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VORTEX SHEET ROLL-UP DUE TO THE MOTION OF A FLAT PLATE

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ABSTRACT

The vortex-blob method is extended to compute vortex sheet separation at a sharp edge. A smoothing parameter controls the amount of detail occuring in the rolled-up spiral. Calculations are presented for the case of vortex sheet roll-up due to an impulsively started flat plate.

INTRODUCTION

Unsteady separation is an important process in fluid dynamics. Separation refers to the process by which a boundary layer leaves the vicinity of a solid surface and enters the interior of the flow field as a free shear layer. Boundary layer separation changes the pressure distribution and thus also affects the drag and lift felt by solid surfaces in the flow. An improved understanding and capability for modeling separation would have important applications in areas such as aerodynamics and ship hydrodynamics.

Many different types of numerical methods have been used to study flow separation, including finite difference, finite element, spectral and vortex methods. The present paper discusses an extension of the vortex-blob method to study a particular example of unsteady separation : vortex sheet roll-up due to the impulsively started motion of a flat plate. This work represents a step towards developing a general numerical method for vortex sheet motion, which could be used to study unsteady separation and the dynamics of free shear layers.

In the model considered here, the vorticity field consists of a bound vortex sheet on the plate and free vortex sheets emanating from the edges of the plate. In order to make the roll-up problem for the free vortex sheets numerically tractable, an artificial smoothing parameter is inserted into the evolution equation. The smoothing parameter enhances the stability of the free vortex sheets and controls the rate at which they roll up. The flow tangency condition along the plate is used to solve for the strength of the bound vortex sheet. The unsteady Kutta condition is used to calculate the circulation shedding rate at the edges of the plate. Numerical results presented here demonstrate the method's ability to model vortex sheet separation and roll-up at a sharp edge.

BACKGROUND

Consider a flat plate at rest which coincides with the horizontal line segment $-1 \leq x \leq 1, y = 0$ for time $t < 0$. The fluid, which is also at rest for $t < 0$, is assumed to be incompressible and inviscid. At $t = 0$, the plate impulsively starts to move in the vertical direction ($u = 0, v = 1/2$). For $t \geq 0$, the flow is required to be tangent to the plate. The problem formulated - impulsively started motion of a flat plate in an incompressible, inviscid fluid - has two possible solutions. The boundary condition enforced at the edges of the plate determines which of the two solutions is realized.

One possibility is a continuous potential flow, i.e. an ideal flow, with no free vortex sheets. The fluid velocity becomes infinite near the edges of the plate, but it remains continuous away from the plate. In a frame of reference moving with the plate, the flow is steady and symmetric with respect to reflection across the line containing the plate. No vorticity is shed in this flow and consequently, there is no wake. Experimental flow visualization documents the separation of thin shear layers at the edges of a flat plate moving normal to the flow (1,2). The ideal flow is therefore a poor approximation to the real flow that occurs behind an impulsively started plate.

The second possibility is a discontinuous potential flow, i.e. a flow with free vortex sheets emanating from the edges of the plate. We would like to think of this solution as the zero viscosity limit for solutions of the viscous flow problem. Fluid viscosity, no matter how small, produces boundary layer separation at the edges of the plate. In the limit of vanishing viscosity, the boundary condition that should be enforced at the edges of the plate is called the "unsteady Kutta condition".

This condition says that circulation is shed at precisely the rate needed to keep the fluid velocity finite near the edges of the plate. The two shed vortex sheets roll up into counter-rotating spirals, amounting to a particular wake model for the real flow behind a moving plate.

The possibility that a free vortex sheet results from taking the zero viscosity limit, in the presence of a sharp-edged boundary, goes back to Prandtl. This idea has great appeal, although satisfactory justification is not yet available. As a step in this direction, the work described here is concerned with developing an improved numerical method for computing vortex sheet roll-up at a sharp edge.

Previous computations have used the point vortex method to represent the free vortex sheets (3,4,5,6,7). The literature has been reviewed by Graham (8) and his paper contains other references as well. Many of these studies have encountered difficulty in obtaining smooth spiral roll-up of the free vortex sheets. The present work seeks to overcome these difficulties by applying Chorin's vortex-blob method (9,10). We shall briefly review some recent work related to this approach.

