VORTEX SHEET ROLL-UP

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ABSTRACT

The equation governing vortex sheet motion is regularized by Chorin's vortex-blob method. Two computed examples of vortex sheet roll-up are presented and discussed. The first example is the Kelvin-Helmholtz problem in which a periodically perturbed vortex sheet rolls up into a double-branched spiral. The second example is an axisymmetric vortex sheet which rolls up at the edge of a circular tube, forming a vortex ring.

1. Introduction

A basic idea in fluid dynamics going back to Prandtl is that a free shear layer converges to a vortex sheet in the zero viscosity limit. From this point of view, the investigation of vortex sheet motion sheds light on the structure of high Reynolds number fluid flow. A prominent feature of vortex sheet motion is "roll-up", i.e. the formation of a spiral in the shape of an evolving vortex sheet. The roll-up phenomenon continues to challenge mathematical analysis. In this article, I will discuss some recent computational work dealing with vortex sheet roll-up.

According to linear theory, a flat vortex sheet of constant strength exhibits the short wavelength Kelvin-Helmholtz instability. To study the sheet's nonlinear motion, one must solve an integro-differential equation[4]

$$\overline{\frac{\partial z}{\partial t}}(\Gamma,t) = \int K(z(\Gamma,t) - z(\Gamma',t))\, d\Gamma' , \tag{1}$$

where $z(\Gamma,t)$ is a complex-valued function representing the vortex sheet, Γ is the circulation parameter along the sheet and t is time. For a periodically perturbed vortex sheet, the kernel is

$$K(z) = \frac{1}{2i}\cot \pi z . \tag{2}$$

Since $K(z)$ is singular, the integral is understood as a Cauchy principal value. Equation (1) is solved subject to an initial condition $z(\Gamma,0) = \Gamma + p(\Gamma,0)$. If the initial perturbation $p(\Gamma,0)$ is an analytic function of Γ, the sheet $z(\Gamma,t)$ remains analytic in Γ for a finite time[6,26]. Asymptotic analysis by Moore[18] indicates that a branch point singularity forms in the vortex sheet at a critical time $t_c > 0$. Although the sheet's curvature at the branch point is infinite, roll-up has not yet occurred at the critical time.

2. Spiral Formation

It is natural to try to extend vortex sheet motion past the critical time. According to a basic theoretical result, the 2-d incompressible Euler equations have a weak solution for all $t > 0$, if the initial velocity corresponds to a vortex sheet with circulation of one sign[8,10]. This result applies to the Kelvin-Helmholtz problem, but it does not provide information about the shape of the vortex sheet for $t > t_c$.

Pullin[20] conjectured that the sheet rolls up immediately for $t > t_c$ into a double-branched spiral with an infinite number of turns. This idea is motivated by the study[22,23] of self-similar vortex sheets which are initially flat and possess a branch point singularity of the form $\Gamma \sim x^p$. For $t > 0$, these sheets have a spiral shape determined by the value of the parameter p. Pullin suggested[20] that this example of self-similar vortex sheet roll-up provides a model for what happens past the critical time in the Kelvin-Helmholtz problem. The self-similar spiral sheets exist for parameter values in the range $0 < p < 1$, although for $p \to 1$ there is some degeneracy[23]. Moffatt[17] has suggested that for a certain value of the parameter p, the self-similar spiral vortex sheet may display Kolmogorov's spectral energy decay rate. It is not known apriori which value of p (if any) should arise in the Kelvin-Helmholtz problem.

The first vortex sheet computation was performed by Rosenhead[24] using the point vortex approximation. The sheet is replaced by a finite set of point vortices $\{z_j(t)\ ,\ j{=}1,...,N\}$ whose motion is determined by the equations

$$\overline{\frac{dz_j}{dt}} = \sum_{\substack{k=1 \\ k\neq j}}^{N} K(z_j - z_k)N^{-1}\ . \tag{3}$$

Rosenhead's method has been the subject of controversy[4] but it is now known[5,13] that for $t < t_c$, the point vortex approximation converges as $N \to \infty$. In order to extend the sheet's motion for $t > t_c$, it is necessary to take a different approach. The key idea is to regularize Equation (1). In Chorin's vortex-blob method[1,7,14], the singular kernel $K(z)$ is replaced by a smooth approximation $K_\delta(z)$. One choice used for free-space problems is

$$K_\delta(z) = K(z)\frac{|z|^2}{|z|^2+\delta^2}\ . \tag{4}$$

For $t > t_c$, the vortex sheet is defined to be the limit of the regularized solutions as the smoothing parameter δ tends to zero. Using vector and parallel processors, filtering to control roundoff error, and adaptive mesh techniques, solutions of the regularized vortex sheet equation have been obtained for smaller values of δ than in previous investigations[15,16]. Figure 1 shows the evolution of a perturbed vortex sheet computed by the vortex-blob method.

