The Riemann Hypothesis: Arithmetic and Geometry

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ABSTRACT

This paper describes basic properties of the Riemann zeta function and its generalizations, indicates some of geometric analogies, and presents various formulations of the Riemann hypothesis. It briefly discusses the approach of A. Connes to a "spectral" interpretation of the Riemann zeros via noncommuative geometry, which is treated in detail by Paula Tretkoff [33].

Contents

1 Introduction

The origin of the Riemann hypothesis was as an arithmetic question concerning the asymptotic distribution of prime numbers. In the last century profound geometric analogues were discovered, and some of them proved. In particular there are striking analogies in the subject of spectral geometry, which is the study of global geometric properties of a manifold encoded in the eigenvalues of various geometrically natural operators acting on functions on the manifold.

This has led to the search for a "geometric" and/or "spectral" interpretation of the zeros of the Riemann zeta function.

One should note that a geometric or spectral interpretation of the zeta zeros by itself is not enough to prove the Riemann hypothesis; the essence of the problem seems to lie in a suitable "positivity property" which must be established. A hope is that there exists such an interpretation in which the positivity will be a natural (and provable) consequence of the internal structure of the "geometric" object.

2 Basics

The Riemann zeta function is an analytic device that encodes information about the ring of integers $\mathbb{Z}$. In particular, it relates to the multiplicative action of $\mathbb{Z}$ on the additive group $\mathbb{Z}$. In its most elementary form, the Riemann zeta function can be defined by the well-known series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

where the domain of convergence is the half-plane $\{s : \Re s > 1\}$. This series was studied well before Riemann, and in particular Euler observed that it can be rewritten in the product form

$$\begin{aligned} \zeta(s) &= \prod_{p \text{ prime}} (1 + p^{-s} + p^{-2s} + \ldots) \\ &= \prod_{p \text{ prime}} (1 - p^{-s})^{-1}. \end{aligned}$$

The zeta function can be extended to a meromorphic function on the entire complex plane. More specifically, if we define the completed zeta function $\hat{\zeta}(s)$ by

$$\hat{\zeta}(s) := \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s),$$

then we have the following.

Theorem 2.1. *The completed zeta function $\hat{\zeta}$ has an analytic continuation to the entire complex plane except for simple poles at $s = 0, 1$. Furthermore, this function $\hat{\zeta}$ satisfies the functional equation*

$$\hat{\zeta}(s) = \hat{\zeta}(1 - s).$$

Proof. With a suitable change of variables, the integral definition of Γ gives

$$\Gamma(\frac{s}{2}) = n^s \pi^{\frac{s}{2}} \int_0^{\infty} e^{-\pi n^2 x} x^{\frac{s}{2}-1} dx, \quad (1)$$

for every $n \in \mathbb{Z}^+$. Rearranging (1) and summing over $n \in \mathbb{Z}^+$, one can show that for all $s \in \mathbb{C}$ with $\Re s > 1$,

$$\begin{aligned} \hat{\zeta}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) &= \int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 x} x^{\frac{s}{2}-1} dx \\ &= \frac{1}{2}\int_0^{\infty} (\theta(x) - 1) x^{\frac{s}{2}} \frac{dx}{x}, \quad (2) \end{aligned}$$

where $\theta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}$. This θ-function satisfies the functional equation

$$\theta(x^{-1}) = \sqrt{x}\theta(x).$$

Now, the integral in (2) can be split as

$$\int_0^\infty (\theta(x)-1)x^{\frac{s}{2}}\frac{dx}{x} \;=\; \int_0^1 (\theta(x)-1)x^{\frac{s}{2}}\frac{dx}{x} \;+\; \int_1^\infty (\theta(x)-1)x^{\frac{s}{2}}\frac{dx}{x}.$$

Applying the change of variables $x \mapsto x^{-1}$ in the first of these, we obtain

$$\hat{\zeta}(s) = \frac{1}{s(s-1)} + \frac{1}{2}\int_1^\infty (\theta(x)-1)(x^{\frac{1-s}{2}} + x^{\frac{s}{2}})\frac{dx}{x}. \tag{3}$$

This integral is uniformly convergent on $\{s : \Re s > \sigma\}$ for any $\sigma \in \mathbb{R}$, and thus is an entire function of s. Therefore, (3) exhibits the meromorphic continuation of $\hat{\zeta}$, and it clearly satisfies the functional equation.

□

It is now natural to define the entire function

$$\xi(s) = \frac{1}{2}s(s-1)\hat{\zeta}(s).$$

The factor of $\frac{1}{2}$ here was introduced by Riemann and has stuck. Hadamard showed that ξ has the product expansion

$$\xi(s) = \prod_\rho (1 - \frac{s}{\rho})e^{\frac{s}{\rho}},$$

where the product is over the zeros of ξ.

The location of the zeros of ξ is of great importance in number theoretic applications of the zeta function. Euler's product formula easily shows that every zero ρ has $\Re(\rho) \leq 1$, and the functional equation then gives that all zeros lie in the closed strip $\{s : 0 \leq \Re(\rho) \leq 1\}$. In fact, it can be shown that all zeros lie within the open strip $\{s : 0 < \Re(\rho) < 1\}$, although this is a non-trivial result.

