = \Gamma$. This gives a system of ordinary differential equations for the points' paths,

$$\frac{dx_j}{dt} = \frac{-1}{2N}\sum_{\substack{k=1\\k\neq j}}^{N}\frac{\sinh 2\pi(y_j-y_k)}{\cosh 2\pi(y_j-y_k)-\cos 2\pi(x_j-x_k)\,+\,\delta^2}, \tag{2a}$$

$$\frac{dy_j}{dt} = \frac{1}{2N}\sum_{\substack{k=1\\k\neq j}}^{N}\frac{\sin 2\pi(x_j-x_k)}{\cosh 2\pi(y_j-y_k)-\cos 2\pi(x_j-x_k)\,+\,\delta^2}. \tag{2b}$$

The term δ^2 is included for later reference; our remarks in this section concern the case when $\delta = 0$. Equations (2a,b) also describe the evolution of N periodic rows of equal strength point vortices. Using a small initial perturbation, Rosenhead integrated these equations with a value of $N = 12$. The interpolating curve that was drawn through the points indicated smooth roll-up around periodic concentrations of vorticity. The validity of these findings was challenged by Birkhoff [5] who viewed Rosenhead's work as inconclusive since convergence as N increases had not been demonstrated. Birkhoff's calculations with $N = 20$ produced irregular point motion, i.e. the interpolating curve became tangled and did not roll up smoothly.

Later investigators have modified the point vortex approximation in various ways. For example, van de Vooren [18] subtracted the singularity in the Cauchy principal value integral and derived equations differing from the point vortex approximation by correction terms $\frac{-1}{4\pi N}\left(\frac{y_\Gamma}{x_\Gamma^2 + y_\Gamma^2}\right)_\Gamma$ and $\frac{-1}{4\pi N}\left(\frac{x_\Gamma}{x_\Gamma^2 + y_\Gamma^2}\right)_\Gamma$ added to the right side of (2a) and (2b). (These Γ-derivatives are evaluated at $\Gamma = \Gamma_j$.) Moore [13] pointed out that this correction gives the leading term in the truncation error of the point vortex approximation. However, this will be true only as long as the derivatives in the correction terms are bounded, i.e. up to the vortex sheet's critical time.

Unlike the situation for a well posed initial value problem, a consistent discretization of an ill posed problem, like the present one, must be unstable. However, as noted by John [8], "Instability of a difference scheme under small perturbations does not exclude the possibility that in special cases the scheme converges towards the correct function, if no errors are permitted in the data or the computation." Conversely, if an ill posed analytic initial value problem is solved with a fixed machine precision, refining the mesh can sometimes increase the computational error. This is because the discretization then resolves shorter wavelength modes which grow faster once they are introduced by roundoff error. This difficulty was overcome for the present problem by using higher machine precision and by filtering out the spurious roundoff error perturbations in wavenumber space [9]. Numerical evidence indicates that the point vortex approximation converges at the rate $O(N^{-1})$ for $0 \le t \le t_c$. Results concerning the vortex sheet's singularity were obtained which agree with the findings of Moore and Meiron et al. It was also observed that the point vortex approximation did not converge as N increases when $t > t_c$.

Summarizing, there are two difficulties which affect computations for the present problem,

(1) growth of spurious roundoff error perturbations due to short wavelength instability,

(2) loss of the discretization's consistency when the exact solution stops being analytic ($t = t_c$).

Were it not for the second difficulty, a higher order accurate discretization (such as van de Vooren's) would be appropriate for dealing with the first difficulty (since a coarser mesh could then be used to attain a given error tolerance). In the next section we apply a method of desingularization to diminish both of these difficulties and to investigate the vortex sheet's evolution past the critical time.

