VISCOUS SIMULATION OF WAKE PATTERNS

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ABSTRACT. Vortex sheet motion is regularized by smoothing the initial data and solving the Navier-Stokes equations. The underlying flow is induced by a vortex sheet containing positive and negative circulation. Wake patterns are obtained which closely resemble previous experimental results and vortex blob simulations.

1. Introduction

A basic premise in fluid dynamics, due to Prandtl, is that a free shear layer converges to a vortex sheet in the zero viscosity limit. From this point of view, one can gain insight into shear layer dynamics at high Reynolds number by studying vortex sheet motion.

The initial value problem for a vortex sheet is ill-posed in the sense of Hadamard, but an analytic solution exists locally [20]. Asymptotic analysis and numerical work indicate that the sheet develops a singularity in finite time [13,16,17,19]. Several methods of regularization have been proposed to extend the sheet's motion past the critical time, into the roll-up regime. The vortex blob method introduces an artificial smoothing parameter to regularize the Birkhoff-Rott integral [1,5,8,14]. Convergence of the smoothing process prior to the critical time has been established [6]. Numerical results indicate that past the critical time, the regularized solutions converge to a well-defined spiral when the smoothing parameter is reduced [14,15].

An important problem is to determine whether the physical regularization due to viscosity gives the same result as the vortex blob regularization. Numerical evidence indicates that the two regularizations do converge to the same limit for a vortex sheet with circulation of one sign, as in a mixing layer [21]. With vortex sheet initial data of this type, there also exists a global weak solution of the Euler equations [10,11].

Another way to regularize the vortex sheet problem is to replace the sheet by a layer of constant vorticity which evolves according to the Euler equations. The initial value problem for the layer is globally well-posed [4,7]. Before the critical

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time, the layer converges to the vortex sheet in the zero thickness limit $H \to 0$ [2,3]. It seems that past the critical time, the finite-thickness regularization does not converge to the same limit as the vortex blob and the viscous regularizations. However, the motion of a piecewise constant vorticity layer does resemble viscous simulation and experiment [12].

The present article concerns the case in which the underlying vortex sheet has circulation of both signs, as in a wake. This work was motivated by the experiments of Couder and Basdevant on the wake of a solid body in a thin soap-film [9]. They found that vortex couples form in a forced wake, with different patterns developing as the forcing parameters were varied. A previous article showed that the experimental wake patterns can be captured by varying the initial conditions in the vortex blob method [15]. The present article extends the investigation by showing that similar patterns are also obtained by solving the Navier-Stokes equations with the vortex blob initial data.

2. Numerical Method

Let $\omega(x,y,t)$ be the vorticity, $\psi(x,y,t)$the stream function and ν the viscosity. The 2-d Navier-Stokes equations in vorticity/stream form are

$$\omega_t + \mathbf{u} \cdot \nabla\omega = \nu\Delta\omega \quad , \tag{1}$$

$$\Delta\psi = -\omega \quad , \tag{2}$$

$$\mathbf{u} = (\psi_y \ , \ -\psi_x) \ . \tag{3}$$

The initial data is obtained from a vortex sheet $(x_0(a), y_0(a))$ having circulation density $\sigma(a)$. Let δ be the artificial smoothing parameter. The Birkhoff-Rott integral [5] is regularized and differentiated to obtain the initial vorticity

$$w_0(x,y) = \pi\delta^2 \int_0^1 \frac{(\ \cosh 2\pi(y - y_0(a)) + \cos 2\pi(x - x_0(a))\ )\ \sigma(a)\ da}{(\ \cosh 2\pi(y - y_0(a)) - \cos 2\pi(x - x_0(a)) + \delta^2)^2} \ . \tag{4}$$

The viscosity was ν=10^{-4} and the smoothing parameter was δ=10^{-1}. With U=$||\sigma||_\infty$=1, the Reynolds number based on initial layer thickness is

$$Re = \frac{U\delta}{\nu} = 10^3 \ . \tag{5}$$

The domain $0 \le x \le 1$, $-1 \le y \le 1$ was discretized by a 256 x 512 uniform mesh. Central differences were used in space. Time integration was performed using the 4th order Runge-Kutta method with time step $\Delta t = 10^{-2}$. The Poisson equation (2) was solved using a routine from FISHPACK [18] with periodic boundary conditions at $x = 0, 1$ and free-slip conditions at $y = \pm 1$. The integral in equation (4) was evaluated by the trapezoid rule with increment $\Delta a = 0{\cdot}0025$. The curve $(x_0(a), y_0(a))$ was passively advected for diagnostic purposes. Bilinear interpolation was used to obtain the curve velocity from the fluid velocity on the mesh. To maintain the curve's resolution, additional points were inserted during the computation [15].