Since $\xi(s)$ is real-valued for real values of s, it is clear that we have

$$\xi(\bar{s}) = \overline{\xi(s)}.$$

Thus if ρ is a zero of ξ, so are $\bar{\rho}$, $1-\rho$ and $1-\bar{\rho}$. Consequently, zeros on the line $\Re(s) = \frac{1}{2}$ occur in conjugate pairs, and zeros off this line occur in quadruples.

The Riemann hypothesis is now stated simply as follows.

Conjecture. *All zeros of $\xi(s)$ lie on the line $\Re(s) = \frac{1}{2}$.*

Riemann confirmed the position of many of the zeros of $\xi(s)$ to be on this critical line by hand, by making use of the symmetry from the functional equation. For if the approximate location of a zero close to the critical line is known, one can consider a small contour C around the zero which is symmetric about the critical line. By estimating the integral

$$\frac{1}{2\pi \mathrm{i}}\int_C \frac{\xi'(s)}{\xi(s)}ds,$$

one can determine the number of zeros (including multiplicity) enclosed within the curve C. If only one such zero exists, symmetry dictates that it must lie on the critical line. To date, no double zeros have been found on the critical line.

The Riemann hypothesis can be reformulated in a number theoretic context as follows. If we define

$$\pi(x) = \sum_{\substack{p \leq x \\ p \text{ prime}}} 1$$

as usual, then the Riemann hypothesis is known to be equivalent to the veracity of the following error term on the Prime Number Theorem:

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(x^{\frac{1}{2}}(\log x)^2).$$

Note that it is a theorem that

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(x \exp(-2\sqrt{\log x})).$$

Although Riemann's zeta function was the original object of interest, it is only one of a much larger set of "L-functions" with similar properties. These functions arise in many applications, and a natural generalizations of the Riemann hypothesis appears to hold for all of them as well. For example:

- Dirichlet L-functions:

  $$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s},$$

  where $\chi : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$ is a character on $(\mathbb{Z}/q\mathbb{Z})^\times$. The character χ is extended to a q-periodic function on $\mathbb{Z}$ where we set $\chi(n) = 0$ for all n with $\gcd(n, q) \neq 1$.

- L-functions of cuspidal automorphic representations of $GL(N)$, cf. [24], [16]. In the case of $GL(2)$ this includes such exotic objects as L-functions attached to Maass cusp forms.

The latter generalize to the "$GL(N)$ case" the "$GL(1)$ case" of the multiplicative group of a field acting on the additive group, which are just the Dirichlet L-functions. The resulting L-functions all have a Dirichlet series representation, which converges for $\Re(s) > 1$. When multiplied by appropriate Gamma-function factors and exponentials one obtains a "completed L-function", which analytically continues to $\mathbb{C}$, except for possible poles at $s = 0, 1$ and which satisies a functional equation relating values at s to values at $1 - s$ of another such L-function. The generalized Riemann hypothesis asserts that all zeros of such L-functions lie on the line $\Re(s) = 1/2$.

This generalization appears to be the most natural context in which to study the Riemann hypothesis. In fact, from a number theoretic point of view, the Riemann zeta function cannot really be segregated from the above generalizations. It seems plausible that a proof of the original Riemann hypothesis will not be found without proving it in these more general circumstances.

At this point, it is also worth noting that much can be achieved in practical situations without the specificity of a proof of the Riemann hypothesis for particular cases. Many results, originally proven under the assumption of some generalized Riemann hypothesis, have more recently been fully proven by using results describing the behaviour of the Riemann hypothesis "on average" across certain families of L-functions. Two such examples are:

- Vinogradov:

  Every sufficiently large odd number can be written as a sum of three primes (a relative of Goldbach's conjecture).

- Cogdell, Piatetskii-Shapiro, Sarnak:

  Hilbert's eleventh problem. Given a quadratic form F over a number field K, which elements of K are represented as values of F?

3 The Explicit Formula

Riemann's original memoir included a formula relating zeros of the zeta function to prime numbers. Early classical forms of the " explicit formula" of prime number theory were found by Guinand [19] and [20]. However it was A. Weil [35], [36], [37], who put the "explicit formula" in a elegant form that connects the arithmetic context of the Riemann hypothesis with objects that appear geometric in nature. This type of connection is central to most of the modern approaches to the Riemann hypothesis.

To state the explicit formula, we require the Mellin transform. For a function $f : (0, \infty) \to \mathbb{C}$, the Mellin transform $\mathcal{M}[f]$ of f is defined by

$$\mathcal{M}[f](s) = \int_0^\infty f(x) x^s \frac{dx}{x} \qquad (s \in \mathbb{C}).$$

This is the Fourier transform on the multiplicative group $\mathbb{R}_{>0}$; if we put $g(u) = f(e^u)$, we see that

$$\begin{aligned} \mathcal{M}[f](s) &= \int_{-\infty}^\infty f(e^u) e^{us} \frac{d(e^u)}{e^u} \\ &= \int_{-\infty}^\infty g(u) e^{-iu(is)} du = \hat{g}(\mathrm{i}s), \end{aligned}$$

where $\hat{g}$ is the Fourier transform of g on the additive group $\mathbb{R}$.