DESINGULARIZATION

With a value of $\delta > 0$ the integrals in (1a,b) are proper and these equations then give a desingularized approximation of vortex sheet evolution. We shall sometimes also refer to (1a,b) as the "δ-equations". The linear dispersion relation becomes,

$$\omega^2 = \frac{k(1-e^{-k\cosh^{-1}(1+\delta^2)})e^{-k\cosh^{-1}(1+\delta^2)}}{4\,\delta\,(2+\delta^2)^{1/2}}. \tag{3}$$

The positive branch of ω is plotted in figure 1 as a function of wavenumber $k \ge 0$ for several values of δ. For a fixed value of $\delta > 0$ there is a wavenumber k_m for which the growth rate

$\omega(k_m)$ is maximum, and in the limit $k \to \infty$, $\omega(k)$ is asymptotic to zero. The δ-equations therefore do not exhibit the severe short wavelength instability of the exact equations. For a fixed wavenumber k, $\omega(k)$ given by (3) converges as $\delta \to 0$ to the linear dispersion relation of Kelvin-Helmholtz instability (the straight line in figure 1). The form of desingularization chosen here does not correspond precisely to a stabilizing physical effect such as surface tension or finite layer thickness. It may be thought of as including "artificial viscosity" although the δ-equations are not dissipative (even with $\delta > 0$ they retain the form of a Hamiltonian system).

With a value of $\delta > 0$, the ordinary differential equations (2a,b) correspond to a trapezoidal rule approximation of the δ-equations (1a,b). For a fixed value of N the solution of equations (2a,b) converges to the solution of the point vortex equations in the limit $\delta \to 0$. As already remarked this limit does not yield information about the vortex sheet when $t > t_c$. Following Anderson [1], for each value of $\delta > 0$ we shall take N large enough to obtain a very accurate solution of the δ-equations (1a,b) for that particular value of δ. By repeating this process at some time $t > t_c$ for several values of δ, it will be possible to examine the limit $\delta \to 0$ of solutions to the δ-equations past the vortex sheet's critical time. We emphasize the experimental nature of this investigation; there is currently no rigorous theory concerning this limiting process and the vortex sheet's long time existence and regularity are open problems.

COMPUTATIONAL RESULTS

The initial condition was chosen to be a perturbation of the equilibrium solution by a linear theory growing eigenfunction for which the singularity forms in the vortex sheet at $t_c = 0{\cdot}375$. The discretized initial condition was $x_j(0) + iy_j(0) = \Gamma_j + 0{\cdot}01(1-i)\sin 2\pi\Gamma_j$. Equations (2a,b) were integrated by the 4th order Runge-Kutta method on a VAX 11/780 computer using, unless otherwise noted, single precision arithmetic.

Solution for $\delta = 0{\cdot}25$

Figure 2 shows a time sequence of the numerical solution using the value $\delta = 0{\cdot}25$ for the smoothing parameter, with $N = 400$ points and a time step $\Delta t = 0{\cdot}05$. The point positions are plotted on the right side of figure 2 and a trigonometric interpolating curve is plotted on the left. The values of N and Δt used were determined empirically and it was checked that using smaller Δt and larger N would not change the plotted curve.

The curve in figure 2 has attained a vertical slope before $t = 1$ and rolls up smoothly at later times. For $t > 2$ there is an inner region or core consisting of turns which become more closely spaced as time progresses. The outer region of the curve becomes elliptical in shape at late times, with the ellipse's major axis tilted slightly from the horizontal. The uneven spacing of the Lagrangian points on the right side of figure 2 indicates that the strain rate along the curve is nonuniform. The "braid" region (centered at integer values of x and connecting the cores) is most strongly stretched. As time progresses, the points travel inward along the curve's arms, being compressed near the ellipse's major axis and stretched near the minor axis.

Convergence in N and δ *past the vortex sheet's critical time*

For a fixed value of $\delta > 0$, the curves interpolating the solution of the ordinary differential equations (2a,b) converge as the number of points N is increased. This convergence occurs at any time, even past the vortex sheet's critical time. Figure 3 illustrates this, showing the results at $t = 4$ for $\delta = 0{\cdot}25$ with $N = 50$, 100 and 200. The time step was small enough to ensure that for each value of N the point positions are an accurate solution of equations (2a,b) to within the plotting resolution. With a small value of N, the interpolating curve is tangled, but as N increases, the tangling disappears. When $N = 200$, the curve's shape has converged to within plotting resolution as may be seen by comparison with the $N = 400$ solution in figure 2.

