Pricing Equity-Linked Pure Endowments via the Principle of Equivalent Utility

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Abstract

We consider a pure endowment contract whose life contingent payout is linked to the performance of a risky stock or index. Because of the additional mortality risk, the market is incomplete; thus, a fundamental assumption of the Black-Scholes theory is violated. We price this contract via the principle of equivalent utility and demonstrate that, under the assumption of exponential utility, the indifference price solves a nonlinear Black-Scholes equation; the nonlinear term reflects the mortality risk and exponential risk preferences in our model. We discuss qualitative and quantitative properties of the premium, including analytical upper and lower bounds.

Keywords: Equity-indexed annuity, indifference price, Hamilton-Jacobi-Bellman equation, Black-Scholes equation, expected utility

1 Introduction

Sales of equity-indexed annuities (EIAs) have increased dramatically in recent years; indeed sales have climbed to over $6.4 billion in 2001 since their

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introduction in 1995, [1]. Under these contracts, the insured makes an initial deposit (or several deposits) and during the deferral period the interest accrual on the fund is linked to the performance of a stock or index, typically the Standard and Poor's (S&P) 500 Index. One factor that contributes to the popularity of these contracts is the fact that they generally carry a guaranteed minimum return; the investor can enjoy the upside potential of equity growth without the downside risk.

In [22], Young considered an equity-linked life insurance policy for which the premium is fixed and the death benefit, not the interest accrual, is modeled directly as a function of the index performance. In this paper we follow the approach of [22] for an equity-linked pure endowment policy. More specifically, we consider a pure endowment policy for which the insured pays a fixed premium P and the benefit level at the end of the deferral period is linked to the stock or index price at that time. Since the benefit is paid only if the insured survives until the end of the deferral period, we can view this policy as a life contingent financial option on the underlying stock or index. In the absence of the mortality risk, the premium for this policy would simply be the Black-Scholes price [4] of the financial option; however, mortality is an additional source of risk that cannot be hedged. Thus, with the mortality risk, we are in an incomplete market setting; one of the fundamental assumptions of the Black-Scholes theory is violated.

In this paper we employ the classical actuarial principle of equivalent utility [5] to price the endowment policy described above. Utility methods have been used for pricing financial options in incomplete markets such as markets with transaction costs [2], [7], [11] and markets in which one cannot trade the underlying asset, but can trade a correlated asset [16]. Young and Zariphopoulou [24], [25] introduced utility methods for dynamic valuation of insurance products for which the insurance risk is independent of the financial risk. In the endowment policy that we consider here, the payment amount is a function of the underlying risky asset, thus, as in [22], the insurance risk is not independent of the financial market.

Under the principle of equivalent utility, one wishes to determine the premium at which the insurer is indifferent between writing and not writing the endowment contract. We formulate this principle mathematically via two stochastic optimal control problems in Section 2. In Section 3, we demonstrate that, under the assumption of exponential utility, the indifference premium P solves a semi-linear Black-Scholes partial differential equation (PDE); the nonlinear term reflects the additional mortality risk and exponential risk preferences in our model.

Though the PDE for P is not explicitly solvable in closed form, we can glean useful information about the qualitative properties of the premium, including analytic upper and lower bounds. More specifically, in Section 4 we prove that the indifference premium P is bounded above by the Black-Scholes price of the "endowment certain" (i.e., the financial option without the mortality risk) and below the the Brennan and Schwartz price [6] of the contract, which incorporates the mortality risk, but not the exponential risk preferences. Thus our premium generalizes both the Black-Scholes price of the contract to include the life contingency and the Brennan and Schwartz price to reflect the insurer's risk preferences.

In Section 5, we compute the indifference premium numerically and demonstrate that the computed solutions obey the ordering predicted in Section 4.

In Section 6, motivated by Musiela and Zariphopoulou [16], we derive an alternative probabilistic representation of the premium that is similar to the arbitrage free representation of derivative prices in complete markets.

We conclude in Section 7 by discussing open questions and directions for future work.

Appendices 1 and 2 contain some of the more routine details of our work.

2 The Model

We consider an insurer who is endowed with initial wealth $W_t = w$ at time $t \geq 0$. The insurer can invest in a risk free bond with rate of return $r > 0$ or a risky stock or index whose price at time t is $S_t = S$. We wish to compute the premium $P = P(w, S, t)$ under which the insurer is indifferent between writing or not writing a single pure endowment policy to (x), a life aged x, whose payout $g(S_T)$ at time T is linked to the underlying stock price at that time. The payment is life contingent; it is made only if (x) survives until time T.

We model the stock price S_s as a geometric Brownian motion, i.e.,

$$\left\{ \begin{array}{ll} dS_s & = \mu S_s + \sigma S_s dB_s \\ S_t & = S \geq 0 \end{array} \right\}. \tag{1}$$

The process B_s is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$

and the constants μ and σ, the mean rate of return and volatility of the stock, respectively, are known. We assume throughout that $\mu > r > 0$.

We assume further that the insurer can trade dynamically between the stock and bond accounts; i.e., the insurer can choose amounts π_s^0 and π_s, $t \leq s \leq T$ to invest in the bond and stock accounts, respectively. The investment strategy satisfies the budget constraint $W_s = \pi_s^0 + \pi_s$, and W_s evolves according to the state dynamics

$$\left\{ \begin{array}{ll} dW_s & = [rW_s + (\mu - r)\pi_s]ds + \sigma \pi_s dB_s \quad t \leq s \leq T \\ W_t & = w \end{array} \right\}. \tag{2}$$

Note that we allow π_s or π_s^0 to be negative; these scenarios correspond to a short position in the stock or a loan against the bond account, respectively.

We assume that the insurer seeks to maximize the expected utility of wealth at time T. In the absence of the endowment contract we define the value function

$$V(w,t) = \sup_{\{\pi_s \in \mathcal{A}\}} E[u(W_T)|W_t = w]. \tag{3}$$

The set $\mathcal{A}$ is the set of admissible policies $\{\pi_s\}$ that satisfy the integrability condition $E\int_t^T \pi_s^2 ds < \infty$ and are $\mathcal{F}_s$ progressively measurable, where $\mathcal{F}_s$ is the augmentation of $\sigma(B_u : t \leq u \leq s)$. The function $u : \mathbf{R} \to \mathbf{R}$ is the insurer's utility function, which measures the insurer's attitudes toward wealth and risk. For now we assume only that u is increasing, smooth, and concave.

Now consider an endowment policy written to (x) with payout Y_T at time T where

$$Y_T = \left\{ \begin{array}{ll} g(S_T) & \text{if } (x) \text{ survives until time } T \\ 0 & \text{otherwise.} \end{array} \right. \tag{4}$$

If the insurer writes this contract, then the insurer seeks to maximize the expected utility of terminal wealth in the presence of the endowment risk. We define the value function in the presence of this risk to be

$$U(w,S,t) = \sup_{\{\pi_s \in \mathcal{A}\}} E[u(W_T - Y_T)|W_t = w, S_t = S]. \tag{5}$$

We wish to compute the premium $P = P(w,S,t)$ under which the insurer is indifferent between writing the endowment and not writing the endowment, i.e., so that

$$U(w+P,S,t) = V(w,t) \tag{6}$$

for any given $(w, S, t) \in \mathbf{R} \times (0, \infty) \times [0, T]$. Ideally, the premium P should be independent of the insurer's wealth w so that all insurers with the same risk preferences will charge the same premium regardless of their wealth. We will see in Section 3 that this is the case under the assumption of exponential utility.

3 The Partial Differential Equation Models for V,U, and P

In this section we discuss terminal value problems for partial differential equations that govern the behavior of the value functions V and U, and we use these results to derive a PDE for the premium $P = P(w, S, t)$. In Sections 4 and 5, we discuss qualitative properties of and quantitative results on the solutions to this equation.

Recall the value functions V and U are defined in (3) and (5). Using the Dynamic Programming Principle and standard arguments from stochastic calculus, one can show that V satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$\left\{ \begin{array}{c} V_t + \max_{\{\pi\}}[(\mu - r)\pi V_w + \frac{1}{2}\sigma^2\pi^2 V_{ww}] + rwV_w = 0 \\ V(w, T) = u(w) \end{array} \right\}. \tag{7}$$

From the Verification Theorem (see, for example, Chapter 14 of [3]) we know that if (7) has a smooth solution V, then it equals the value function as defined in (3) and one can recover the optimal investment strategy π_t^* from the first order condition in (7). Moreover, the concavity of the utility function u and the linearity of the state equation (2) in W_s and π_s dictate that V is concave in wealth. Thus, the maximum in (7) is attained at

$$\pi_t^* = -\frac{(\mu - r)}{\sigma^2} \frac{V_w(W_t^*, t)}{V_{ww}(W_t^*, t)} \tag{8}$$

where W_t^* is the optimally controlled wealth. We can rewrite (7)

$$\left\{ \begin{array}{c} V_t - \dfrac{(\mu - r)^2}{2\sigma^2} \dfrac{V_w^2}{V_{ww}} + rwV_w = 0 \\ V(w, T) = u(w) \end{array} \right\}. \tag{9}$$

We can use similar techniques to derive the HJB for U. We assume throughout that $\lambda_x(t)$, the force of mortality for (x) at time t (or the intensity of the mortality process for (x)), is given. In the work that follows, we employ standard actuarial notation: ${}_hp_{x+t}$ is the probability that an individual survives until age $x+t+h$ given that she has survived until age $x+t$. Moreover

$$ {}_hp_{x+t} = e^{-\int_t^{t+h} \lambda_x(s)ds} \quad \text{and} \quad {}_hq_{x+t} = 1 -_h p_{x+t}. \tag{10} $$

Following as in Section 4.1 of [24] and Section 14.3 of [3], we first employ the Dynamic Programming Principle: If one follows the optimal investment strategy on $[t,T]$, one's expected utility is at least as great as if one invests arbitrarily on $[t,t+h)$ and then optimally on $[t+h,T]$. (Here we are assume that h is sufficiently small so that $t+h<T$.)