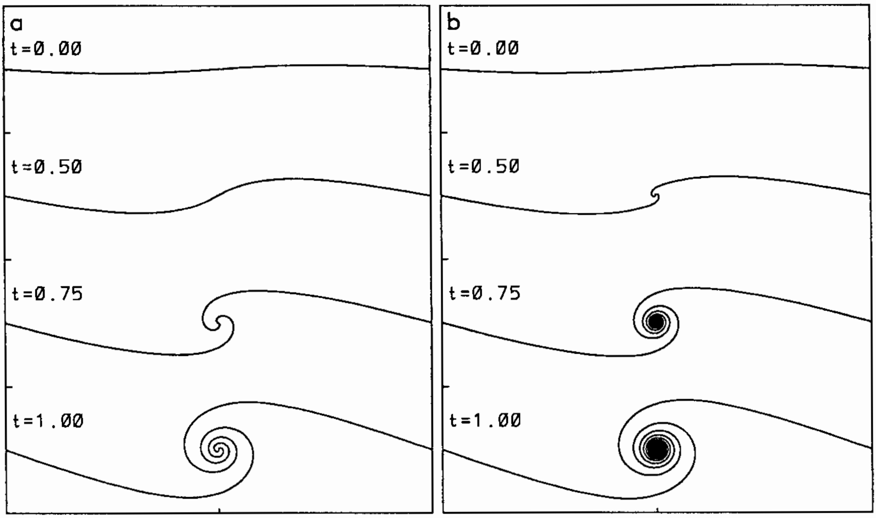

Figure 1. Roll-up of the Kelvin-Helmholtz vortex sheet, computed by the vortex-blob method. The critical time is $t_c \sim 0.37$. The values of the smoothing parameter are: (a) $\delta = 0.10$, (b) $\delta = 0.03$.

The numerical evidence[14,15] supports Pullin's conjecture. At a fixed time $t > t_c$, the regularized solutions converge to a well-defined spiral in the limit $\delta \to 0$. The results also indicate that the core of the spiral behaves like

$$\Gamma \sim r \quad , \quad r \sim \theta^{-1} \ , \tag{5}$$

where Γ, r, θ are measured from the spiral center. This imples that the spiral which forms in the Kelvin-Helmholtz problem resembles a self-similar solution with parameter value $p = 1$. It should be noted that Hobson[11] obtained a different value, $p \sim 0.8$. The numerical evidence supporting $p = 1$ will be presented in a forthcoming article[16]. A $p = 1$ spiral can be thought of as arising in the Kelvin-Helmholtz problem from the linear circulation distribution $\Gamma \sim x$ in the unperturbed vortex sheet. However, this finding requires further clarification since $p \to 1$ is a singular limit for the self-similar family studied by Pullin[23].

Another topic of current interest is the effect of different methods of regularization. Baker and Shelley[3] studied the motion of a layer of constant vorticity. They observed that in the limit of zero thickness, the layer converges to the vortex sheet for $t < t_c$. For $t > t_c$, the shape of the layer does not seem to match the spiral found using the vortex-blob method. Jacobs and Pullin[12] studied the motion of several superimposed layers of constant vorticity. Their results were in qualitative agreement with other simulations and experimental visualization of rolling-up shear layers. Tryggvason, Dahm and Sbeih[27] compared solutions of the Navier-Stokes equations to vortex-blob simulations and obtained good agreement for the shape of the spiral. Their work suggests that for the Kelvin-Helmholtz problem, the zero smoothing limit of the vortex-blob method agrees with the zero viscosity limit of the Navier-Stokes equations.

The computational work discussed above is helping to clarify the issue of spiral formation in the Kelvin-Helmholtz problem. However, most of the numerical findings are not yet supported by rigorous proof, especially in the roll-up regime. In practice, one wants to know whether vortex-blob results computed with $\delta > 0$ are close to an actual viscous flow. This issue has been addressed by using the vortex-blob method to simulate a laboratory experiment, as discussed in the next section.

3. Axisymmetric Vortex Sheet Roll-Up

Didden[9] performed an experiment in which a piston ejects fluid from a circular tube, leading to the formation of a vortex ring. Nitsche[19] simulated the experiment using an axisymmetric vortex-blob method which incorporates separation at the edge of the tube. Figure 2 shows the results of the simulation. In the experiment, the piston velocity was held constant until time $t = 1.6$, when the piston stopped moving. Before the shutoff time, the piston pushes fluid from the tube and a primary vortex ring travels away from the edge of the tube. Past the shutoff time, a secondary vortex ring forms and travels slowly upstream inside the tube. The primary ring continues to travel away from the edge.

