Math 676, Homework 4 (11-th hour)

(Here is one more problem. This problem is mandatory, one of the five problems to turn in.)

11. [S-Unit Theorem] Let K be a number field, and let S be a finite set of prime ideals in O_K. Call an integer α in O_K an S-unit if it is a unit or else its ideal factorization (α) involves only prime ideals in S, and denote the multiplicatively closed set of all S-units by $U_S^\times$. Form the localization $O_{K,S} = (U_S^\times)^{-1} O_K$.

(a) Show that $O_{K,S}$ is a Dedekind domain.

(b) Show that the multiplicative group $O_{K,S}^\times$ has at least $r + s - 1 + |S|$ independent generators.

(c) Show that the multiplicative group $O_{K,S}^\times$ is a direct product of a torsion part $(O_{K,S})_{tors} = (O_K)_{tors}$ which are the roots of unity in K and a free abelian part with structure $\mathbb{Z}^{r+s-1+|S|}$.

[Hint: Enlarge the logarithm map to go to $O_{K,S} \to \mathbb{R}^{r+s+|S|}$ by adding some extra coordinates of type $x_k(\alpha) := -ord_P(i(\alpha)) \log |N_{K/\mathbb{Q}}(P)|$ for $P \in S$. Show that the embedding is additive and discrete. Show that the image of the map fall in a hyperplane, to be determined.]