In employing the Dynamic Programming Principle, we must consider whether the investor survives from time t until time $t+h$. If $(x+t)$ survives for another h years until time $t+h$, which happens with probability ${}_hp_{x+t}$, the insurer still faces the endowment risk on the time interval $[t+h,T]$. In this case, by the definition of U in the (5), the maximum expected utility derived by investing optimally on $[t+h,T]$ is $U(W_{t+h},S_{t+h},t+h)$.

However, if $(x+t)$ dies in the time interval $[t,t+h]$, which happens with probability ${}_hq_{x+t}$, the insurer is no longer at risk for the endowment payout. In this case, by the definition of V in (3), the maximum expected utility derived by investing optimally on $[t+h,T]$ is $V(W_{t+h},t+h)$.

Thus we have

$$ \begin{aligned} U(w,S,t) \geq \;& {}_hp_{x+t}E[U(W_{t+h},S_{t+h},t+h)|W_t=w,S_t=S] \\ & +{}_hq_{x+t}E[V(W_{t+h},t+h)|W_t=w]. \end{aligned} \tag{11} $$

Applying Itô's formula and recalling that $W_t=w$ and $S_t=S$, yields

$$ \begin{aligned} U(w,S,t) \geq \;& {}_hp_{x+t}U(w,S,t) +_h q_{x+t}V(w,t) \\ & +{}_hp_{x+t}E^{w,S,t}\left[\int_t^{t+h}(U_t+(rW_\xi+(\mu-r)\pi_\xi)U_w+\mu S_\xi U_S \right. \\ & \left. +\tfrac{1}{2}\sigma^2\pi_\xi^2U_{ww}+\sigma^2\pi_\xi S_\xi U_{wS}+\tfrac{1}{2}\sigma^2S^2U_{SS})d\xi\right] \\ & +{}_hq_{x+t}E^{w,t}\left[\int_t^{t+h}(V_t+(rW_\xi+(\mu-r)\pi_\xi)V_w+\tfrac{1}{2}\sigma^2\pi_\xi^2V_{ww})d\xi\right]. \end{aligned} \tag{12} $$

In the integrals above, we have suppressed the independent variable (W_ξ,S_ξ,ξ) and the notation $E^{w,S,t}$ means that the expectation is conditioned on the information $W_t=w, S_t=S$.

Rearranging terms, dividing both sides by h, letting $h \to 0$ and recalling that as $h \to 0$,

$$ {}_hp_{x+t} \to 1, \quad {}_hq_{x+t} \to 0 \tag{13} $$

and

$$ \frac{{}_hq_{x+t}}{h} \to \lambda_x(t), \tag{14} $$

yields

$$ \begin{aligned} 0 \; &\geq \lambda_x(t)(V-U) + U_t + [rw + (\mu - r)\pi_t]U_w + \mu S U_S \\ &+\tfrac{1}{2}\sigma^2\pi_t^2 U_{ww} + \sigma^2 \pi_t S U_{wS} + \tfrac{1}{2}\sigma^2 S^2 U_{SS}. \end{aligned} \tag{15} $$

Again, we have suppressed the independent variables (w, t) and (w, S, t) of V and U, respectively. Finally, we note that along the optimal path $\pi_t = \pi_t^*$, equality holds in (11), and therefore in (15). The definition of U in (5) prescribes the value of U at $t = T$; thus, we have the terminal value problem

$$ \left\{ \begin{array}{c} U_t + \max_{\{\pi\}}[(\mu - r)\pi U_w + \frac{1}{2}\sigma^2\pi^2 U_{ww} + \sigma^2\pi S U_{wS}] \\ +rwU_w + \mu S U_S + \frac{1}{2}\sigma^2 S^2 U_{SS} + \lambda_x(t)(V-U) = 0 \\ U(w,S,T) = u(w - g(S)) \end{array} \right\}, \tag{16} $$

where V solves the HJB given in (7).

We note that this is simply a formal derivation; we did not justify the implicit assumptions on the regularity of U and V and the interchanging of limit and expectation. For a rigorous derivation of the HJB equation, we refer the reader to [8].

Again, invoking the Verification Theorem and the concavity of U, the maximum in (16) is attained at

$$ \pi_t^* = -\frac{(\mu - r)}{\sigma^2}\frac{U_w(W_t^*, S_t, t)}{U_{ww}(W_t^*, S_t, t)} - \frac{S U_{wS}(W_t^*, S_t, t)}{U_{ww}(W_t^*, S_t, t)}, \tag{17} $$

and we rewrite (16) as

$$ \left\{ \begin{array}{c} U_t - \dfrac{[(\mu - r)U_w + \sigma^2 S U_{wS}]^2}{2\sigma^2 U_{ww}} + rwU_w + \mu S U_S + \frac{1}{2}\sigma^2 S^2 U_{SS} + \lambda_x(t)(V-U) = 0 \\ U(w,S,T) = u(w - g(S)) \end{array} \right\} \tag{18} $$

Thus we see that the value functions U and V solve the fully nonlinear, partially coupled system (9),(18). Under certain assumptions on the utility function $u(w)$, it is possible to solve (9) and determine V and the optimal investment strategy π_t^* in closed form. Specifically, if we assume constant absolute risk aversion (CARA or exponential utility) or constant relative risk aversion (CRRA or power utility), we can obtain V in closed form. Here the absolute and relative risk aversion are measured by $-\frac{u''(w)}{u'(w)}$ and $-w\frac{u''(w)}{u'(w)}$, respectively, [17]. In Merton's seminal paper [14], he obtained closed form expressions for the value function and optimal investment and consumption strategies under CARA and CRRA utility. In a similar vein, Siegmann and Lucas determine the optimal investment strategy and contribution level for a pension fund, [18].

Ideally, we would like to solve the system for U and V explicitly and recover the premium P via the indifference relationship (6), but in general, the complexity of the system (9),(18) prohibits this. We demonstrate below that under the assumption of exponential utility, we can derive a more tractable PDE for the premium P from which we can glean useful qualitative and quantitative information about P's behavior.

Assumption 3.1 *In the work that follows, we assume that the insurer exhibits constant absolute risk aversion (CARA), i.e., that the insurer's utility of wealth is given by*

$$u(w) = -\tfrac{1}{\alpha}e^{-\alpha w}, \quad \alpha > 0. \tag{19}$$

In this case, by direct calculation, one can verify that the solution V to (9) is given by

$$V(w,t) = -\frac{1}{\alpha}\exp(-\alpha w e^{r(T-t)})\exp\left(-\frac{(\mu-r)^2}{2\sigma^2}(T-t)\right); \tag{20}$$

the details of this calculation are in Appendix 1. We observe that by (8), the optimal investment strategy is given by

$$\pi_t^* = \frac{(\mu-r)}{\sigma^2}\frac{e^{-r(T-t)}}{\alpha}. \tag{21}$$

We note that π^* is not stochastic and is independent of wealth. This is generally the case in lognormal stock dynamics under exponential utility because the absolute risk aversion is independent of wealth. Observe that, consistent with our intuition, as the absolute risk aversion α increases, the amount invested in the stock decreases. Finally, we note the similarity between our optimal investment strategy above and the optimal risky portfolio share derived in Merton's paper [14].

Remark 3.2 *In the work that follows, we demonstrate that, under exponential (CARA) utility, the value function $U(w, S, t)$ is a "separable" function of w and S, and it follows that the indifference premium P is independent of the insurer's wealth. Under power (CRRA) utility, one can determine V in closed form; however, U is no longer separable with respect to w and S. Thus, we can not treat the PDE (18) using the methods of this paper. We will elaborate on this point in Appendix 2.*

We wish to calculate the indifference premium $P(w, S, t)$, which is related to U and V by (6). As value functions often inherit the structure of the underlying utility function u, which in this case is exponential, we conjecture that U is of the form

$$U(w, S, t) = V(w, t)e^{\eta(S,t)}, \tag{22}$$

i.e., that U is a separable function of w and S, or more specifically, that the ratio of the maximum expected utility with the endowment contract to the maximum expected utility without the endowment contract is independent of the insurer's wealth. Computing the appropriate derivatives of U, plugging in to (18), exploiting the fact that V solves (9) and is given by (20), and simplifying yields the PDE

$$\eta_t + rS\eta_S + \frac{1}{2}\sigma^2 S^2 \eta_{SS} + \lambda_x(t)[e^{-\eta} - 1] = 0; \tag{23}$$

the details of this calculation are in Appendix 2. From (22) and the terminal conditions on V and U, we have

$$\eta(S, T) = \alpha g(S). \tag{24}$$

Recall that the indifference premium P satisfies (6). Using our conjectured solution (22), we have

$$V(w + P, t)e^{\eta(S,t)} = V(w, t). \tag{25}$$

Also, since $V(w,t)$ is given by (20), we have

$$\exp(-\alpha P e^r (T-t))\exp(\eta(S,t)) = 1 \tag{26}$$

and hence

$$P(S,t) = P(w,S,t) = \frac{1}{\alpha}\eta(S,t)e^{-r(T-t)}. \tag{27}$$

We observe that the premium is independent of the insurer's wealth level w. This highly desirable property of *universality* is an artifact of the constant absolute risk aversion inherent in exponential utility.