The convolution operation associated with the Mellin transform is

$$f * g(x) = \int_0^\infty f(\frac{x}{y}) g(y) \frac{dy}{y},$$

so that

$$\mathcal{M}[f * g](s) = \mathcal{M}f(s)\mathcal{M}[g](s).$$

We also have an involution

$$\tilde{f}(x) = \frac{1}{x} f(\frac{1}{x}),$$

giving

$$\mathcal{M}[\tilde{f}](s) = \mathcal{M}[f](1-s).$$

The "explicit formula" is a family of assertions, for a set of "test functions". We consider the family of *nice test functions* to consist of $f : (0, \infty) \to \mathbb{C}$ such that f is piecewise C^2, compactly supported and has the averaging property at discontinuities:

$$f(x) = \frac{1}{2}[\lim_{t\to x^+} f(t) + \lim_{t\to x^-} f(t)].$$

The "spectral side" $W_{spec}(f)$ of the explicit formula consists of three terms

$$W_{spec}(f) := W^{(2)}(f) - W^{(1)}(f) + W^{(0)}(f),$$

in which

$$\begin{aligned} W^{(2)}(f) &= \mathcal{M}[f](1), \\ W^{(1)}(f) &= \sum_{\rho \text{zeros of } \xi} \mathcal{M}[f](\rho). \\ W^{(0)}(f) &= \mathcal{M}f(0). \end{aligned}$$

The "arithmetic" side of the explicit formula consists of terms corresponding to the finite primes p plus the "infinite prime" (the real place),

$$W_{arith}(f) := W_\infty(f) + \sum_{p \quad prime} W_p(f)$$

in which

$$W_p(f) := \log p \left(\sum_{n=1}^{\infty} f(p^n) + \tilde{f}(p^n) \right),$$

and for $p = \infty$,

$$W_\infty(f) := (\gamma + \log p) f(1) + \int_1^\infty [f(x) + \tilde{f}(x) - \frac{2}{x^2} f(1)] \frac{x dx}{x^2 - 1}.$$

Theorem 3.1 (Explicit Formula)**.** *For any nice test function $f : (0, \infty) \to \mathbb{C}$ there holds*

$$W_{spec}(f) = W_{arith}(f).$$

The "explicit formula" has a formal resemblance to a fixed point formula of Atiyah-Bott-Lefschetz type; Here the "spectral side" has the form of a generalized Euler characteristic, in which the term $W^{(j)}(f)$ should measure the contribution of the trace of an operator on j-th cohomology group of an unknown object, while the "arithmetic side" would be viewed as contributions coming from the fixed points of an map on an unknown object. This resemblance has been noted by many authors, starting with Andre Weil, whose first proof of the Riemann hypothesis in the one-variable function field case was based on exactly this interpretation.

The statement of the explicit formula takes on the form

"spectral term" = "arithmetic term".

Note that the "spectral" side of the "explicit formula" is expressed in terms of the Mellin transform of f, while the terms in the "arithmetic" side are expressed directly in terms of values of f; the proof below shows that the "arithmetic" terms do have an expression in terms of the Mellin transform of f.

Using the explicit formula, Weil was able to reformulate the Riemann hypothesis as a positivity statement.

Theorem 3.2 (Weil's Positivity Statement). *The Riemann hypothesis is equivalent to*

$$W^{(1)}(f * \tilde{\bar{f}}) \geq 0,$$

for all nice test functions f.

Remark 3.3. For two "nice" functions f and g, we can define the intersection product

$$\langle f_1, f_2 \rangle := W^{(1)}(f_1 * \tilde{\bar{f}}_2).$$

The conjectural Castelnovo inequality states that

$$\hat{f}(0)\hat{f}(1) \geq \frac{1}{2} W_{spec}(f * \tilde{\bar{f}}).$$

This connects with Weil's positivity statement above.

The "explicit formula" was originally given by Weil [35] in terms of the Fourier transform; the Mellin transform version given here can be found in Patterson [30], whos proves it for a wide class of test functions; one needs the test functions to have Mellin transforms $\mathcal{M}[f](s)$ that are holomorphic in a region $-\epsilon < \Re(s) < 1 + \epsilon$. A proof for a certain explicit set of test functions is given in [5], [4]. There are a number of proofs of the "explicit formula", all based on similar ideas, which we indicate below.

Proof sketch of explicit formula. Consider the logarithmic derivative of the completed zeta function $\hat{\zeta}$,

$$\frac{\hat{\zeta}'(s)}{\hat{\zeta}(s)} = \frac{d}{ds}[\log \hat{\zeta}(s)].$$

We assume the function $\mathcal{M}[f](s)$ extends to an analytic function in the closed strip $-\epsilon < \Re(s) < 1 + \epsilon$ and has rapid enough decay vertically. We evaluate in two ways the contour integral

$$-\frac{1}{2\pi \mathrm{i}} \int_{\square_T} \mathcal{M}[f](s) \frac{\hat{\zeta}'(s)}{\hat{\zeta}(s)} ds, \tag{4}$$

around a closed box $\square_T$ on the vertical lines $\Re(s) = 1 + \frac{1}{2}\epsilon$, and $\Re(s) = -\frac{1}{2}\epsilon$, going from height $-iT$ to height $+iT$, oriented counterclockwise, and then letting the height of the box $T \to \infty$. Firstly, taking the logarithm of the Hadamard factorization for $\xi(s)$ gives

$$\log \hat{\zeta}(s) = \log 2 - \log s - \log(s-1) + \sum_{\rho \text{ zeros of } \xi}{}' \left(\log(1 - \frac{s}{\rho}) \right),$$

where the prime indicates the zeros must be summed in pairs $\rho, 1-\rho$. Differentiating,

$$\frac{d}{ds}[\log \hat{\zeta}(s)] = -\frac{1}{s} - \frac{1}{s-1} + \sum_{\rho \text{ zeros of } \xi}{}' \frac{1}{s-\rho}.$$

