Math 676, Homework 5

Do 5 problems of set of 11 problems .

1 [Cassels, Chap. 1, Problem 2].

Let $S_k(n) = 1^k + 2^k + \cdots + (n-1)^k$, and let B_k denote the k-th Bernoulli number, defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k.$$

(Note that aside from B_1 all $B_{2l+1} = 0$.) Let $p \geq 5$ and let $k = 2l \geq 4$. Show that

$$|p^{-m} S_{2l}(p^{-m}) - B_{2l}|_p \leq p^{-2m+a}$$

where $a = 1$ if $p-1$ divides $(2l-2)$ and $a = 0$ otherwise.

[Hint: Read Cassels, Chapter 1.3, and use the von-Staudt Clausen Theorem and its proof.]

2. [Cassels, Chap 1, Problem 3]

For each positive m let $N_m = \lfloor (1+\sqrt{3})^{2m+1} \rfloor$. Prove that the 2-adic valuation of the integer N_m is

$$|N_m|_2 = 2^{-m-1}.$$

[Hint: Show N_m satisfies a linear recurrence.]

3. [Cassels, Chap. 2, problem 22] A function $f(x)$ on a topological space is *locally constant* if every y has a neighborhood $U(y)$ on which $f(x) = f(y)$ for all $x \in U(y)$.

(a) Give an example of a $\mathbb{Q}_p$-valued function $f(x)$ on domain $\mathbb{Z}_p$ which is both continuous and locally constant, but not constant.

(b) If the $\mathbb{Q}-p$-valued function on domain $\mathbb{Z}_p$ is both continuous and locally constant, show that the set of x with $f(x) = b$ is both open and closed. Deduce that f takes only finitely many values on this domain.

(c) Show that any continuous $\mathbb{Q}_p$-valued function on domain $\mathbb{Z}_p$ is the uniform limit of continuous, locally constant functions.

4. [Cassels, Chap 4, problem 2]

For what $a \in \mathbb{Z}$ is $5x^2 = a$ solvable in $\mathbb{Z}_5$? For what a in $\mathbb{Q}_5$ is it solvable?

5. [Cassels, Chap 4, problem 4]

Show that each of the following functions has a zero in $\mathbb{Z}_p$ for every prime p.

(i) $(X^2-2)(X^2-17)(X^2-34)$.

(ii) $(X^3-37)(X^2+3)$.

[Note: For (ii) you may need to use the law of quadratic reciprocity.]

6. [Cassels, Chap. 4, Problem 5]

Show that the cubic polynomial

$$F(x) = 5x^3 - 7x^2 + 3x + 6$$

has a root α in the 7-adic integers $\mathbb{Z}_7$ with $|\alpha - 1|_7 < 1$. Find an $a \in \mathbb{Z}$ such that

$$|\alpha - a|_7 \leq \frac{1}{7^4}.$$

7. Let $f(x) = x^3 - x + 1$.

(a) Factorize $f(x)$ modulo 5, and then factorize it modulo 5^3. What can you say about the factorization of $f(x)$ over the 5-adic field $\mathbb{Q}_5$?

(b) Factorize $f(x)$ modulo 101, and then factorize it modulo $(101)^3$. What can you say about the factorization of $f(x)$ over the 101-adic field $\mathbb{Q}_{101}$?

[A calculator may be needed for (b). Or else, use a computer package.]

8. [Cassels, Chap. 4, Problem 20] Let p be prime and let for a positive integer let its base p expansion be

$$m = a_0 + a_1 p + \cdots + a_J p^J \quad (J \geq 0), \quad 0 \leq a_j \leq p-1.$$

Also write

$$m! = p^M N, \quad gcd(N,p) + 1.$$

(a) Show that

$$(p-1)M = m - \sum_j a_j$$

(b) Show that

$$N \equiv (-1)^M \prod_j (a_j)! \pmod{p}.$$

9. Let $p \geq 3$ and let $\exp(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}$, and let $\log(1-x) = -\sum_{n=2}^{\infty} \frac{1}{n} x^n$. Let $p \geq 3$.

(a) Determine the radius of convergence of $\exp(x)$ around $x = 0$ in the p-adic variable $x \in \mathbb{Q}_p$, i.e. for which values $|x|_p$ does the series converge?

(b) Determine the radius of convergence of $\log(1-x)$ similarly in $\mathbb{Q}_p$.

(c) For which values $x \in \mathbb{Q}_p$ can you conclude $\exp(\log(1-x)) = 1 - x$, and for which values $x \in \mathbb{Q}_p$ can you conclude $\log(1 - \exp(1-x)) = x$?

[Hint: See problem 8. These functions are sometimes denoted $\exp_p(x)$ and $\log_p(x)$.]

10. [Cassels, Chap. 4, Problem 25]

(i) Let K be a field of characteristic p which is complete with respect to the valuation $|\cdot|_\nu$ and suppose that its residue class field k is finite with q elements. Show that K contains the finite field $\mathbb{F}_q$ having q elements.

(ii) If, in addition, $|\cdot|_\nu$ is a discrete valuation, show that one can write K as $K = \mathbb{F}_q((\pi))$ as a Laurent series field in a (non-unique) uniformizer π.

11. [Cassels, Chap. 4, Problem 12]

Let $u(0) = u(1) = 1$ and consider the recurrence

$$u(n+2) = 5u(n+1) - 11u(n) \quad (n \geq 0)$$

(a) By working in $\mathbb{Q}_5$, or otherwise, show that $u_n = 1$ only for $n = 0, 1$.

(b) Is it possible that $u(n) = 0$ for some $n \geq 2$?