Thus, if we can compute a positive solution η to the terminal value problem (23), (24), we have the positive premium P via (27).

Using (23) and (27), we can show that the premium P solves the terminal value problem

$$\left\{ \begin{array}{c} P_t + rSP_S + \frac{1}{2}\sigma^2 S^2 P_{SS} + \frac{\lambda_x(t)}{\alpha e^{r(T-t)}}[\exp(-\alpha P e^{r(T-t)}) - 1] - rP = 0 \\ P(S,T) = g(S) \end{array} \right\}. \tag{28}$$

Note that this is a nonlinear Black-Scholes equation; the nonlinear term reflects the fact that we have incorporated mortality risk and exponential risk preferences into our model. If $\lambda \equiv 0$, i.e., if there is no mortality risk, the premium P is the Black-Scholes price, as we predicted in Section 1. In the sections that follow, we present results on the qualitative and quantitative properties of P. All of the results can be obtained easily by analyzing the problem (28), but in some cases the calculations are slightly cleaner if we work instead with the problem (23),(24) and then compute P via (27). For clarity of exposition and to simplify calculations, we will use this latter approach.

4 Qualitative Properties of the Premium

In this section we discuss the qualitative properties of the indifference premium P. More specifically, we discuss ordering of solutions to (28) as we vary the parameters $\lambda_x(t), \alpha$, and σ, and we obtain analytical upper and lower bounds on P. Moreover, we observe that each of these results is consistent with our financial intuition. In Section 5, we demonstrate that numerically computed solutions exhibit the monotonicity and obey the bounds predicted in this section.

We obtain our results via comparison principles for parabolic PDEs. To use these theorems, we first transform the equations (23) and (28) from terminal value problems for degenerate parabolic equations to initial value problems for uniformly parabolic equations via the standard transformation

$$S = e^y \quad \text{and} \quad t = T - \tau. \tag{29}$$

We let

$$\eta(S,t) = \tilde{\eta}(y,\tau) \quad \text{and} \quad P(S,t) = \tilde{P}(y,\tau). \tag{30}$$

Computing the appropriate derivatives of η and P and plugging into (23) and (28) yields the transformed problems

$$\left\{ \begin{array}{c} \tilde{\eta}_\tau = \frac{1}{2}\sigma^2\tilde{\eta}_{yy} + (r - \frac{1}{2}\sigma^2)\tilde{\eta}_y + \lambda_x(T-\tau)[e^{-\tilde{\eta}} - 1] \\ \tilde{\eta}(y,0) = \alpha g(e^y) = \alpha\tilde{g}(y) \end{array} \right\} \tag{31}$$

and

$$\left\{ \begin{array}{c} \tilde{P}_\tau = \frac{1}{2}\sigma^2\tilde{P}_{yy} + (r - \frac{1}{2}\sigma^2)\tilde{P}_y + \frac{\lambda_x(T-\tau)}{\alpha e^{r\tau}}[\exp(-\alpha\tilde{P}e^{r\tau}) - 1] - r\tilde{P} \\ \tilde{P}(y,0) = g(e^y) = \tilde{g}(y) \end{array} \right\} \tag{32}$$

where $\tilde{g} := g \circ exp$.

Denote the nonlinear terms in (31) and (32) by

$$\begin{array}{l} h_1(y,\tau,\tilde{\eta},\tilde{\eta}_y) = (r - \frac{1}{2}\sigma^2)\tilde{\eta}_y + \lambda_x(T-\tau)(e^{-\tilde{\eta}} - 1) \\ h_2(y,\tau,\tilde{P},\tilde{P}_y) = (r - \frac{1}{2}\sigma^2)\tilde{P}_y + \frac{\lambda_x(T-\tau)}{\alpha e^{r\tau}}(\exp(-\alpha\tilde{P}e^{r\tau}) - 1) - r\tilde{P} \end{array} \tag{33}$$

and observe that when $\tilde{\eta} > \overline{\eta}$ and $\tilde{P} > \overline{P}$, the nonlinearities satisfy the one-sided Lipschitz conditions

$$\left\{ \begin{array}{l} h_1(y,\tau,\tilde{\eta},\tilde{\eta}_y) - h_1(y,\tau,\overline{\eta},\overline{\eta}_y) \le c_1(y,\tau)(\tilde{\eta} - \overline{\eta}) + b_1(y,\tau)|\tilde{\eta}_y - \overline{\eta}_y| \\ h_2(y,\tau,\tilde{P},\tilde{P}_y) - h_2(y,\tau,\overline{P},\overline{P}_y) \le c_2(y,\tau)(\tilde{P} - \overline{P}) + b_2(y,\tau)|\tilde{P}_y - \overline{P}_y| \end{array} \right\} \tag{34}$$

for some $b_i, c_i \ge 0$, $i = 1, 2$.

The results that follow rely on a standard comparison theorem for semilinear parabolic PDEs. We state it below for completeness; see, for example, Chapter IV, Section 28 of [20].

Theorem 4.1 *Let $G = \mathbf{R} \times [0,T]$ and let $a = a(y,\tau) \in (0,M)$ for some $0 < M < \infty$. Suppose $v, w \in \mathcal{C}(\overline{G}) \cap \mathcal{C}^2(G)$ and that there exists $K > 0$ such that*

$$|v(y,\tau)|, |w(y,\tau)| \le Ke^{Ky^2} \quad \text{in } G. \tag{35}$$

Finally, suppose

$$\left\{ \begin{array}{rcll} v_\tau - av_{yy} - h(y,\tau,v,v_y) & \le & w_\tau - aw_{yy} - h(y,\tau,w,w_y) & \text{in } G \\ v & \le & w & \text{on } \mathbf{R} \times \{\tau = 0\} \end{array} \right\} \tag{36}$$

where h obeys the one-sided Lipschitz condition in (34).

Then $v \le w$ in G. Moreover, the solution to the initial value problem

$$\left\{ \begin{array}{rcll} u_\tau & = & au_{yy} + h(y,\tau,u,u_y) & \text{in } G \\ u(y,0) & = & \phi(y) & \text{on } \mathbf{R} \times \{\tau = 0\} \end{array} \right\} \tag{37}$$

is unique among $\mathcal{C}(\overline{G}) \cap \mathcal{C}^2(G)$ functions that satisfy the growth condition (35).

In the theorems that follow, we show that as we vary the model parameters, the premium response is consistent with our expectations.

4.1 Impact on premium of changing mortality assumptions

Theorem 4.2 *Suppose $\lambda_x^1(t) \ge \lambda_x^2(t)$ for all $t \in [0,T]$ and let $P^i(S,t)$ be positive solutions to the problem (28) with $\lambda_x = \lambda_x^i$, $i = 1,2$. Then $P^1(S,t) \le P^2(S,t)$ for all $(S,t) \in [0,\infty) \times [0,T]$.*

Heuristically, this means that the premium is lower under higher mortality. This is consistent with our intuition as, under higher mortality, an endowment payout is less likely.

Proof: Let $\lambda_x^1(t) > \lambda_x^2(t)$ for all $t \in [0,T]$ and let $\tilde{\eta}^i$ be positive solutions to the problem (31) with $\lambda_x = \lambda_x^i$, $i = 1,2$. Define the operator

$$Lu := u_\tau - \frac{1}{2}\sigma^2 u_{yy} - (r - \frac{1}{2}\sigma^2)u_y - \lambda_x^2(T-\tau)[e^{-u} - 1]. \tag{38}$$

Since $\tilde{\eta}^2$ solves the problem (31) with the parameter λ_x^2, we have that $L\tilde{\eta}^2 = 0$. Moreover, since $\tilde{\eta}^1$ solves the problem (31) with the parameter λ_x^1, we have

$$\begin{aligned} L\tilde{\eta}^1 &= \tilde{\eta}^1_\tau - \tfrac{1}{2}\sigma^2\tilde{\eta}^1_{yy} - (r - \tfrac{1}{2}\sigma^2)\tilde{\eta}^1_y - \lambda_x^2(T-\tau)[e^{-\tilde{\eta}^1} - 1] \\ &= \tilde{\eta}^1_\tau - \tfrac{1}{2}\sigma^2\tilde{\eta}^1_{yy} - (r - \tfrac{1}{2}\sigma^2)\tilde{\eta}^1_y \\ &\quad -\lambda_x^1(T-\tau)[e^{-\tilde{\eta}^1} - 1] + \lambda_x^1(T-\tau)[e^{-\tilde{\eta}^1} - 1] - \lambda_x^2(T-\tau)[e^{-\tilde{\eta}^1} - 1] \\ &= 0 + (e^{-\tilde{\eta}^1} - 1)[\lambda_x^1(T-\tau) - \lambda_x^2(T-\tau)] \\ &\leq 0 \end{aligned} \tag{39}$$

since $\tilde{\eta}^1 > 0$ and $\lambda_x^1 \geq \lambda_x^2$.