case	$x_0(a)$	$y_0(a)$	$\sigma(a)$
1	a	$0.20 \sin 2\pi a$	$\sin 2\pi a$
2	$a + 0.10 \sin 2\pi a$	$0.20 \sin 2\pi a$	$\sin 2\pi a$
3	$a + 0.05 \sin 4\pi a$	$0.05 \sin 4\pi a$	$\sin 4\pi a \quad ; \quad 0.0 \leq a \leq 0.5$
		$-0.05 \sin 4\pi a$	$-\sin 4\pi a \quad ; \quad 0.5 \leq a \leq 1.0$

TABLE I
Initial data: vortex sheet $(x_0(a), y_0(a))$, circulation density $\sigma(a)$.

3. Numerical Results

Table 1 contains (x_0, y_0) and σ for the 3 cases computed. The initial conditions lie within a restricted class of sinusoidal perturbations. The various amplitudes were chosen to obtain close agreement with the experimental results [9]. Computed results on the time interval $0 \leq t \leq 8$ are shown in Figures 1,2,3. The passive curve is plotted on the left and vorticity contours are plotted on the right.

4. Discussion

The passive curves obtained here are in close agreement with experimental wake patterns [9] and vortex blob simulations [15]. This is consistent with the previously observed close agreement between the viscous and vortex blob regularizations for a vortex sheet with circulation of one sign [21]. Such results indicate that the vortex blob method gives the physically correct extension of vortex sheet motion. Close agreement between these two regularizations can only hold for a finite time. The Navier-Stokes equations are dissipative and the circulation decays to zero as $t \to \infty$. On the other hand, the vortex blob method is conservative and the circulation does not decay. A more detailed study of these issues is in preparation.

Acknowledgements

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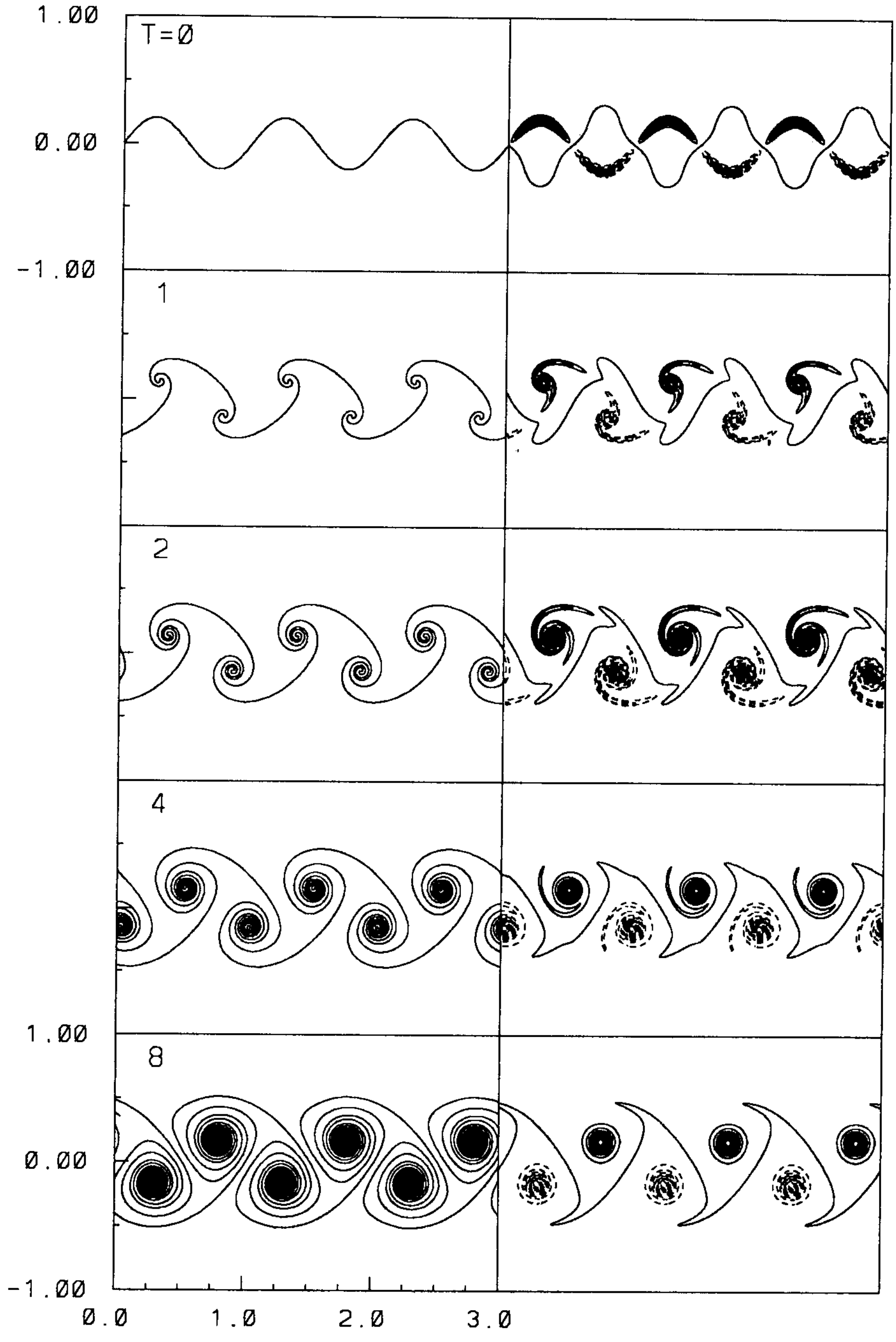