The results in Figure 2 agree well with Didden's flow visualization[9]. The main discrepancy between simulation and experiment is in the initial circulation shedding rate. This is attributed to the neglect of viscous effects in the simulation. The simulation did reproduce other detailed features of the experiment. In particular, both simulation and experiment found that the trajectory of the vortex ring before the shutoff time is curved, $x_c \sim t^{3/2}$, $r_c \sim t^{2/3}$. Here, x_c, r_c are the axial and radial coordinates of the ring center, measured from the edge of the tube. On the other hand, 2-d similarity theory[21,25] predicts that the trajectory is a straight line, $x_c \sim t^{2/3}$, $r_c \sim t^{2/3}$. Auerbach[2] listed several factors, among them viscous effects, which could explain this discrepancy. However, the agreement between simulation and experiment seems to eliminate the possibility that viscous effects cause the ring trajectory to depart from the predictions of 2-d similarity theory. Still, there is currently no theoretical explanation for the observed behaviour $x_c \sim t^{3/2}$.

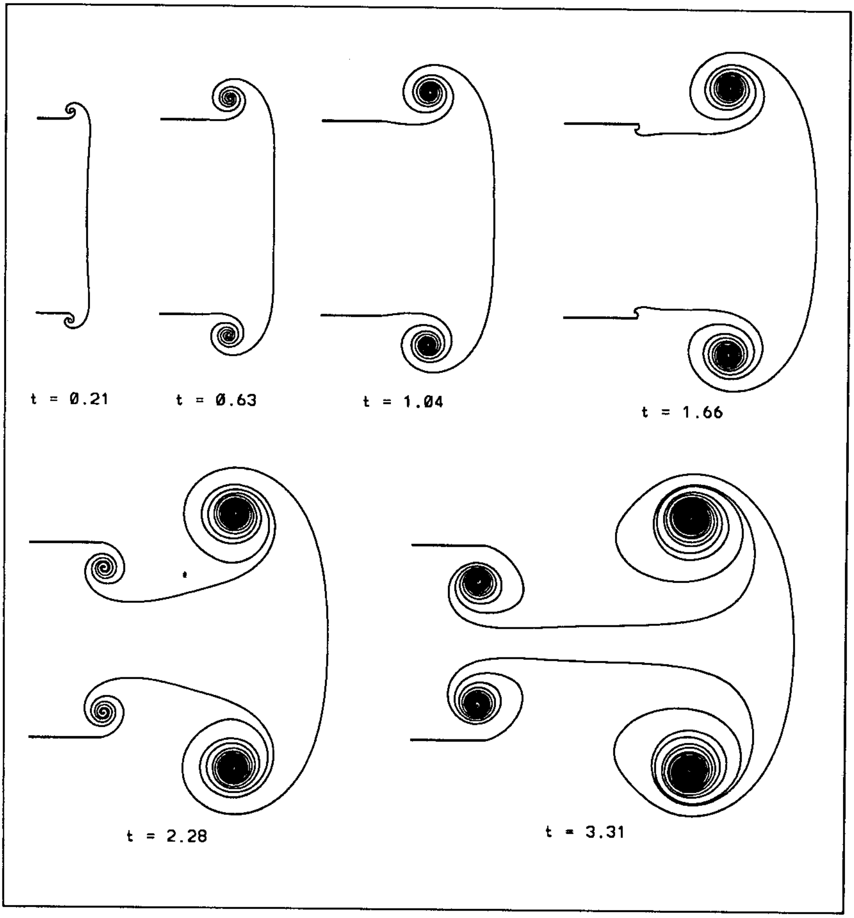

Figure 2. Roll-up of an axisymmetric vortex sheet at the edge of a circular tube. These results are reproduced from Nitsche's simulation[19] of Didden's experiment[9].

4. Conclusion

The vortex-blob method has been used to extend vortex sheet motion past the critical time, into the physically important roll-up regime. For the Kelvin-Helmholtz problem, the regularized solutions converge to a well-defined spiral. Good agreement between computation and experiment has been obtained for the case of axisymmetric vortex sheet roll-up at the edge of a circular tube,

The numerical results presented here deal only with laminar flow. Experimental evidence indicates that turbulent flow contains large-scale coherent vortex structures which evolve deterministically. It is tempting to speculate that these structures could be modeled as evolving vortex sheets. Testing this idea will require further development of numerical methods for vortex sheet roll-up.

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