Thus we have $L\tilde{\eta}^1 \leq L\tilde{\eta}^2$. In addition, since $\tilde{\eta}^1$ and $\tilde{\eta}^2$ both satisfy the initial condition in (31), we have that $\tilde{\eta}^1(y,0) = \tilde{\eta}^2(y,0) = \alpha\tilde{g}(y)$, and we can conclude from Theorem 4.1 that $\tilde{\eta}^1 \leq \tilde{\eta}^2$ in $\mathbf{R} \times [0,T]$.

Now, defining $\eta^i(S,t) = \tilde{\eta}^i(y,\tau)$ via the transformation in (29),(30) for $i = 1,2$, we have that $\eta^i(S,t)$ solves (23), (24) with $\lambda_x = \lambda_x^i$ and that $\eta^1(S,t) \leq \eta^2(S,t)$ on $[0,\infty] \times [0,T]$. Finally, by (27), it follows that the premium satisfies $P^1(S,t) \leq P^2(S,t)$. ■

We can use an argument similar to the proof of Theorem 4.2 to establish an upper bound on the indifference premium P. First consider the problem (31) with $\lambda_x \equiv 0$, i.e., consider the problem

$$\left\{ \begin{array}{c} \tilde{\eta}_\tau = \frac{1}{2}\sigma^2\tilde{\eta}_{yy} + (r - \frac{1}{2}\sigma^2)\tilde{\eta}_y \\ \tilde{\eta}(y,0) = \alpha g(e^y) = \alpha\tilde{g}(y) \end{array} \right\}. \tag{40}$$

Solving (40) via the Fourier transform, we find that the solution $\tilde{\eta}^{\lambda 0}$ is given by

$$\tilde{\eta}^{\lambda 0}(y,\tau) = \frac{\alpha}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} exp\left(-\frac{(y - x + (r - \frac{1}{2}\sigma^2)\tau)^2}{2\sigma^2\tau} \right) g(e^x)dx. \tag{41}$$

Now let $\tilde{\eta}$ solve (31) for some $\lambda_x \neq 0$. An argument similar to the proof of Theorem 4.2 yields that $\tilde{\eta} \leq \tilde{\eta}^{\lambda 0}$ on $\mathbf{R}\times[0,T]$. Employing the transformations (29),(30), and (27), we have that $P \leq P^{\lambda 0}$, where

$$P^{\lambda 0}(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} exp\left(-\frac{(\ln S - x + (r - \frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)} \right) g(e^x)dx. \tag{42}$$

The premium $P^{\lambda 0}(S,t)$ above for the "endowment certain" can also be obtained by setting $\lambda_x \equiv 0$ in (28) and solving via the Feynman-Kač formula; see, for example, Chapter 4 of [3]. In this case, we write $P^{\lambda 0}$ in the familiar form

$$P^{\lambda 0}(S,t) = e^{-r(T-t)}\tilde{E}[g(S_T)|S_t = S] \tag{43}$$

where the expectation $\tilde{E}$ is taken with respect to the risk neutral measure and the S dynamics are given by

$$\left\{ \begin{array}{ll} dS_s & = rS_s + \sigma S_s d\tilde{B}_s \\ S_t & = S \end{array} \right\}, \tag{44}$$

where $\tilde{B}_s$ is a standard Brownian motion with respect to the risk neutral measure. Writing the expectation above in integral form and changing variables, one can verify that the representations (42) and (43) are equivalent.

We state this result succinctly in the following.

Corollary 4.3 *If $P(S,t)$ is a positive solution to (28) for some $\lambda_x \neq 0$, then $P(S,t) \leq P^{\lambda 0}(S,t)$ where $P^{\lambda 0}(S,t)$ is given in (42) or (43) above.*

To close this subsection, we observe that $P^{\lambda 0}$ is the price of an "endowment certain," i.e., a contract with no life contingency. More specifically, it is the Black-Scholes price of a derivative with (certain) payout $g(S_T)$. Thus the fact that $P^{\lambda 0}$ serves as an upper bound on the premium for the life contingent contract is consistent with our financial and actuarial intuition.

4.2 Impact on the premium of changing risk aversion assumptions

Theorem 4.4 *Suppose $\alpha^1 > \alpha^2$ and let $P^i(S,t)$ be positive solutions to (28) with $\alpha = \alpha^i, i = 1, 2$. Then $P^1(S,t) \geq P^2(S,t)$ for all $(S,t) \in [0,\infty] \times [0,T]$.*

Heuristically, this means that the premium increases as absolute risk aversion increases. Again, this is consistent with our intuition.

Proof: We proceed as in the proof of Theorem 4.2. Let $\alpha^1 > \alpha^2$ and let $\tilde{P}^i$ be positive solutions to (32) with $\alpha = \alpha^i, i = 1, 2$. Define the operator

$$Lu = u_\tau - \frac{1}{2}\sigma^2 u_{yy} - (r - \frac{1}{2}\sigma^2)u_y + ru - \frac{\lambda_x(T-\tau)}{\alpha^1 e^{r\tau}}[exp(-\alpha^1 u e^{r\tau}) - 1]. \tag{45}$$

Since $\tilde{P}^1$ solves (32) with $\alpha = \alpha^1$, we have that $L\tilde{P}^1 = 0$. Also,

$$\begin{aligned} L\tilde{P}^2 &= \tilde{P}^2_\tau - \tfrac{1}{2}\sigma^2 \tilde{P}^2_{yy} - (r - \tfrac{1}{2}\sigma^2)\tilde{P}^2_y + r\tilde{P}^2 - \tfrac{\lambda_x(T-\tau)}{\alpha^1 e^{r\tau}}[exp(-\alpha^1 \tilde{P}^2 e^{r\tau}) - 1] \\ &= \tilde{P}^2_\tau - \tfrac{1}{2}\sigma^2 \tilde{P}^2_{yy} - (r - \tfrac{1}{2}\sigma^2)\tilde{P}^2_y + r\tilde{P}^2 - \tfrac{\lambda_x(T-\tau)}{\alpha^2 e^{r\tau}}[exp(-\alpha^2 \tilde{P}^2 e^{r\tau}) - 1] \\ &\quad + \tfrac{\lambda_x(T-\tau)}{\alpha^2 e^{r\tau}}[exp(-\alpha^2 \tilde{P}^2 e^{r\tau}) - 1] - \tfrac{\lambda_x(T-\tau)}{\alpha^1 e^{r\tau}}[exp(-\alpha^1 \tilde{P}^2 e^{r\tau}) - 1] \\ &= 0 + \tfrac{\lambda_x(T-\tau)}{e^{r\tau}}[\tfrac{exp(-\alpha^2 \tilde{P}^2 e^{r\tau}) - 1}{\alpha^2} - \tfrac{exp(-\alpha^1 \tilde{P}^2 e^{r\tau}) - 1}{\alpha^1}] \end{aligned} \tag{46}$$

where the last equality follows from the fact that $\tilde{P}^2$ solves (32) with $\alpha = \alpha^2$. Since $\tilde{P}^2, \alpha^1$,and α^2 are positive, one can verify that $f(\alpha) := \frac{exp(-\alpha \tilde{P}^2 e^{r\tau}) - 1}{\alpha}$ is increasing in α on $(0, \infty)$, thus $\alpha^1 > \alpha^2$ implies that

$$L\tilde{P}^2 = \frac{\lambda_x(T-\tau)}{e^{r\tau}}[f(\alpha^2) - f(\alpha^1)] < 0. \tag{47}$$

Thus we have that $L\tilde{P}^2 \leq L\tilde{P}^1$ on $\mathbf{R} \times [0, T]$ and since $\tilde{P}^1$ and $\tilde{P}^2$ both satisfy the initial condition in (32), $\tilde{P}^1(y, 0) = \tilde{P}^2(y, 0) = \tilde{g}(y)$, and it follows from Theorem 4.1 that $\tilde{P}^1 \geq \tilde{P}^2$ on $\mathbf{R} \times [0, T]$. Defining $P^i(S, t)$ via the transformation in (29), (30), we have that $P^i(S, t)$ solves (28) with $\alpha = \alpha^i$ and that $P^1(S, t) \geq P^2(S, t)$ on $[0, \infty) \times [0, T]$. ■

We can obtain an analytical lower bound on the premium as follows. By arguments similar to the proof of Theorem 4.4 and Corollary 4.3, we can show that $P(S, t) \geq P^{\alpha 0}(S, t)$ where $P^{\alpha 0}$ is a positive solution of (28) with $\alpha = 0$ (in the limit). Again, this is consistent with our intuition; the premium will be lower in the absence of risk aversion.

Setting $\alpha = 0$ in the limit in (28) yields the linear terminal value problem

$$\left\{ \begin{array}{c} P_t + rSP_S + \frac{1}{2}\sigma^2 S^2 P_{SS} - (\lambda_x(t) + r)P = 0 \\ P(S, T) = g(S), \end{array} \right\}. \tag{48}$$

Solving this problem via the Feynman-Kač formula (or by the Fourier transform on the transformed uniformly parabolic problem), we find that the solution $P^{\alpha 0}$ is given by

$$P^{\alpha 0}(S, t) = {}_{T-t}\, p_{x+t} e^{-r(T-t)} \tilde{E}[g(S_T) | S_t = S] = {}_{T-t}\, p_{x+t} P^{\lambda 0}(S, t) \tag{49}$$

where where ${}_{T-t}p_{x+t} = e^{-\int_t^T \lambda_x(s)ds}$ is the probability that (x) survives from time t until time T, conditional on (x) surviving to age $x + t$.