Fig. 1. The initial curve contains a transverse sinusoidal perturbation. The curve rolls up into an array of counter-rotating vortices resembling the unforced experimental wake [9,15]

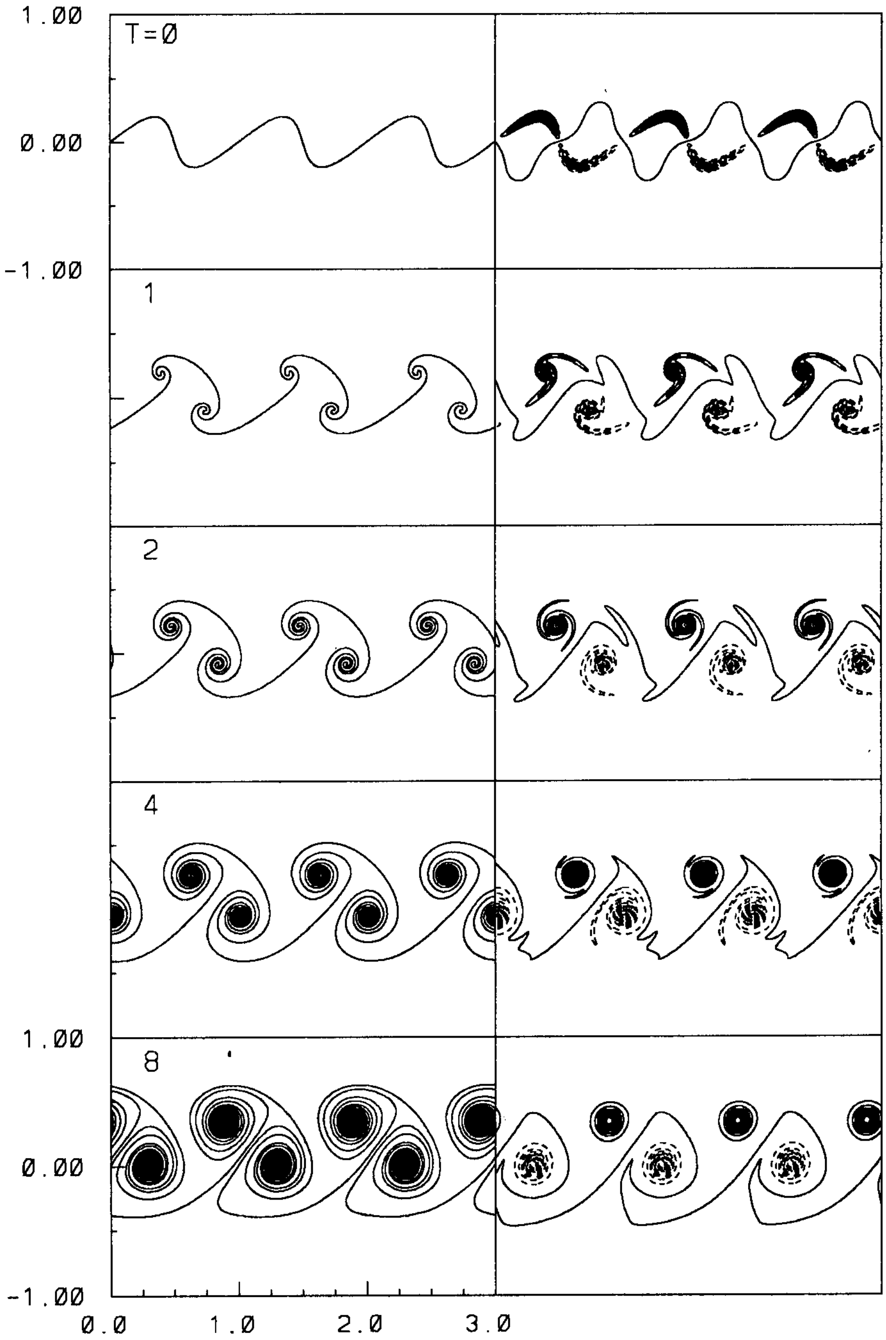

Fig. 2. A longitudinal perturbation has been added to the shape of the initial curve. Vortex couples propagate into the region $y > 0$ as in a forced wake experiment [9,15].