Putting aside for the moment issues concerned with solid boundaries, difficulties arise in computing vortex sheet roll-up in free space because of Kelvin-Helmholtz instability (11). The instability causes a singularity to form in an evolving vortex sheet at a finite critical time $t = t_c$ (12,13,14). At the critical time, the sheet has infinite curvature at some point, but the sheet's slope remains bounded and there is no sign of roll-up. It has been found that the point vortex approximation converges as the number of points is increased, as long as the vortex sheet remains an analytic curve. However past t_c, the numerical evidence indicates that the point vortex approximation does not converge.

Chorin's vortex-blob method has been applied to overcome this obstacle and to capture the physically important roll-up phenomenon (15,16). In this method, an artificial smoothing parameter $\delta > 0$ is introduced to desingularize the vortex sheet equation. The resulting equation is called the "δ-equation" and the exact vortex sheet equation is formally recovered in the limit $\delta \to 0$. On the discrete level, the basic computational element becomes a vortex-blob, i.e. a smoothed point vortex. Convergence of this method, before the critical time, has been proven by Caflisch and Lowengrub (17). The particular smoothing used does not correspond precisely to a physical mechanism. However, linear stability analysis shows that the δ-equation does not exhibit a short wavelength instability.

The δ-equation is numerically tractable and its solutions exhibit regular spiral roll-up for $t > t_c$. Numerical evidence indicates that, past the critical time, in the limit $\delta \to 0$, the smoothed solutions converge to a tight spiral with an infinite number of turns (15,16). Similar results for $t > t_c$ have also been observed in vortex-in-cell calculations by Tryggvason (18). The conclusion from these studies is that vortex sheet roll-up past the critical time can be captured by taking an appropriate limit of smoothed approximations. Another alternative, suggested by Baker and Shelley (19), is to consider a layer of constant vorticity in the limit of increasing vorticity and vanishing layer thickness.

The work described above deals with vortex sheet roll-up in free space. A natural problem is to extend the δ-equation methodology to compute vortex sheet separation and roll-up past the sharp edge of a flat plate. Two issues arise :

1) satisfying the flow tangency condition on the plate,

2) enforcing the unsteady Kutta condition at the edges of the plate.

Many different numerical approaches to these issues have been investigated in the past (8). Prior work on this topic using the vortex-blob method has been done by Chou (20). These previous studies have not obtained smooth spiral roll-up, and the effect of the various numerical parameters has not been well documented. The goal of the present work is to find a δ-equation model for improved calculations of vortex sheet roll-up past a sharp edge.

NUMERICAL METHOD

The flow contains a bound vortex sheet on the plate and free vortex sheets emanating from the edges of the plate. The free vortex sheets are represented by a collection of vortex-blobs, whereas the bound vortex sheet is represented by a collection of point vortices. The strength of the bound vortex sheet $\sigma(x,t)$ adjusts to satisfy the flow tangency condition on the plate. Some previous studies have used a conformal mapping to determine the strength of the bound vortex sheet, but the present method instead solves an integral equation of the first kind for $\sigma(x,t)$. Each complete time step contains the following sub-steps :

1) The free vortex-blobs are convected.

2) New vortex-blobs are shed at each edge of the plate.

3) The total amount of shed circulation is updated using the unsteady Kutta condition.

4) The bound vortex sheet strength is computed.

In more detail, let (x_j, y_j) be the position of a vortex element (either a point vortex or a vortex-blob) having strength Γ_j. The velocity of the jth vortex-blob is given by,

$$\left(\frac{dx_j}{dt}, \frac{dy_j}{dt}\right) = \sum_{k \neq j} \frac{(-(y_j - y_k),(x_j - x_k))\Gamma_k}{2\pi((x_j - x_k)^2 + (y_j - y_k)^2 + \delta^2)} . \qquad (1)$$

The quantity δ is the artificial smoothing parameter which allows the free vortex sheet to roll up into a smooth spiral. The sum in equation (1) is taken over all vortex elements, with the understanding that δ is set equal to zero when the index k corresponds to one of the bound point vortices.