We state this observation succinctly in the following.

Corollary 4.5 *If $P(S,t)$ is a positive solution to (28), then $P(S,t) \geq P^{\alpha 0}(S,t)$, where $P^{\alpha 0}$ is given in (49).*

We observe that $P^{\alpha 0}(S,t)$ is analogous to the Brennan and Schwartz price for an equity-linked life insurance contract, [6]. As this price does not incorporate the insurer's risk aversion, we expect that the Brennan and Schwartz premium should be lower than P, as our model assumes exponential utility. Thus the result of Corollary 4.5 is consistent with our financial intuition.

In summary, combining the results of Corollaries 4.3 and 4.5, we have that

$$P \in [P^{\alpha 0}, P^{\lambda 0}] = [{}_{T-t}p_{x+t} P^{\lambda 0}, P^{\lambda 0}] \tag{50}$$

where ${}_{T-t}p_{x+t} \in (0,1)$. We demonstrate in Section 5 that under reasonable mortality assumptions, (50) yields tight bounds on the indifference premium P.

4.3 Impact on the premium of changing volatility assumptions

Theorem 4.6 *Let $\sigma_1 > \sigma_2$ and let P^i be positive solutions to (28) with $\sigma = \sigma_i$. If P^1 or P^2 satisfies $P^i_{SS} \leq 0$ on $[0,\infty) \times [0,T]$, then $P^2 \geq P^1$. However, if P^1 or P^2 satisfies $P^i_{SS} \geq 0$ on $[0,\infty) \times [0,T]$, then $P^1 \geq P^2$.*

Proof: Define $\tilde{P}(y,\tau) = P(S,t)$ via the transformation in (29),(30). Since

$$\begin{aligned} P_S(S,t) &= e^{-y}\tilde{P}_y(y,\tau) \\ P_{SS}(S,t) &= e^{-2y}(\tilde{P}_{yy} - \tilde{P}_y), \end{aligned} \tag{51}$$

we see that $P_{SS}(S,t)$ has the same sign as $(\tilde{P}_{yy} - \tilde{P}_y)(y,\tau)$. Moreover, $\tilde{P}^i$ is a positive solution to (32) with $\sigma = \sigma_i$.

Define the operator

$$Lu = u_\tau - \frac{1}{2}\sigma_2^2 u_{yy} - (r - \frac{1}{2}\sigma_2^2)u_y - h_2(u) \tag{52}$$

where $h_2(u) = h_2(y,\tau,u,u_y)$ is defined in (33).

Since $\tilde{P}^2$ solves (32) with $\sigma = \sigma_2$, we have that $L\tilde{P}^2 = 0$. Also,

$$\begin{aligned} L\tilde{P}^1 &= \tilde{P}^1_\tau - \tfrac{1}{2}\sigma_2^2\tilde{P}^1_{yy} - (r - \tfrac{1}{2}\sigma_2^2)\tilde{P}^1_y - h_2(\tilde{P}^1) \\ &= \tilde{P}^1_\tau - \tfrac{1}{2}\sigma_1^2\tilde{P}^1_{yy} - (r - \tfrac{1}{2}\sigma_1^2)\tilde{P}^1_y - h_2(\tilde{P}^1) \\ &\quad + \tfrac{1}{2}\sigma_1^2\tilde{P}^1_{yy} + (r - \tfrac{1}{2}\sigma_1^2)\tilde{P}^1_y - \tfrac{1}{2}\sigma_2^2\tilde{P}^1_{yy} - (r - \tfrac{1}{2}\sigma_2^2)\tilde{P}^1_y \\ &= 0 + \tfrac{1}{2}\tilde{P}^1_{yy}(\sigma_1^2 - \sigma_2^2) + \tfrac{1}{2}\tilde{P}^1_y(\sigma_2^2 - \sigma_1^2) \\ &= \tfrac{1}{2}(\sigma_1^2 - \sigma_2^2)(\tilde{P}^1_{yy} - \tilde{P}^1_y), \end{aligned} \tag{53}$$

where the third equality follows from the fact that $\tilde{P}^1$ solves (32) with $\sigma = \sigma_1$.

Now, if $P^1_{SS} \le 0$, then $L\tilde{P}^1 \le 0$. From Theorem 4.1, $\tilde{P}^1 \le \tilde{P}^2$ and hence $P^1 \le P^2$. However, if $P^1_{SS} \ge 0$, then $L\tilde{P}^1 \ge 0$. From Theorem 4.1, $\tilde{P}^1 \ge \tilde{P}^2$ and hence $P^1 \ge P^2$.

Suppose instead that we know the concavity of P^2 *a priori*. We can use the same argument as above to obtain the desired result. We simply define the operator L as above, but with σ_1 in place of σ_2. We then exploit the fact that $L\tilde{P}^1 = 0$ and compute $L\tilde{P}^2$. The condition on the concavity of $\tilde{P}^2$ induces the appropriate sign on $L\tilde{P}^2$ and the desired result follows. ■

Similarly, we can derive an analytical lower bound on the premium provided the payout $g(S_T)$ satisfies an appropriate growth condition.

Theorem 4.7 *Let $P(S,t)$ be a positive solution of (28) with $\sigma > 0$ and let $g''(S) \ge 0$. Then $P(S,t) \ge P^{\sigma 0}(S,t)$ where*

$$P^{\sigma 0}(S,t) := \frac{1}{\alpha} e^{-r(T-t)} \ln[{}_{T-t}p_{x+t}(e^{\alpha g(Se^{r(T-t)})} - 1) + 1] \tag{54}$$

Proof: Given $P(S,t)$ and $P^{\sigma 0}(S,t)$, we compute $\tilde{\eta}(y,\tau), \tilde{\eta}^{\sigma 0}(y,\tau)$, and $\eta^{\sigma 0}(S,t)$ via the transformations (29),(30),(27). Specifically,

$$\tilde{\eta}^{\sigma 0}(y,\tau) = \ln[{}_\tau p_{x+T-\tau}(e^{\alpha\tilde{g}(y+r\tau)} - 1) + 1] \tag{55}$$

and

$$\eta^{\sigma 0}(S,t) = \ln[{}_{T-t}p_{x+t}(e^{\alpha g(Se^{r(T-t)})} - 1) + 1]. \tag{56}$$

A straightforward but tedious calculation shows that $\tilde{\eta}^{\sigma 0}$ solves (31) with $\sigma = 0$, therefore, $P^{\sigma 0}$ as defined above solves (28) with $\sigma = 0$.

Now, define $Lu = u_\tau - \frac{1}{2}\sigma^2 u_{yy} - (r - \frac{1}{2}\sigma^2)u_y - \lambda_x(T-t)[e^{-u} - 1]$. Since P solves (28), we have that $L\tilde{\eta} = 0$. Moreover,

$$L\tilde{\eta}^{\sigma 0} = -\frac{1}{2}\sigma^2 \tilde{\eta}^{\sigma 0}_{yy} + \frac{1}{2}\sigma^2 \tilde{\eta}^{\sigma 0}_{y} = -\frac{1}{2}\sigma^2 (\tilde{\eta}^{\sigma 0}_{yy} - \tilde{\eta}^{\sigma 0}_{y}) = -\frac{1}{2}\sigma^2 S^2 \eta^{\sigma 0}_{SS}. \tag{57}$$

By direct calculation, we see that the sign of $\eta^{\sigma 0}_{SS}$ is the same as the sign of the quantity

$$Q := [{}_{T-t}p_{x+t}(e^{\alpha g(Se^{r(T-t)})} - 1) + 1]g''(Se^{r(T-t)}) + \alpha g'(Se^{r(T-t)})^2(1 -{}_{T-t}\, p_{x+t}), \tag{58}$$

which is nonnegative if $g'' \geq 0$. In this case we have $L\tilde{\eta}^{\sigma 0} \leq 0 = L\tilde{\eta}$, which implies that $\tilde{\eta}^{\sigma 0} \leq \tilde{\eta}$ and hence $P^{\sigma 0} \leq P$. ■

Remark 4.8 *Heuristically, we've shown that the premium for an endowment written on a volatile stock exceeds that for an endowment written on a less volatile stock, provided the payout g increases "fast enough" with S.*

Remark 4.9 *The requirement $g'' \geq 0$ in the statement of Theorem 4.7 is stronger than we need. In (58), to guarantee the appropriate sign on Q, and hence on $L\tilde{\eta}^{\sigma 0}$, we need only require that*

$$g'' \geq -\frac{\alpha g'^2(1 -{}_{T-t}\, p_{x+t})}{{}_{T-t}p_{x+t}(e^{\alpha g} - 1) + 1}, \tag{59}$$

i.e., that g'' is not "too negative." Thus we can broaden this theorem to include payout functions that are concave on part of their domain.

5 Numerical solutions to the Premium Equation

Because of the nonlinear term in (28), we cannot solve for the premium P in closed form. In Section 4, we described P's qualitative properties. More specifically, we derived analytical upper and lower bounds on P and demonstrated that as we vary the parameters λ and α, the change in P is consistent with our intuition. In this section, we compute the approximate premium P via a numerical finite difference scheme.