Adding up the residues of the poles these contribute in the box (as $T \to \infty$) gives the geometric term; the terms $W^{(0)}(f)$ and $W^{(2)}(f)$ come from the poles of $\hat{\zeta}(s)$ at $s = 0$ and $s = 1$, respectively.

Secondly, the Euler product form gives

$$\frac{\hat{\zeta}'(s)}{\hat{\zeta}(s)} = \frac{d}{ds}\left(\log \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) - \sum_{p \text{ a prime}} \log(1 - p^{-s})\right).$$

The derivative of the sum is

$$-\frac{1}{2}\log \pi + \frac{1}{2}\frac{\Gamma'(\frac{s}{2})}{\Gamma(\frac{s}{2})} - \sum_{p \text{ a prime}} \frac{(\log p)p^{-s}}{1 - p^{-s}}.$$

This is substituted in the right vertical side of the box integral and evaluated for each term separately. The integral, with $\Re(s) > 1$, evaluates to

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{M}[f](s)\left(\sum_{n=1}^{\infty}(\log p)p^{-ns}\right) ds = (\log p)\sum_{n=1}^{\infty} f(p^n),$$

since each term separately is an inverse Mellin transform. For the left vertical side integral with $\Re(s) < 0$, we use the functional equation and obtain similarly the contribution $(\log p)\sum_{n=1}^{\infty} \tilde{f}(p^n)$. The horizontal sides contribute zero in the limit $T \to \infty$.

The contour integral for the Gamma function term requires delicate care to convert the answer to the form for $W_\infty(f)$ given above; see [5], [4], [30]. $\square$

For spectral and trace formula interpretations of the "explicit formula", see Goldfeld [17], [18] Haran [21], [22], Hejhal [23], as well as the recent work of Connes [9], [10], [11]. A number of other interesting viewpoints on the "explicit formula" appear in Burnol [6] and Deninger [13].

4 The Function Field Case

We now consider the Riemann hypothesis for function fields over finite fields, or equivalently, for zeta functions attached to complete nonsingular projective varieties. For function fields of one variable the Riemann hypothesis was formulated in E. Artin's 1923 thesis, in analogy with the number field case. It was proved for genus one function fields by H. Hasse in 1931, and It was then proved for all one-variable function fields by A. Weil in the 1940's. Weil's key idea was to introduce an underlying geometric object—a projective variety—which allows the translation of the problem to a problem in algebraic geometry. Finally in 1973 Deligne proved the Riemann hypothesis for the zeta functions of complete nonsingular projective varieties of any dimension.

Let $\mathbb{F}_q$ be the finite field with $q = p^k$ elements for some prime p and some $k \in \mathbb{Z}^+$, and let K be a function field in one variable T over $\mathbb{F}_q$. Let O_K denote the ring of integers of K. We exclude one prime from O_K which we define to be the "prime at infinity". For instance, if $K = \mathbb{F}_q(T)$, then we can let $O_K = \mathbb{F}_q[T]$, the ring of polynomials in T, where the prime $\frac{1}{T}$ is excluded as the prime at infinity.

We now define a zeta function for K by

$$\zeta_K(s) = \sum_{I \in I_K} (N(I))^{-s},$$

where I_K denotes the set of ideals contained in the ring of integers O_K, and

$N(I) = \#(O_K/I)$ is the norm of I. In our example above, all ideals $I = (f)$ are principal, generated by a monic polynomial $f(T)$, with norm

$$N((f)) = q^{-\deg f}.$$

Therefore we have

$$\begin{aligned} \zeta_K(s) &= \sum_{\substack{f \text{ monic} \\ \text{polynomials over } \mathbb{F}_q}} q^{-(\deg f)s} \\ &= \sum_{n=1}^{\infty} q^n q^{-ns}, \end{aligned}$$

since there are q^n monic polynomials of degree n over $\mathbb{F}_q$. Thus,

$$\zeta_K(s) = \frac{1}{1 - q^{1-s}}.$$

We complete this to a function $\hat{\zeta}_K(s)$ by including a term corresponding to the prime at infinity. In the present example, we obtain

$$\hat{\zeta}_K(s) = \left(\frac{1}{1 - q^{1-s}}\right)\left(\frac{1}{1 - q^{-s}}\right).$$

Now $\hat{\zeta}(s)$ satisfies the functional equation

$$q^{-s}\hat{\zeta}_K(s) = q^{-(1-s)}\hat{\zeta}_K(1-s),$$

Here the additional non-vanishing factor q^{-s} plays the role of a "conductor," analogous to the conductor term appearing in the functional equation of a Dirichlet L-function.