The effect of decreasing δ at a fixed time ($t = 1$) greater than the vortex sheet's critical time is shown in figure 4 which plots the interpolating curve for several values of δ between 0·2 and 0·05. These calculations used $N=400$ and $\Delta t=0{\cdot}05$, except for the $\delta=0{\cdot}05$ case which used $\Delta t=0{\cdot}01$ and which was performed in double precision arithmetic.

As δ decreases with $t = 1$ in figure 4, more turns appear in the core. For $\delta = 0{\cdot}05$, the core region is tightly packed and an enlarged view (figure 4) shows that each branch of the spiral contains five complete revolutions. Figure 6 contains information about how the core region behaves as δ decreases, keeping $t = 1$ fixed. The curves' x-axis intercepts are plotted as a function of δ for a single branch of the spiral ($0 \le \Gamma \le 0{\cdot}5$). For example, with $\delta = 0{\cdot}1$ each spiral branch has four x-axis intercepts (see figure 5; the intercept at $x = 0{\cdot}5$ is not included in figure 6). The outermost intercept approaches a value near $x = 0{\cdot}6$ as $\delta \to 0$, and the next intercept on the spiral branch approaches a value near $x = 0{\cdot}45$. A well defined spiral shape is emerging at $t = 1$ in the limit $\delta \to 0$.

Past the vortex sheet's critical time, the curve becomes a more complicated object as δ decreases and an accurate approximation becomes more expensive, requiring larger values of N and smaller Δt. For example, the double precision $\delta = 0{\cdot}05$ calculation required about five hours to run on the VAX 11/780. This should not be taken to mean that computations using values of δ smaller than 0·05 are infeasible. The difficulty in resolving such solutions could be overcome to a large extent by using a faster computer or by using more efficient adaptive Γ and t meshes. Another difficulty affecting computations with a small value of δ is related to, though less severe than, the loss of computational accuracy due to roundoff error that occurs in point vortex calculations for the present problem. In [10] it was shown that computational accuracy can be improved by using higher machine precision or a spectral filter.

DISCUSSION

The desingularization method used here mitigates the difficulties which have caused previous studies of vortex sheet roll-up from analytic initial data to fail. For $\delta=0$, short wavelength instability restricts the number of points which can be used for a given machine precision. This restriction is loosened when $\delta>0$ since then the short wavelength modes are not violently unstable. Since putting $\delta > 0$ also apparently mollifies the vortex sheet's singularity formation, the point vortex approximation ($\delta = 0$) is more appropriate for studying that phenomenon [9]. Alternatively, the point vortex approximation does not converge beyond the vortex sheet's critical time and smoothing appears to be needed in order to study the sheet's later evolution.

Evidently for any fixed value of $\delta>0$, the solution of the δ-equations becomes, for some $t>t_c$, a spiral with a finite number of turns. As δ approaches zero, more turns appear at a fixed time $t > t_c$. These computational results are consistent with Pullin's conjecture, previously mentioned. Further work is needed to understand the relation between the singularity which forms in the vortex sheet at t_c and the spiral structure which is apparently present at later times.

The desingularized equations come from replacing the velocity field of a periodic row of point vortices by that of "vortex blobs". Convergence of vortex blob methods has been proven for problems involving a smooth vorticity field (see [2] and the references contained therein). In that case the initial value problem considered is well posed and vortex blobs ensure that the discretization is consistent and stable. This contrasts with the situation for vortex sheet evolution where the effect of using vortex blobs is to replace the ill posed initial value problem with a sequence of problems that are better behaved (though still unstable).

The curves computed with a fixed value of $\delta > 0$ resemble pictures of material curves in shear flow which have been obtained by other investigators. For example, figure 2 displays features similar to finite difference solutions of the Navier-Stokes equations [7] and to the flow visualization obtained by Roberts et al. (see [19] p.85). It would be interesting to know precisely in what sense the solution of the δ-equations (1a,b) with a fixed value of $\delta > 0$ approximates a solution of the Euler or the Navier-Stokes equations.