Because of the degeneracy of the equation (28), we chose to implement an implicit scheme (explicit in the nonlinear part) on the uniformly parabolic problem (32) defined on $\mathbf{R} \times [0, T]$. Of course, for numerical implementation, we must truncate the spatial domain and work on, say, $[-M, M] \times [0, T]$. Two delicate questions arise then:

1. What boundary conditions should we impose at $y = \pm M$?
2. How large must M be so that the error introduced by the imposition of the artificial boundary is not significant in our domain of interest?

We address the first question by deriving the correct boundary condition for (28) at $S = 0$ (and hence the correct condition for (32) at $y = -\infty$) from the definition of the value functions V and U given in (3) and (5) and the indifference relation (6). Assume that the payout function g is bounded and continuous. At $S = 0$, we observe that wealth evolves deterministically and terminal wealth W_T given $W_t = w$ is $W_T = we^{r(T-t)}$; thus, under exponential utility, from (3) we have

$$V(w,t) = \sup_{\{\pi_s\}} E[u(W_T)|W_t = w] = u(we^{r(T-t)}) = -\frac{1}{\alpha} e^{-\alpha w e^{r(T-t)}}. \tag{60}$$

Moreover, $U(w, 0, t) = \sup_{\{\pi_s\}} E[u(w_T - Y_T)|W_t = w, S_t = 0]$ where

$$Y_T = \begin{cases} g(S_T) & \text{if } (x) \text{ survives until time } T \\ 0 & \text{otherwise.} \end{cases} \tag{61}$$

Since the probability of a payout is ${}_{T-t}p_{x+t}$, we have

$$\begin{aligned} U(w,0,t) &= {}_{T-t}p_{x+t} u(W_T - g(0)) + {}_{T-t}q_{x+t} u(W_T) \\ &= -\tfrac{1}{\alpha} e^{-\alpha w e^{r(T-t)}} ({}_{T-t}p_{x+t} e^{\alpha g(0)} + {}_{T-t}q_{x+t}) \end{aligned} \tag{62}$$

where ${}_{T-t}q_{x+t} = 1 - {}_{T-t}p_{x+t}$ is the probability that the insured dies during the deferral period. Finally, the indifference relation (6) yields

$$P(0,t) = \frac{1}{\alpha} e^{-r(T-t)} \ln[{}_{T-t}p_{x+t}(e^{\alpha g(0)} - 1) + 1] \tag{63}$$

and we can compute $\tilde{P}(-\infty, \tau)$ via the transformation (29), (30).

Alternatively, one could derive this condition by setting $S = 0$ in (23), solving the resulting ODE for $\eta(0, t)$, and transforming to $\tilde{P}(-\infty, \tau)$ via (27),(29), and (30). Moreover, this argument is applicable in deriving the boundary condition at $y = \infty$ provided that g is bounded.

In [12], the authors address the second question above by determining pointwise bounds on the near field error and determining a suitable location for the artificial boundary for a *linear* Black-Scholes problem. As we have not done similar analysis for the nonlinear problem yet, we set the artificial boundary very conservatively. In the examples that follow, our spatial domain of interest is $S \in [0, 100]$. We solved the problem (32) in the transformed coordinates $y \in [-20, 20]$ and $y \in [-50, 50]$. We remark that these domains are far more conservative than Kangro and Nicolaides prescription and the "rules of thumb" that they cite, [12]. Moreover, although the solutions changed for very large S when we changed from $y \in [-20, 20]$ to $y \in [-50, 50]$, the change was negligible for $S \in [0, 100]$.

5.1 Numerical Experiments

In the experiments that follow, we set the investment horizon, volatility, and risk free rate to be $T = 20$, $\sigma = 0.2$, and $r = 0.06$, respectively. Moreover, we set the payout function

$$g(S) = \begin{cases} 7.5 & 0 \le S \le 10 \\ 0.75S & 10 \le S \le 90 \\ 67.5 & 90 \le S \le 100; \end{cases} \tag{64}$$

thus, the payout is linear in the stock price with a "stop loss" for the insurer and downside protection for the insured.

Experiment 1: *Premium evolution over time*
In this example, we show the premium for a 50-year-old female for deferral periods $T - t = 5, 10, 15$ and 20 years. We set the insurer's risk aversion coefficient $\alpha = 0.1$ and, as in [10] and [15], we assume Gompertz mortality

$$\lambda_x(t) = \frac{1}{\beta} e^{\frac{x+t-m}{\beta}}. \tag{65}$$

with parameters $\beta = 8.75$ and $m = 92.63$. Figure 1 shows the evolution of the premium over time. For example, at the horizon ($T - t = 0$), the

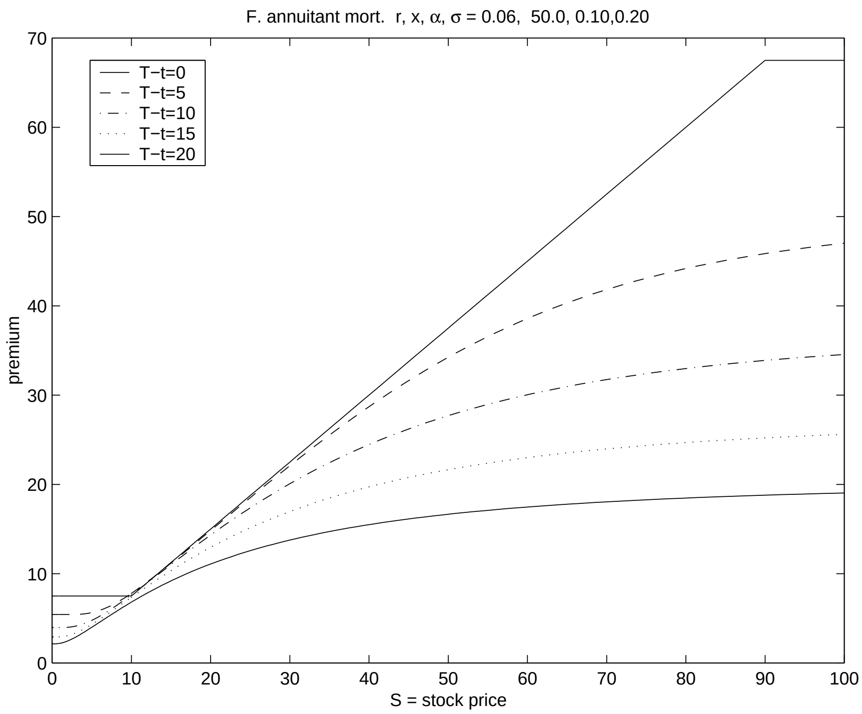

Figure 1: Experiment 1. Premium for several different deferral periods. As we expect, the premium is lower for longer deferral periods.

premium is exactly the payout function $g(S)$; five years prior to the horizon, the premium is given by the dashed curve, etc. We observe that, consistent with our intuition, the premium is lower for longer deferral periods.

Experiment 2: *Impact of changing mortality assumptions*
In Theorem 4.2 and Corollary 4.3, we proved that the premium is bounded above by $P^{\lambda 0}$, the Black-Scholes price for an "endowment certain" ($\lambda = 0$), and that the premium decreases as mortality increases. Moreover, from Corollary 4.5, we know that $P^{\alpha 0}$, the Brennan and Schwartz [6] premium for an insurer with no risk aversion ($\alpha = 0$), serves as a lower bound on the premium. We demonstrate this result in Figure 2 for deferral periods of 5, 10, 15, and 20 years. In each graph, the solid curve shows $P^{\lambda 0}$, the premium under zero mortality. The next two curves are the premium with constant mortality $\lambda = 0.04$ and $\lambda = 0.09$, respectively. (Of course, the assumption of constant mortality is an oversimplification; we simply wish to demonstrate that our numerical solutions exhibit the behavior guaranteed by Theorem

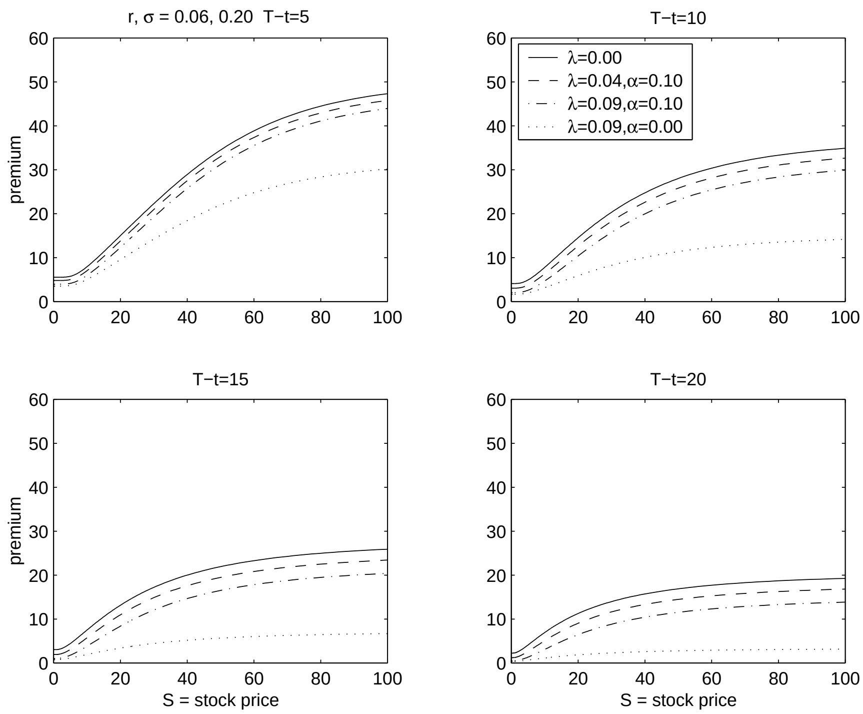

Figure 2: Experiment 2. Consistent with our intuition, premium decreases as mortality increases.