Weil made several celebrated conjectures about these zeta functions, all of which are now proven. A major implication of the Weil conjectures is that $\hat{\zeta}(s)$ can be expressed in terms of L-functions arising from the structure of an underlying geometric object, namely a non-singular projective variety V having K as a function field. Essentially, $\hat{\zeta}_K$ can be written as a quotient of L-functions arising from the cohomology of V:

$$\hat{\zeta}_K(s) = \frac{L(s, H^1)}{L(s, H^0)L(s, H^2)}.$$

For instance, consider the line

$$X_1 + X_2 = 1$$

over $\mathbb{F}_q$. We projectivize to obtain

$$V(\mathbb{F}_q) = \mathbb{P}^1(\mathbb{F}_q) \;:\; X_1 + X_2 = X_0,$$

with $(X_1, X_2, X_3) \neq (0,0,0)$, subject to the usual equivalence relation

$$(\lambda X_1, \lambda X_2, \lambda X_3) \sim (X_1, X_2, X_3), \text{ for all } \lambda \in \mathbb{F}_q^{\times}.$$

Thus V consists of q points of the form $(x, 1-x, 1)$ for $x \in \mathbb{F}_q$, plus the point at infinity $(1, -1, 0)$. We now extend this space to the projective space $V(\overline{\mathbb{F}_q})$ over the algebraic closure $\overline{\mathbb{F}_q}$ of $\mathbb{F}_q$.

Such a projective variety has a natural dynamical system induced by the Frobenius automorphism,

$$\begin{array}{rcl} \mathrm{Frob} : V & \longrightarrow & V \\ (x_1, x_2, x_3) & \longmapsto & (x_1^q, x_2^q, x_3^q). \end{array}$$

Associated to this dynamical system is a dynamical zeta-function, defined by

$$\hat{\zeta}_{\mathrm{dyn}}(T) = \exp\left(\sum_{n=1}^{\infty} \frac{T^n}{n} \#\mathrm{Fix}(\mathrm{Frob}^n)\right),$$

where $\#\mathrm{Fix}(\mathrm{Frob}^n)$ is the number of fixed points of Frob^n. For instance, the example above yields

$$\begin{aligned} \hat{\zeta}_{\mathrm{dyn}}(T) &= \exp\left(\sum_{n=1}^{\infty} \frac{T^n}{n}(q^n+1)\right) \\ &= \exp\left(-\log(1-Tq) - \log(1-q)\right) \\ &= \frac{1}{(1-q)(1-Tq)}. \end{aligned}$$

Therefore, putting $T = q^{-s}$, we get

$$\hat{\zeta}_{\mathrm{arith}}(s) = \hat{\zeta}_{\mathrm{dyn}}(q^{-s}).$$

This connection is remarkable. As a heuristic analogy, a similar situation arises in statistical mechanics. Associated to a one-dimensional system, Ruelle defined a (two-variable) statistical mechanics zeta function by

$$\zeta(s, T) = \exp\left(\sum_{n=1}^{\infty} \frac{T^n}{n} p_n(s)\right).$$

Here, $p_n(s)$ is the partition function of a finite system Σ_n of "size" n , given by

$$p_n(s) = \sum_{\sigma \in \Sigma_n \text{ states}} e^{-sH(\sigma)},$$

where H is the Hamiltonian function. Here Σ_n could represent a system on the line with periodic boundary conditions of period n.

An analogous result in statistical mechanics to the number theoretic statement above is the following, cf [26, Theorem 3.1].

Theorem 4.1. *For "uniformly expanding maps" on* $[0,1]$,

$$\zeta(s, 1) = \zeta(0, \beta^{-s}),$$

where $\beta = \exp(\text{entropy})$.

Here a "uniformly expanding map" $f : [0,1] \to [0,1]$ is a (possibly discontinuous) C^1- map all of whose pieces are linear with slopes $\pm\beta$ with $\beta > 1$; an example is $f(x) = \beta x \pmod 1$. For the function field zeta function above, the Frobenius automorphism acts like a uniformly expanding map with entropy $\log q$.

Can Weil's ideas be extended to the number field case of the Riemann hypothesis? This suggests, in particular, three questions. Firstly, what is the "geometrical" or "dynamical" zeta function which should be considered in the number field case? Secondly, how is this geometrical object related to the arithmetic zeta function? Thirdly, what property in the number field case is the analogue of the Castelnuovo positivity property that provides a "geometric" explanation of the truth of the Riemann hypothesis in the function field case? For possible ideas in these directions, see Connes [9], [10], [11], and Deninger [12], [14].

5 The Number Field Case and Non-commutative Geometry

Polya and Hilbert postulated the following idea as a possible approach to the Riemann hypothesis. Suppose some geometric considerations can lead to the construction of a Hilbert space $\mathcal{H}$ and an unbounded operator D, such that

$$\text{Spectrum}(D) = \{\rho : \xi(\rho) = 0\}.$$

One might then hope to be able to understand the location of the zeros using operator theory—ideally, by showing that

$$(D - \frac{1}{2})^* = -(D - \frac{1}{2}).$$

This hope is plausible, at least in philosophy, for several reasons. First, there is some analogy with the work of Selberg. Selberg's work considered the Laplace operator, which has form

$$\Delta = (D - \frac{1}{2})^2.$$

The behaviour of "primes" (prime geodesics) is related to the spectrum of this operator via Selberg's trace formula. Second, work of Montgomery indicates that the distribution of the zeros of $\xi(s)$ compares well with results on the distribution of eigenvalues of random matrices, cf. [2], [25]. This has been strikingly supported by numerical computations of Odlyzko [29]. This suggests that spectral considerations lie beneath the theory of the zeta function.