The main result presented here is the numerical demonstration that the desingularization approach converges past the vortex sheet's critical time, if the mesh is refined and the smoothing parameter is reduced in the proper order. This is an improvement over previous numerical studies of periodic vortex sheet roll-up. It is desireable that a rigorous theoretical justification of the procedure be provided in the future. The desingularization approach taken here is capable of generalization to other singular boundary integral problems and an application to the vortex sheet shed by an elliptically loaded wing is in progress.

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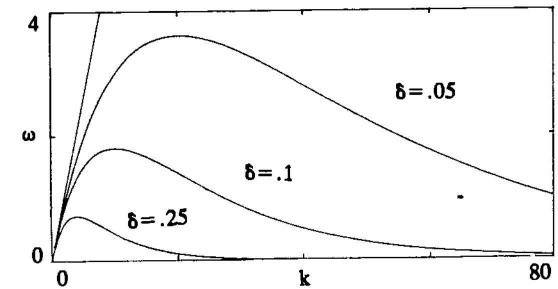

Fig. 1 Linear dispersion relation of eqs. (1a,b).

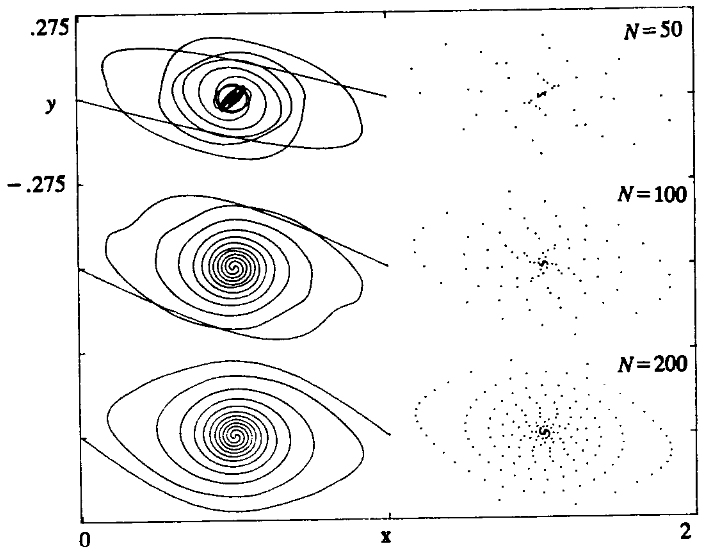

Fig. 3 Convergence in N past t_c. δ = .25 , t = 4

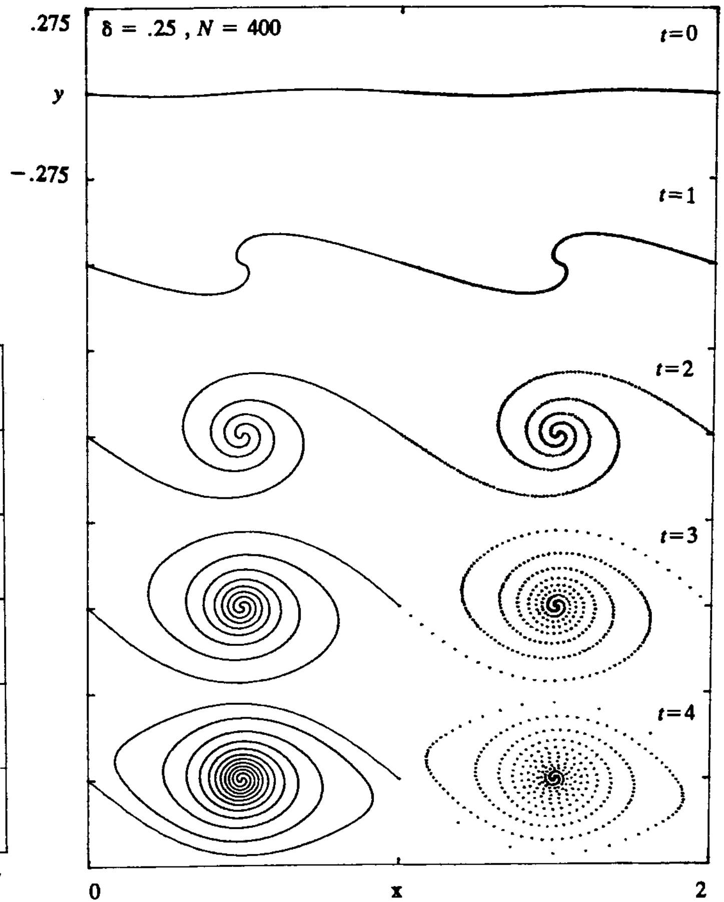

Fig. 2 Desingularized vortex sheet evolution.

