Math 676, Homework 6

Do 5 problems out of set of 10 problems. The 11-th problem is an extra problem, not counted as one of the 5 problems.

1. (Skolem-Mahler-Lech Theorem fails in characteristic p)
Let $K = \mathbb{F}_p(T)$ be the rational function field over the finite field $\mathbb{F}_p$, p prime.

(a) Show that

$$u_n = (T+1)^n - T^n, \quad n \geq 0.$$

satisfies a linear recurrence relation over K.

(b) Show that $u_n = 1$ precisely when $n = p^k$ for some $k \geq 0$.
(This is an infinite set which is not a finite union of arithmetic progressions, so the Skolem-Mahler-Lech theorem doesn't hold.)

[*Note:* Harm Derksen (U.Mich.) recently found (Invent. Math. **168** (2007), 175-224) a modified formulation of the Skolem-Mahler-Lech theorem valid in characteristic p.]

2. (Eisenstein's Condition) [Cassels, II. 2, II.3] A power series $f(T) = \sum_{j \geq 0} a_j z^j \in \mathbb{Q}[[T]]$ satisfies *Eisenstein's condition* if there is some integer w such that $a_j w^j \in \mathbb{Z}$ for all $j \geq 0$. Let $g_j(T) \in \mathbb{Q}[[T]]$ for $0 \leq j \leq J$ be not all zero. Suppose $f(T) \in \mathbb{Q}[[T]]$ satisfies a polynomial equation over $\mathbb{Q}[[T]]$:

$$(*) \qquad \sum_{j=0}^{J} g_j(T)(f(T))^j = 0.$$

(a) Prove that if all the $g_j(T)$ satisfy Eisenstein's condition, then $f(T)$ does also.

(b) Show that $f(T) = \sum_{n=1}^{\infty} \frac{1}{n} T^n$ does not satisfy any equation of form (*) over $K[[T]]$ with all $g_j(T) \in \mathbb{Q}[T]$, i.e. it is not the power series expansion at $T = 0$ of an algebraic function defined over $\mathbb{Q}$.

3. (Algebraic functions) [Cassels, IV.2] Let K be complete with respect to a non-archimedean valuation $|\cdot|_\nu$. Suppose that $g_j = g_j(T) \in K[[T]]$ $(0 \leq j \leq J)$ are not all 0 and that $f(T) \in K[[T]]$ satisfies

$$\sum_{j=0}^{J} g_j(T)(f(T))^j = 0.$$

(a) If all the $g_j(T)$ converge in an open neighborhood $|\alpha|_\nu < \delta$ of the origin $T = 0$, show that $f(T)$ converges in a (possibly smaller) open neighborhood of 0.

(b) Does statement (a) hold also for archimedean valuations?

[Hint: For (a), (b) see problem 2.]

5. (Content of integer polynomials) [Cassels, VI.1]
For $f(X_1, ..., X_n) \in \mathbb{Z}[X_1, \cdots, X_n]$ define the content $c(f)$ to be the greatest common divisor on its coefficients. If also $g(X_1, ..., X_n) \in \mathbb{Z}[X_1, \cdots, X_n]$ show

$$c(fg) = c(f)c(g).$$

5. (Strassmann theorem zeros) [Cassels VI. 2]
Let K be a complete valuation ring with nonarchimedean valuation, and let $f(z) = \sum_{n=0}^{\infty} f_n z^n \in K[[z]]$, be not identically zero. Set N so that $|f_N|_\nu = \max\{|f_n|_\nu\}$, and $|f_n|_\nu < |f_N|_\nu$ for all $n > N$. Let $f^{(k}(z)$ denote the k-th derivative of $f(z)$, taken formally.

(a) If $b \in O_{K,\nu}$ has $f(b) = 0$, call it an *integral zero*. Show that necessarily some $f^{(k)}(b) \neq 0$.

(b) Define the *multiplicity of an integral zero* b to the lowest k such that $f^{(k)}(b) \neq 0$. Show that the sum of the multiplicities of all integral zeros of b is at most N.