4.2.) The dotted curve shows the lower bound $P^{\alpha 0}$, the premium with the risk aversion parameter α equal to zero.

Experiment 3: *Impact of changing risk aversion assumptions*
In Theorem 4.4, we proved that premium increases as risk aversion increases. We verify this in Figure 3, which shows the premium for $\alpha = 0, \alpha = 0.1$, and $\alpha = 1$ under constant mortality $\lambda = 0.04$ for deferral periods of 5, 10, 15, and 20 years. Again, we include the $\lambda = 0$ premium as an upper bound.

Experiment 4: *Impact of changing volatility assumptions*
In Theorem 4.6 we proved that if the premium is concave, then it decreases with volatility, but if it is convex, then it increases with volatility. We demonstrate this phenomenon in Figure 4, which shows the premium for deferral periods of 5, 10, 15, and 20 years for $\sigma = 0.2$ and $\sigma = 0.4$. Our mortality assumption is the same as in Experiment 1.

Experiment 5: *Premium Bounds*

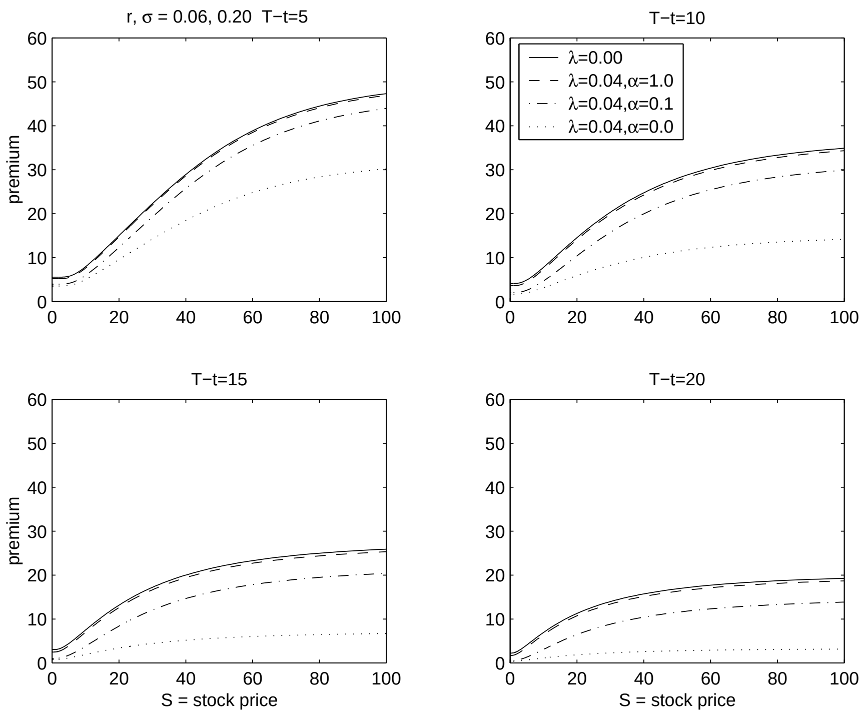

Figure 3: Experiment 3. Consistent with our intuition, premium increases as risk aversion increases.

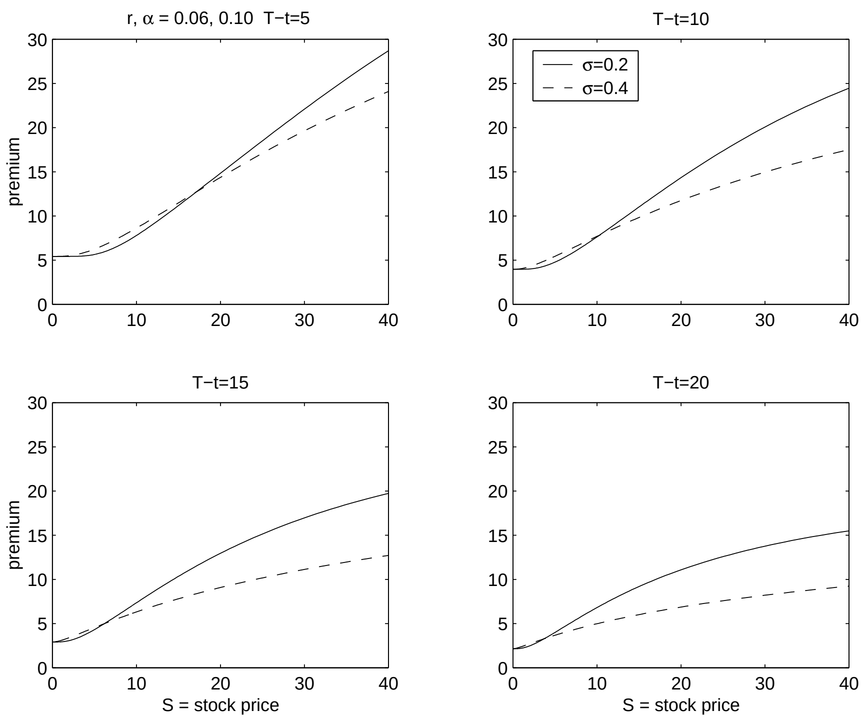

Figure 4: Experiment 4. When the premium is convex, it increases with volatility. When it is concave, it decreases with volatility.

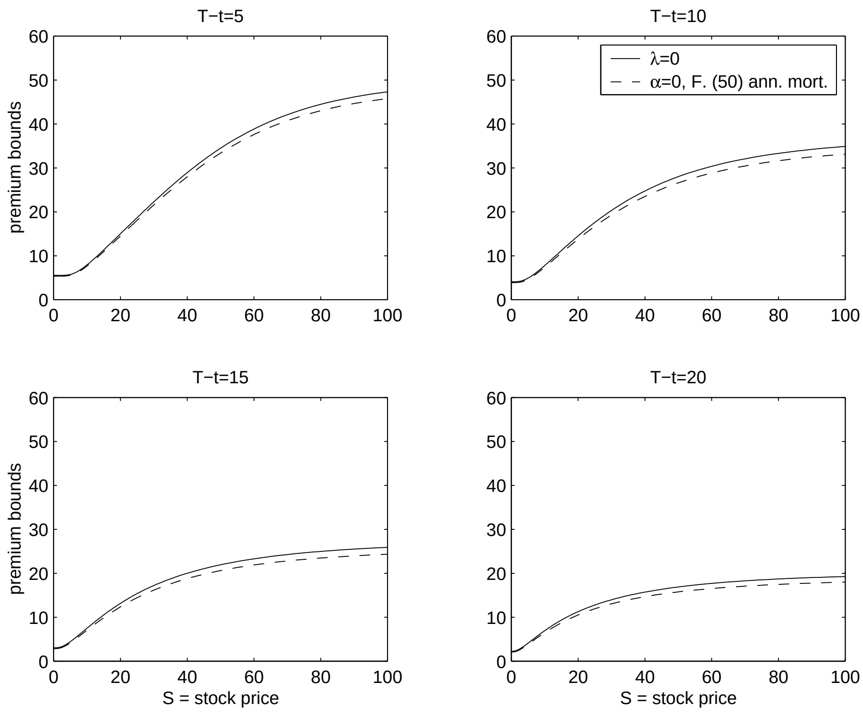

Figure 5: Experiment 5. In this example, we obtain tight bounds on the premium via Corollaries 4.3 and 4.5 without solving the nonlinear PDEs.

We illustrate the strength of Corollaries 4.3 and 4.5 in Figure 5. We set $\alpha = 0.1$ and assume Gompertz mortality for a female aged 50 as in Experiment 1. Corollaries 4.3 and 4.5 ensure that the premium P satisfies

$$P^{\alpha 0}(S,t) \leq P(S,t) \leq P^{\lambda 0}(S,t) \tag{66}$$

where $P^{\alpha 0}$ is given by (49) using Gompertz mortality (65) and $P^{\lambda 0}$ is given by (42). We remark that the integrals in (49) and (42) can be evaluated in closed form (in terms of error functions) for piecewise linear g or numerically for more complicated g. As Figure 5 demonstrates, we obtain reasonably tight bounds on the premium P without actually solving the nonlinear PDE (28) for P.

6 A Probabilistic Representation of the Premium

In this section, we develop an alternative probabilistic representation of the premium P. We aim for a representation similar to the arbitrage free representation of derivative prices in complete models as the expectation of the discounted payoff under the martingale measure. Following as in [16], we reformulate the PDE (28) that governs P as the HJB of a stochastic control problem.

Because the calculations are simpler, for conciseness of exposition, we will work with the PDE (23) to derive the probabilistic representation of η and then compute P via (27). One can easily verify that starting with the PDE (28) for P yields the same expression.