The bound vortex sheet strength $\sigma(x,t)$ satisfies a singular integral equation of the first kind along the plate,

$$\frac{1}{2\pi} \int_{-1}^{1} \frac{\sigma(\tilde{x},t)d\tilde{x}}{(x - \tilde{x})} = -v(x,t) . \qquad (2)$$

The right hand side in equation (2) is the normal velocity at a point x on the plate, that is induced by the free vortex sheets. One reason for representing the bound vortex sheet by point vortices, instead of vortex-blobs, is to accurately satisfy the flow tangency condition on the plate. Another reason is that the integral equation could not be solve for a general right hand side if the kernel were desingularized. The bound point vortices are placed on the plate at positions $x_j = \cos\theta_j, \theta_j = j\pi/n$ and the integral equation is discretized by collocation at the midpoint of each interval. The total amount of bound circulation is set equal to the negative of the total amount of free circulation, in accordance with Kelvin's theorem. The system of linear equations for the strength of the bound point vortices is solved by Gaussian elimination.

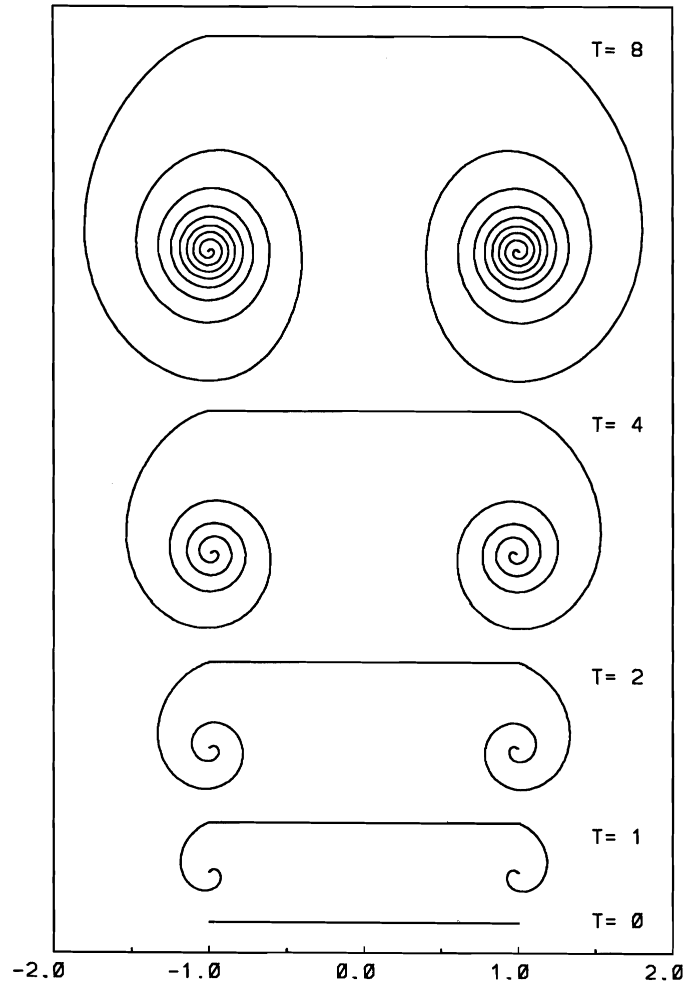

Fig. 1 Time evolution of smoothed vortex sheet roll-up. ($0 \le t \le 8$, smoothing parameter $\delta = 0.2$)