6. (Strassmann theorem zeros-2) [Cassels, VII.3] Let K be a complete valuation ring with nonarchimedean valuation, and let $f(z) = \sum_{n=0}^{\infty} f_n z^n \in K[[z]]$, be not identically zero. Set N so that $|f_N|_\nu = \max\{|f_n|_\nu\}$, and $|f_n|_\nu < |f_N|_\nu$ for all $n > N$. Let $f^{(k}(z)$ denote the k-th derivative of $f(z)$, taken formally. Show that there is some finite algebraic extension L/K such that the number of integer zeros of $f(z) = 0$ in the ring $O_{L,\nu}$ of the unique valuation extending ν to L is exactly N, counting these zeros with multiplicity.

7. (Quadratic extensions of 2-adics) [Cassels, VII. 5]
(a) Show that there are precisely 7 different quadratic extensions of the 2-adic field $\mathbb{Q}_2$, namely $K = \mathbb{Q}_2(\sqrt{d})$ for $d = -1, \pm 2, \pm 5, \pm 10$.

(b) Show that in all 7 cases that the (unique) extended 2-adic valuation is given by

$$|a + b\sqrt{d}|_2 := (|a^2 - db^2|_2)^{\frac{1}{2}}.$$

(c) Show that the only one of these fields K that is unramified is $\mathbb{Q}_2(\sqrt{5})$.

(d) Show that $O_{K,2} = \mathbb{Z}_2[1, \frac{1}{2}(1+\sqrt{5})]$ when $d = 5$, and that in the other six cases $O_{K,2} = \mathbb{Z}_2[1, \sqrt{d}]$.

(e) Compute $Disc(K/\mathbb{Q}_2)$ and show it is 1 precisely for the unramified field above.

8. (Completely ramified extension of $\mathbb{Q}_2$) [Cassels, VII. 13] (a) Show that $K = \mathbb{Q}_2(\delta)$ is completely ramified of degree 4 where $\delta^2 = 1 + \gamma$, $\gamma^2 = -2$.

(b) Show that the integers O_K are given by $\mathbb{Z}_2[1, \gamma, \delta, \gamma\delta]$.

9. (Unramified extension of $\mathbb{Q}_p$) [Cassels VII. 10] Let θ be a root of $G(X) = X^p - X - 1 \in \mathbb{Q}_p[X]$. Show that $\mathbb{Q}_p(\theta)$ is normal over $\mathbb{Q}_p$ and show that it is an unramified extension of degree p.

(b) Let $a \in \mathbb{Q}_p$ be a unit, and let ϕ be a root of $X^p - X + a = 0$. Show that $\mathbb{Q}_p(\phi) = \mathbb{Q}_p(\theta)$.

10. (Mixed ramified extension of $\mathbb{Q}_3$)[Cassels VII. 14] Show that an extension K of $\mathbb{Q}_3$ of degree 6 is given by $K = \mathbb{Q}_3(\delta, \theta)$ where $\delta^3 = 3, \theta^2 = 2 - \delta$.

(a) Show that the ramification parameter $e = 3$ and the residue class degree $f = 2$.

(b) Find an unramified extension L of $\mathbb{Q}_3$ of degree 2 with $L \subset K$.

11. [Bonus Problem]
Let K be a field with a (nontrivial) nonarchimedean valuation $|\cdot|_\nu$. Here the valuation is not necessarily complete or discrete. Let X be transcendental over K

(a) Recall that for $c > 1$ the valuations $||f(X)||_c$ defined for $f(X) = f_0 + f_1 X + \cdots + f_n X^n \in K[X]$ by

$$||f(x)||_c = \max_{0 \le j \le n} c^j |f_j|_\nu,$$

extends uniquely to $K(X)$, and agrees with $|\cdot|_\nu$ on K. Show these valuations are all inequivalent.

(b) Do all valuations on $K(X)$ that agree with $|\cdot|_\nu$ on K arise this way?