Denote the nonlinear term in (23) by

$$\beta(\eta) = 1 - e^{-\eta} \tag{67}$$

and define the convex dual of β by

$$\hat{\beta}(y) = \max_{\{\eta\}} [\beta(\eta) - y\eta]. \tag{68}$$

Simple calculations yield that

$$\hat{\beta}(y) = 1 - y + y \ln y \geq 0 \tag{69}$$

and one can verify that

$$\beta(\eta) = \min_{\{y>0\}} [\hat{\beta}(y) + y\eta]. \tag{70}$$

Since $-\min_z f(z) = \max_z(-f(z))$, we can rewrite (23) as

$$\left\{ \begin{array}{c} \eta_t + rS\eta_S + \frac{1}{2}\sigma^2 S^2 \eta_{SS} + \lambda_x(t) \max_{\{y>0\}} [-\hat{\beta}(y) - y\eta] = 0 \\ \eta(S,T) = \alpha g(S) \end{array} \right\}. \tag{71}$$

Consider the process $\tilde{S}_s$ whose evolution is governed by the geometric Brownian motion

$$\left\{ \begin{array}{ll} d\tilde{S}_s & = r\tilde{S}_s + \sigma \tilde{S}_s d\tilde{B}_s \\ \tilde{S}_t & = S \geq 0 \end{array} \right\}. \tag{72}$$

Define the value function

$$\begin{aligned} \eta(S,t) &= \sup_{\{y_s\}} \tilde{E}[\alpha g(\tilde{S}_T) e^{-\int_t^T \lambda_x(s) y_s ds} \\ &\quad - \int_t^T \hat{\beta}(y_s)\lambda_x(s) e^{-\int_t^s \lambda_x(u) y_u du} ds | \tilde{S}_t = S] \end{aligned} \tag{73}$$

where the expectation $\tilde{E}$ is taken with respect to the risk neutral measure. Using the Dynamic Programming Principle and Itô's formula as in the derivation of (16) in Section 3, one can show that (71) is the HJB associated with the value function above. As the derivation is similar to the derivation of (16) from (5) in Section 3, we omit it. Moreover, the Verification Theorem ensures that a solution η of (71) coincides with this value function. Finally, from (27), we have that

$$\begin{aligned} P(S,t) &= e^{-r(T-t)} \sup_{\{y_s\}} \tilde{E}[g(\tilde{S}_T) e^{-\int_t^T \lambda_x(s) y_s ds} \\ &\quad - \frac{1}{\alpha} \int_t^T \hat{\beta}(y_s)\lambda_x(s) e^{-\int_t^s \lambda_x(u) y_u du} ds | \tilde{S}_t = S]. \end{aligned} \tag{74}$$

The first term in this expression is similar to the pricing formula when α equals zero, the Brennan-Schwarz price. The control y_s acts as a multiplier on the force of mortality $\lambda_x(s)$. In other words, $y_s\lambda_x(s)$ is a controlled force of mortality. Thus, we can think of this first term as a type of actuarial present value in a proportional hazards rate model, [21]. The second term acts as a penalty to the first in computing the supremum, so we have expressed the premium as a type of penalized Brennan and Schwarz price with a penalty that depends on the risk aversion of the insurer.

7 Conclusion

By using the principle of equivalent utility we showed that, under the assumption of exponential utility, the indifference price P of the endowment contract is governed by a nonlinear Black-Scholes equation. We examined the qualitative and quantitative behavior of P and demonstrated that in special cases ($\alpha = 0$ and $\lambda = 0$), our premium reduces to the Brennan and Schwartz and Black-Scholes prices, respectively, and that, under reasonable mortality assumptions, these prices yield tight lower and upper bounds on P. Moreover, we derived an expression for P as a controlled expectation with respect to the risk neutral measure.

Several interesting questions remain open. In this paper, the benefit level was determined by the stock price at the horizon T. In future work, we will consider contracts whose payouts are similar to Asian or look-back options, for which the benefit levels are tied to the index performance over the life of the contracts. In addition, the assumptions of constant volatility and risk free rate are an oversimplification; given the long term nature of these contracts, we should extend our model to incorporate stochastic volatility and risk free rate as in [9], [13], and [23]. Moreover, one might model the stock price evolution as a jump-diffusion process. Finally, we will examine the application of utility methods in determining participation rates and minimum guarantees on the type of contract considered in [19], where the interest accrual, not the benefit level, is driven by the index performance.

Appendix 1

In this section, we present the details of the results in (20).

Recall that the value function V solves the PDE

$$\left\{ \begin{array}{c} V_t - \frac{(\mu - r)^2}{2\sigma^2} \frac{V_w^2}{V_{ww}} + rwV_w = 0 \\ V(w,T) = u(w) \end{array} \right\}. \tag{75}$$

As value functions often inherit the structure of the underlying utility, we conjecture that V is of the form

$$V(w,t) = -\frac{1}{\alpha} e^{f(t)w + h(t)}. \tag{76}$$

The condition at $t = T$ in (75) dictates that

$$f(T) = -\alpha \quad \text{and} \quad h(T) = 0. \tag{77}$$

Computing derivatives, we have

$$V_t = V(wf' + h'), \quad V_w = Vf, \quad V_{ww} = Vf^2, \quad \text{and} \quad \tfrac{V_w^2}{V_{ww}} = V. \tag{78}$$

Plugging into (75) and rearranging terms yields

$$w(f' + rf) + h' - \frac{(\mu - r)^2}{2\sigma^2} = 0. \tag{79}$$

To ensure that f and h are independent of w, as conjectured, we require that

$$f' + rf = 0 \quad \text{and} \quad h' - \tfrac{(\mu - r)^2}{2\sigma^2} = 0. \tag{80}$$

Solving the differential equations above subject to the conditions in (77) yields

$$f(t) = -\alpha e^{r(T-t)} \quad \text{and} \quad h(t) = -\tfrac{(\mu-r)^2}{2\sigma^2}(T-t). \tag{81}$$

Finally, from (76) we have that

$$V(w,t) = -\frac{1}{\alpha}\exp(-\alpha w e^{r(T-t)})\exp\left(-\frac{(\mu-r)^2}{2\sigma^2}(T-t)\right). \tag{82}$$

Appendix 2

In this section, we present the details of the derivation of the PDE (23). For convenience, we remind the reader that the value function U solves the PDE

$$\left\{ \begin{array}{c} U_t - \dfrac{[(\mu - r)U_w + \sigma^2 S U_{wS}]^2}{2\sigma^2 U_{ww}} + rwU_w + \mu S U_S + \frac{1}{2}\sigma^2 S^2 U_{SS} + \lambda_x(t)(V - U) = 0 \\ U(w,S,T) = u(w - g(S)) \end{array} \right\} \tag{83}$$

Because value functions often inherit the structure of the underlying utility function, we conjecture that our solution to (18) is of the form

$$U(w,S,t) = V(w,t)e^{\eta(S,t)}. \tag{84}$$

Computing the appropriate derivatives of U yields

$$\begin{array}{ll} U_t = e^\eta(V\eta_t + V_t) & U_S = Ve^\eta \eta_S \\ U_w = V_w e^\eta & U_{SS} = Ve^\eta(\eta_{SS} + \eta_S^2) \\ U_{ww} = V_{ww}e^\eta & U_{wS} = V_w e^\eta \eta_S. \end{array} \tag{85}$$

Plugging these quantities in to (83), we have

$$\left\{ \begin{array}{c} Ve^\eta\eta_t + e^\eta V_t - \frac{[(\mu-r)V_w e^\eta + \sigma^2 S V_w e^\eta \eta_S]^2}{2\sigma^2 V_{ww} e^\eta} + rwV_w e^\eta \\ +\mu S V e^\eta \eta_S + \frac{1}{2}\sigma^2 S^2 V e^\eta[\eta_{SS} + \eta_S^2] + \lambda_x(t)V[1 - e^\eta] = 0 \end{array} \right\} \tag{86}$$

Expanding the numerator in the fraction above, dividing both sides by e^η, and exploiting the fact that V solves (75), we have

$$\begin{array}{c} V\eta_t + \mu S V \eta_S + \frac{1}{2}\sigma^2 S^2 V[\eta_{SS} + \eta_S^2] \\ +\lambda_x(t)V[e^{-\eta} - 1] - \frac{V_w^2}{V_{ww}}[(\mu - r)S\eta_S + \frac{1}{2}\sigma^2 S^2 \eta_S^2] = 0 \end{array} \tag{87}$$

Now, recall from (78) in Appendix 1 that $\frac{V_w^2}{V_{ww}} = V$. Dividing both sides of the equation by V and simplifying yields

$$\eta_t + rS\eta_S + \frac{1}{2}\sigma^2 S^2 \eta_{SS} + \lambda_x(t)(e^{-\eta} - 1) = 0, \tag{88}$$

as desired. Note also that (84) and the conditions at $t = T$ in (75) and (83) dictate that

$$\eta(S,T) = \alpha g(s); \tag{89}$$

thus, η is indeed independent of w, as conjectured.

In the case of power utility (CRRA) or more general utility functions, it is tempting to employ the same methods, i.e., to conjecture that the solution U to 83 is of the form $U(w,S,t) = f(S,t)V(w,t)$. For power utility, one can derive a semi-linear PDE for f that is independent of wealth. However, the condition at $t = T$ is $f(S,T) = \frac{u(w-g(S))}{u(w)}$; thus for power utility, $f(S,T)$ depends on wealth and therefore contradicts our conjecture that f is independent of wealth. Thus, we do not obtain a more tractable PDE as we did in the exponential utility case. For power and more general utility, one must solve the prohibitively complex fully nonlinear, partially coupled PDE system (75), (83) and extract the argument P for which $U(w+P,S,t) = V(w,t)$.

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