Connes' recent idea is that the philosophy of Polya and Hilbert might be realized using non-commutative geometry. In the function field case above, a space is produced from the action of the Frobenius automorphism on the underlying variety V. Spectral methods in this context yield Weil's proof of the Riemann Hypothesis for function fields. In the number field case, Connes' proposal is that appropriate space is generated by the action of the multiplicative group $k^\times$ of the number field on the adèle space[1] A. The space $A/k^\times$ is extremely badly behaved from the classical point of view, but the hope is that it

[1] More generally, we should be looking at the action of $\mathrm{GL}_n(k)$ on $M_n(A)$.

may be handled effectively as a non-commutative space. The development of this idea is the focus of Paula Tretkoff's paper [33].

6 Equivalent Forms of the Riemann Hypothesis

To conclude this survey, we present four equivalents of the Riemann hypothesis. These demonstrate connections of the Riemann hypothesis with other areas of mathematics, and some of them have a geometric flavor.

6.1 Ergodicity of Horocycle Flows

Consider the group $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ acting on the hyperbolic plane $\mathbb{H}$ in the familiar way, so that the group is generated by the isometries

$$z \mapsto z+1$$

$$\text{and} \qquad z \mapsto -\frac{1}{z}$$

of the upper half-plane model of $\mathbb{H}$. Let us denote by h_t the horocycle in the upper half-plane having constant imaginary part $y = t$. We look at the projection of this horocycle onto the quotient space $\mathbb{H}/\Gamma$. Since h_t is invariant under the mapping $z \mapsto z+1$, this is a periodic horocycle, and we can restrict our attention to the segment of h_t lying within the vertical strip $\{z : 0 \le z \le 1\}$. Let γ_t denote the image of this segment in $\mathbb{H}/\Gamma$. The length of γ_t is $\frac{1}{t}$, and in particular, $\mathrm{length}(\gamma_t) \to \infty$ as $t \to 0$. Furthermore, γ_t satisfies the following ergodic property as $t \to 0$.

Theorem 6.1. *For any "nice" open set S in $\mathbb{H}/\Gamma$,*

$$\frac{\mathrm{length}(\gamma_t \cap S)}{\mathrm{length}(\gamma_t)} \longrightarrow \frac{\mathrm{vol}(S)}{\mathrm{vol}(\mathbb{H}/\Gamma)}$$

as $t \to 0$.

Here $\mathrm{vol}(\mathbb{H}/\Gamma) = \frac{\pi}{3}$. Here "nice" can be taken to be that the boundary $\partial(S) = \bar{S}\backslash S$ has finite 1-dimensional Hausdorff measure, see Verjovsky [34]. A connection on the rate of convergence to ergodicity was noted by Zagier [38, pp. 279–280]. The Riemann hypothesis is equivalent to the following bound on the rate of convergence of the above ([32, p. 738]); here a "smooth test function" is needed.

Theorem 6.2. *The Riemann hypothesis holds if and only if, for any "nice" test function $f \in C^\infty_{00}(S\mathbb{H}/\Gamma)$, where $S\mathbb{H}/\Gamma$ is the unit tangent bundle over $\mathbb{H}/\Gamma$, for $t \to 0$ there holds*

$$\frac{1}{t}\int_{\gamma_t} f(z)d\nu_t z = \frac{\int_{\mathbb{H}/\Gamma} f(z)d\mu z}{\mathrm{vol}(\mathbb{H}/\Gamma)} + O(t^{\frac{3}{4}+\epsilon}),$$

for any $\epsilon > 0$. Here ν_t is the arc-length measure on the horocycle at height t and μ is Poincare measure on $(S\mathbb{H}/\Gamma)$, which gives it volume $2\pi\mathrm{vol}(\mathbb{H}/\Gamma)$.

A subtlety in this criterion is that if a test function is used that is not sufficiently smooth, then slower rates of convergence can hold even if the Riemann hypothesis is valid. See Verjovsky [34] for an example involving the characteristic function of a open set S lifted to the unit tangent bundle.

6.2 Brownian motion

Gnedenko and Kolmogorov observed that the Riemann zeta function arises naturally in relation to Brownian motion, see [3]. Consider, for example, the case of "pinned Brownian motion" on $\mathbb{R}$. This is a standard Brownian motion on the line $B_t \in \mathbb{R}$, $t \geq 0$, started at $B_0 = 0$ and conditioned on the property $B_1 = 0$. Now let

$$Z = \max_{0 \leq t \leq 1} B_t - \min_{0 \leq t \leq 1} B_t$$

be the length of the range of B_t. Then the expectation of Z^s is known to be

$$E[Z^s] = \xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s).$$

In another Brownian system, we obtain the following equivalent of the Riemann hypothesis [1].

