Math 775, Homework 1-part 1

Problems with a star might be harder.

1. (Values of L-functions at $s = 0$.)
Let χ be a Dirichlet character (mod m).

(i) Show that

$$L(s,\chi) = m^{-s} \sum_{c=1}^{m} \chi(a)\zeta(s, \frac{c}{m}),$$

where $\zeta(s, c)$ is the Hurwitz zeta function

$$\zeta(s,c) = \sum_{n=0}^{\infty} \frac{1}{(n+c)^s}.$$

(ii) Show that if χ is a non-principal character, then

$$L(0,\chi) = \frac{-1}{m} \sum_{c=1}^{m} \chi(c)\, c.$$

(iii) Show that if χ is a non-principal character, then

$$L'(0,\chi) = L(0,\chi) \log m + \sum_{c=1}^{m} \chi(c) \log \Gamma(\frac{c}{m}).$$

Note. A point here is that the formula (ii) at $s = 0$ is *much simpler* than that of Dirichlet's formula at $s = 1$, and does not depend on the value of $\chi(-1)$.

2. (Products of Γ-values)

Let $\mathcal{F}_n$ denote the set of Farey fractions of order n, which consists of all rationals $r = \frac{a}{b}$ with $0 < r < 1$, which in lowest terms have $0 \le a \le b \le n$.

(i) Prove the Gauss multiplication formula

$$(2\pi)^{\frac{1}{2}(1-n)} n^{nz-\frac{1}{2}} \prod_{k=0}^{n-1} \Gamma(z + \frac{k}{n}) = \Gamma(nz).$$

(ii) Prove that

$$\prod_{\substack{k=1 \\ (k,n)=1}}^{n} \Gamma(\frac{k}{n}) = (2\pi)^{\varphi(n)/2} g(n)$$

in which $g(n) = \frac{1}{\sqrt{p}}$ if $n = p^k$ is a prime power, and $g(n) = 1$ otherwise. Here $\varphi(n)$ is Euler's totient function.

(iii) Deduce that the least common multiple

$$[1, 2, ..., n] = \prod_{\substack{r \in \mathcal{F}_n \\ 0<r<1}} \frac{\pi}{\Gamma(r)^2}.$$

(iv) Deduce that the least common multiple

$$[1, 2...., n] = \frac{1}{2} \prod_{\substack{r \in \mathcal{F}_n \\ 0 < r \leq \frac{1}{2}}} (2 \sin \pi r).$$

3. (Bound for $\log \zeta(s)$)

Let $s = \sigma + it$, and suppose $|s - 1| \geq 1$. Show that, uniformly for $-1 \leq \sigma \leq 2$, one has

$$\log \zeta(s) = \sum_{\rho, |\gamma - t| \leq 1} \log(s - \rho) + O\Big(\log(|t| + 2)\Big),$$

where $\log \zeta(s)$ is defined by continuous variation from $\sigma + it$ to $\infty + it$, with $\log(\infty + it) = 0$, and imposing $-\pi < Im(\log(s - \rho) < \pi$ at zeros.

4. (Bound for $\psi(x)$ implies bound for $\pi(x)$)

Show that the estimate

$$|\psi(x) - x| \leq C_1 x^{1/2} (\log x)^2$$

implies the existence of a constant C_2 with

$$|\pi(x) - Li(x)| \leq C_2 x^{1/2} \log x.$$

5. (Upper bound for inverse zeta and L-function zero sum)

(i) Derive the estimate for the zeta zeros

$$\sum_{\substack{|Im(\rho)| < T \\ 0 \leq Re(\rho) \leq 1}} \frac{1}{|\rho|} = O((\log T)^2),$$

using results from Davenport Chapter 15, say.

(ii) Derive for a Dirichlet L-function $L(s, \chi)$ with $\chi(\bmod\ m)$ a non-principal primitive character an upper bound for

$$\sum_{\substack{0 < |Im(\rho_\chi)| < T \\ 0 \leq Re(\rho_\chi) \leq 1}} \frac{1}{|\rho_\chi|}$$

How does your estimate depend on the character χ and the conductor m?

(iii) Derive a similar estimate for a Dirichlet L-function $L(s, \chi)$ with $\chi(\bmod\ m)$ a non-principal character, possibly imprimitive. How much worse does it get?

(iv) What happens to your estimate in (ii) if real zeros with $0 \leq Re(s) \leq 1$ are allowed?

(v) What happens to your estimate in (iii) if real zeros with $0 \leq Re(s) \leq 1$ are allowed?