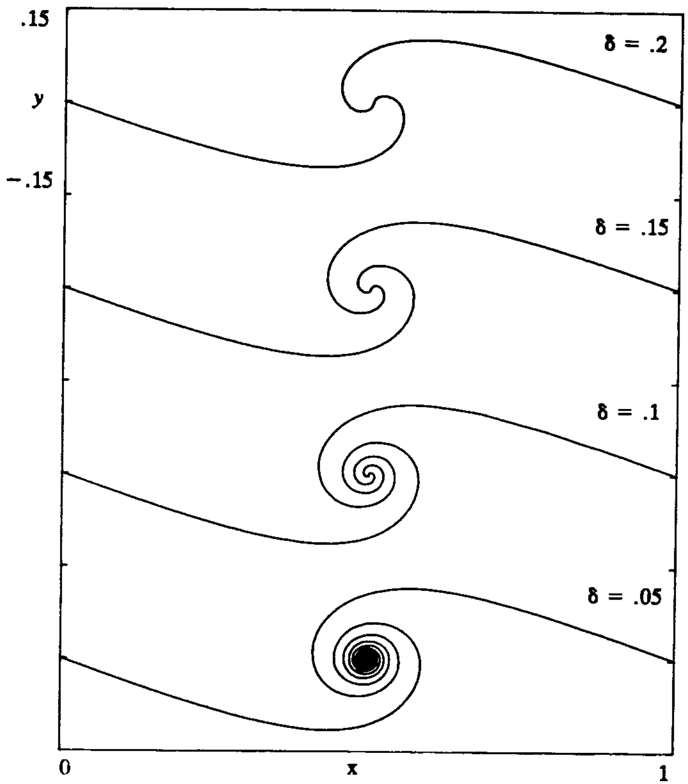

Fig. 4 Convergence in δ past t_c. $t = 1$

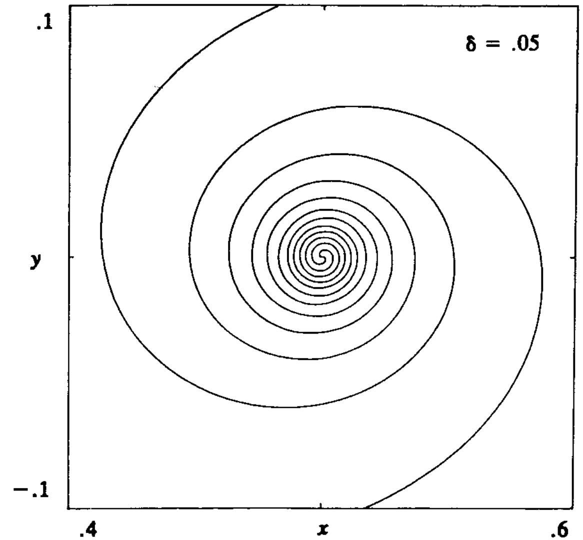

Fig. 5 Closeup of core from Fig. 4.

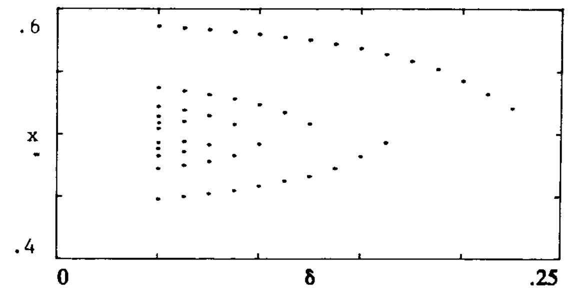

Fig. 6 Convergence of x-intercepts as $\delta \to 0$