Theorem 6.3 (Balazard, Saias, Yor)**.** *Consider two-dimensional Brownian motion in the plane, starting at* $(0,0)$*. Let* $(\frac{1}{2}, W)$ *be the first point of contact with the line* $X = \frac{1}{2}$*. Then the Riemann hypothesis is equivalent to*

$$E\left[\log|\zeta(W)|\right] = 0.$$

This statement is actually a restatement of the following integral.

Theorem 6.4. *The Riemann hypothesis is equivalent to*

$$\int_0^\infty \frac{\log|\zeta(\frac{1}{2} + \mathrm{i}t)|}{\frac{1}{4} + t^2} dt \;=\; 0.$$

Note that it is known unconditionally ([1], [7, Theorem 1.5]) that

$$\frac{1}{2\pi}\int_{\Re(s)=1/2} \frac{\log|\zeta(s)|}{|s|^2}|ds| = \frac{1}{\pi}\int_0^\infty \frac{\log|\zeta(\frac{1}{2} + \mathrm{i}t)|}{\frac{1}{4} + t^2} dt \quad = \sum_{\substack{\rho \text{ zeros of } \xi \\ \Re\rho > \frac{1}{2}}} \log\frac{\rho}{1-\rho}.$$

6.3 Li's Positivity Criterion

Define for $n \geq 0$ the Li coefficient

$$\lambda_n := \frac{1}{(n-1)!}\frac{d^n}{ds^n}(s^n \log \xi(s)))|_{s=1}.$$

Note that these quantities are given at $s = 1$, which can be computed in the absolute convergence region $\Re(s) > 1$ of the Euler product, as a limit $s \to 1^+$. These coefficients have a power series interpretation as:

$$\frac{1}{(z-1)^2}\frac{\xi'(\frac{1}{1-z})}{\xi(\frac{1}{1-z})} = \sum_{n=0}^{\infty} \lambda_{n+1} z^n.$$

The Riemann hypothesis can be rephrased as the positivity of these coefficients ([28]).

Theorem 6.5 (Li)**.** *The Riemann hypothesis is equivalent to*

$$\lambda_n \geq 0 \quad \textit{for all} \quad n \geq 1.$$

In fact, this criterion is related to Weil's explicit formula, since it can be shown ([5]) that

$$\lambda_n = W^{(1)}(\phi_n * \tilde{\bar{\phi}}_n)$$

for a certain sequence of test functions (ϕ_n); this sequence of test functions falls outside the class of test functions considered in §3 but the "explicit formula" can be justified for them, in a slightly modified form.

6.4 An Elementary Formulation

Because the Riemann hypothesis is such a fundamental question, it seems appropriate to give a completely elementary statement of it. In Lagarias [27] it is shown that the following problem is equivalent to the Riemann hypothesis.

Problem. *Let* $H_n = \sum_{j=1}^{n} \frac{1}{j}$ *be the nth harmonic number, and let* $\sigma(n) = \sum_{d|n} d$ *be the sum of the divisors of* n*. Prove that for each* $n \geq 1$*,*

$$\sigma(n) \leq H_n + \exp(H_n)\log(H_n),$$

with equality only for $n = 1$*.*

This problem essentially encodes a necessary and sufficient condition for the Riemann hypothesis due to Guy Robin [31].

References

[1] M. Balazard, E. Saias, and M. Yor, Notes sur la fonction ζ de Riemann, **2**, Adv. Math. **143** (1999), 284–287.

[2] M. V. Berry and J. P. Keating, The Riemann zeros and eigenvalue asymptotics, SIAM Rev. **41** (1999), 236–266.

[3] P. Biane, J. Pitman and M. Yor, Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions, Bull. Amer. Math. Soc. **38** (2001), 435–465.

[4] E. Bombieri, Remarks on Weil's quadratic functional in the theory of prime numbers I, Rend. Math. Acad. Lincei, Ser IX, **11** (2000), 183–233.

[5] E. Bombieri and J. C. Lagarias, Complements to Li's criterion for the Riemann hypothesis, J. Number Theory **77** (1999), 274–287.

[6] J.-F. Burnol, Sur les formules explicites I. Analyse invariante, C. R. Acad. Sci. Paris Sér I. Math., **331** (2000), 423–428.

[7] J.-F. Burnol, An adelic causality problem related to abelian L-functions, J. Number Theory, **87** (2001), 253–269.

[8] J. Cogdell, On sums of three squares, J. Number Theory, Bordeaux **15** (2003) 33–44.

[9] A. Connes, Trace formula in noncommutative geometry and the zeros of the Riemann zeta function, Selecta Math. (N. S.) **5** (1999), 29–106.

[10] A. Connes, Trace formula on the adéle class space and Weil positivity, in: *Curent developments in mathematics, 1997*(Cambridge, MA), Intl. Press, Boston, MA 1999, pp. 5–64.

[11] A. Connes, Noncommutative geometry and the Riemann zeta function, in: *Mathematics: Frontiers and Perspectives*, Amer. Math. Soc., Providence 2000, pp. 35–54.

[12] C. Deninger, Evidence for a Cohomological Approach to Analytic Number Theory, Proc. First European Congress of Mathematicians, Vol. I (Paris 1992), 491–510.