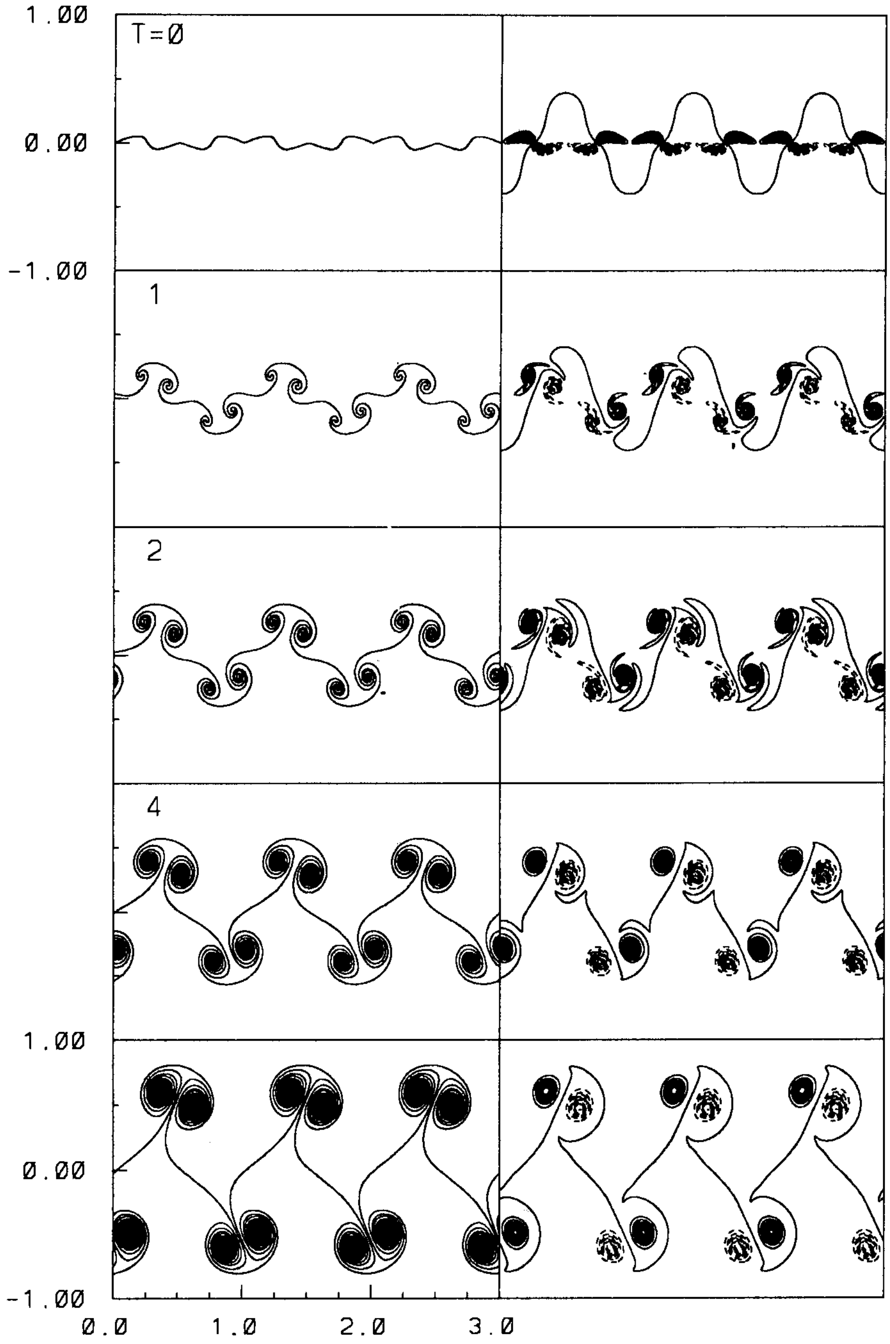

Fig. 3. The initial perturbation is reduced in scale and a reflection about $y = 0$ is added. Vortex couples propagate into the regions $y > 0$ and $y < 0$ as in another forced wake experiment [9,15]).

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