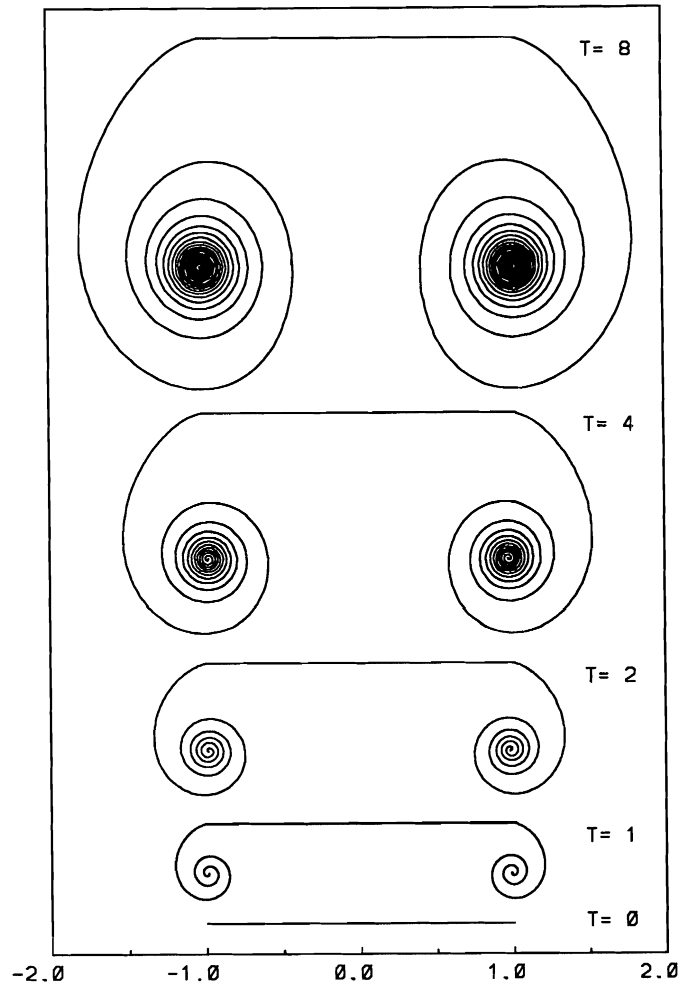

Fig. 2 Time evolution of smoothed vortex sheet roll-up. ($0 \le t \le 8$, smoothing parameter $\delta = 0.1$)

The total circulation $\Gamma(t)$ in each free vortex sheet is determined by the unsteady Kutta condition,

$$\frac{d\Gamma}{dt} = \frac{1}{2}(U_-^2 - U_+^2) = \overline{U}\sigma \ . \tag{3}$$

Here, U_- and U_+ are the one-sided limiting velocities at the edge, $\overline{U}$ is the average of these velocities, and σ, the vortex sheet strength, is the difference. The average velocity $\overline{U}$ is induced by the free vortex sheets and $\sigma \approx \Delta\Gamma/\Delta x$ is approximated by a finite difference formula applied to the bound circulation.

The fourth order Runge-Kutta method was used to solve the ordinary differential equations (1) and (3), for the motion of the vortex-blobs and the total shed circulation. Computer cpu costs were reduced by using a variable time-step size. Accuracy in the sheet's shape was maintained by inserting new vortex-blobs when the curve was sufficiently stretched. Vortex shedding was initiated using the one-sided velocity at the edges of the plate. Further description and validation of the numerical method, including a comparison with Pullin's study of a self-similar vortex sheet (21), will be presented elsewhere.

NUMERICAL RESULTS

In these computations the plate moves vertically with uniform speed $v = 1/2$. Figure 1 was obtained using the smoothing parameter value $\delta = 0.2$. Counter-rotating vortices are shed and roll up smoothly at the two edges of the plate. As time progresses the vortices grow in size, forming a wake behind the plate. The outer turns of the curve become elliptically deformed due to straining by the neighboring vortex. Figure 2 shows the results obtained with the smoothing parameter value $\delta = 0.1$. The overall shape and size of the solution is only slightly changed with this smaller amount of smoothing. However, with $\delta = 0.1$, the vortex sheet rolls up more quickly and more tightly than with $\delta = 0.2$.

Figure 3 is a plot of the computed velocity field at time $t = 8$ in a frame of reference that is fixed at infinity. The velocity vectors are plotted on a regular grid with a small square at the base of each vector. Also plotted, as a sequence of points, are the point vortices and vortex-blobs that represent the free and bound vortex sheets. The plate pushes away the fluid in its path. The counter-rotating vortices behind the plate create a jet that is directed at the rear of the plate. Velocity magnitudes