[13] C. Deninger, Lefshetz trace formulas and explicit formulas in analytic number theory, J. reine Angew **441**(1993), 1–15.

[14] C. Deninger, Some analogies between number theory and dynamical systems on foliated spaces, Proc. Intl. Congress Math., Vol. I (Berlin 1998), Doc. Math. **1998**, Extra Vol. I, pp. 163–186.

[15] C. Deninger, On dynamical systems and their possible significance for arithmetic geometry, in: *Regulators in analysis, geometry and number theory*, Prog. Math. Vol. 171, Birkhäuser Boston, Boston 2000, pp. 29–87.

[16] S. Gelbart and S. D. Miller, Riemann's zeta function and beyond, Bull. Amer. Math. Soc. **41** (2004), 59–112.

[17] D. Goldfeld, Explicit formulae as trace formulae, in: *Number theory, trace formulas and discrete groups (Oslo, 1987)*, Academic Press: Boston 1989, pp. 281–288.

[18] D. Goldfeld, A spectral interpretation of Weil's explicit formula, in: *Explicit Formulas*. Lecture Notes in Math. No. 1593, Springer-Verlag: Berlin 1994, pp. 136–152.

[19] A. P. Guinand, Summation formulae and self-reciprocal functions (III), Quart. J. Math. **13** (1942), 30–39.

[20] A. P. Guinand, A summation formula in the theory of prime numbers, Proc. London Math. Soc. **50** (1948), 107–119.

[21] S. Haran, Index theory. potential theory and the Riemann hypothesis, in: *L-Functions and Arithmetic (Durham 1989)*, London Math. Soc. Tract, No. 153, Cambridge Univ. Press: Cambridge 1991, pp. 257–270.

[22] S. Haran, On Riemann's zeta function, in: *Dynamical, Spectral and Arithmetic Zeta Functions*, (M. L. Lapidus and M. van Frankenhuysen, Eds.), Contemp. Math. Vol. 290, AMS: Providence, RI 2002.

[23] D. Hejhal, The Selberg trace formula and the Riemann zeta function, Duke Math. J. **43** (1976), 41–82.

[24] H. Jacquet, Principal L-functions over GL(N), in: *Representation theory and automorphic forms (Edinburgh 1996),* Proc. Symp. Pure Math. Vol. 61, Amer. Math. Soc.: Providence 1997, pp. 321–329.

[25] N. Katz and P. Sarnak, Zeros of zeta functions and symmetry, Bull. Amer. Math. Soc. **36** (1999), 1–26.

[26] J. C. Lagarias, Number theory zeta functions and dynamical zeta functions, in *Spectral Problems in Geometry and Arithmetic (Iowa City, IA 1997)*,(T. Branson, Ed.), Contemp. Math., No. 237, 1999, pp. 45–86.

[27] J. C. Lagarias, An elementary problem equivalent to the Riemann hypothesis, Amer. Math. Monthly, **109** (2002), 534–543.

[28] Xian-Jin Li, The positivity of a sequence of numbers and the Riemann hypothesis, J. Number Theory **65** (1997), 325–333.

[29] A. M. Odlyzko, On the distribution of spacings between zeros of the zeta function, Math. Comp. **48** (1987), 273–308.

[30] S. J. Patterson, *An introduction to the theory of the Riemann Zzta function*, Cambridge University Press: Cambridge 1988.

[31] G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann, J. Math. Pures Appl. **63** (1984), 187–213.

[32] P. Sarnak, Asymptotic behavior of periodic orbits of the horocycle flow and Eisenstein series, Comm. Pure Appl. Math. **34** (1981), 719–739.

[33] P. B. Tretkoff, Noncommutative geometry and number theory, pp. 143-189 in: *Surveys in Noncommutative Geometry* (N. Higson and J. Roe, Eds.), Clay Mathematics Proceedings **6**, Amer. Math. Soc. : Providence, RI, Clay mathematics Institute, Cambridge, MA 2006.

[34] A. Verjovsky, Arithmetic geometry and dynamics in the unit tangent bundle of the modular orbifold, in: *Dynamical Systems (Santiago 1990)*, Pitman Res. Notes Math. No. 285 , Longman Sci. Tech., Harlow, 1993, pp. 263–298.

[35] A. Weil, Sur les "formules explicites" de la théorie des nombres premiers, Comm. Sem. Math. Univ. Lund 1952 (Marcel Riesz volume) Tom. Supp., 252–265. [Oeuvres Scientifiques, Vol II, pp. 48–61].

[36] A. Weil, Fonction zêta et distributions, Sem. Bourbaki No. 312, Juin 1966. [Oeuvres Scientifiques, Vol III, pp. 158–163].

[37] A. Weil, Sur les formules explicites de la théories des nombres, Izv. Akad. Nauk SSSR, Ser. Mat. **36** (1972), 3–18. [Oeuvres Scientifiques, Vol III, pp. 249–264].

[38] D. Zagier, Eisenstein series and the Riemann zeta function, in: *Automorphic forms, representations and arithmetic (Bombay 1979)*, Tata Inst. Fund. Res. Studies in Math., Vol. 10, Tata Inst. Fund. Res., Bombay 1981, pp. 275–301.