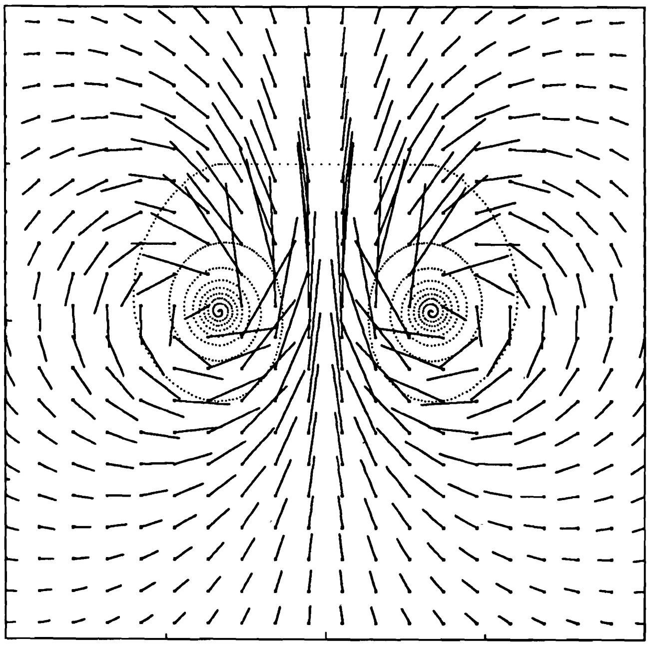

Fig. 3 Velocity field in a frame of reference fixed at infinity.

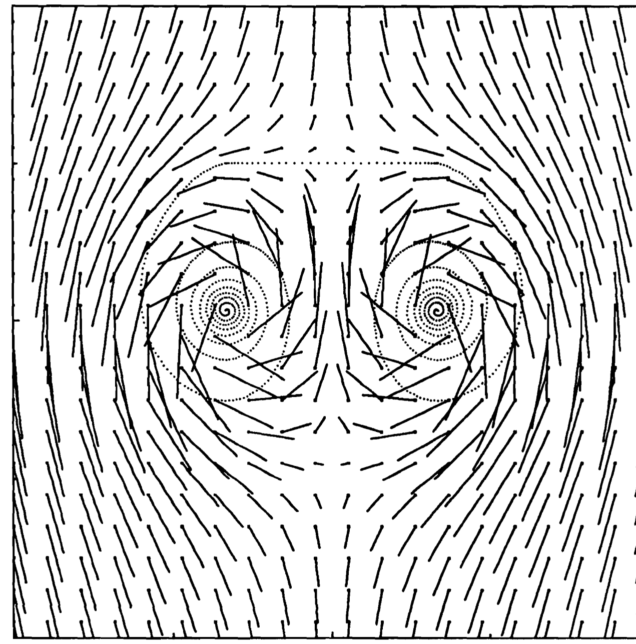

Fig. 4 Velocity field in a frame of reference moving with the plate.

in this jet exceed the uniform plate velocity $v = 1/2$. Figure 4 is a plot of the computed velocity field in a frame of reference that is moving with the plate. The velocity field has two stagnation points along the wake centerline ($x = 0$) in this reference frame - one at the center of the plate and one at the back of the wake.

DISCUSSION

The numerical method presented here provides a smooth approximation for vortex sheet roll-up past a sharp edge. The shape of the inner part of the spiral depends upon the value of the smoothing parameter and upon the numerical procedure for initiating the flow. However, the outer spiral turns are less sensitive to the numerical parameters. A comparison with Pullin's (21) solution for self-similar vortex sheet roll-up past a semi-infinite flat plate is underway, in order to assess the method's validity in the limit $\delta \to 0$. The smoothing parameter has been inserted for a purely practical reason - to obtain a convergent sequence of approximations to the infinite spiral vortex sheet. It would be interesting to know whether the smoothing parameter used here is at all correlated with the boundary layer thickness upstream from the separation point in a real flow.

The vortex sheet model for separation at a sharp edge makes no explicit mention of fluid viscosity, but the effect of viscosity is implicitly accounted for by applying the unsteady Kutta condition. The standard model for flow separation consists of the Navier-Stokes equations together with the no-slip boundary condition along solid surfaces. However, it is difficult to compute accurate solutions of the Navier-Stokes equations at high Reynolds number. The reason for pursuing the vortex sheet model is the possibility that it may provide a useful alternative to complement the Navier-Stokes and other models. Comparison with other numerical solutions and laboratory experiment is needed in order to assess the validity of the vortex sheet model for separation at a sharp edge.

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