Math 775, Homework 1-part 2

Problems with a star might be harder.

6. (Lemma 17.1 revisited)
Let

$$I(y,T) = \int_{c-iT}^{c+iT} \frac{y^s}{s} ds$$

(i) Prove that for $y = 1$, $c > 0$ and $T > 0$, that

$$|I(1,T) - \frac{1}{2}| \leq cT^{-1}$$

(ii) Prove for $y > 1$, $c > 0$ and $T > 0$, that

$$|I(y,T) - 1| \leq y^c \min(1, \frac{1}{T|\log y|}).$$

Also prove for $0 < y < 1$ that

$$|I(y,T)| \leq y^c \min(1, \frac{1}{T|\log y|}).$$

(iii) Comparing (i) and (ii), can you explain where the discontinuity of the integral comes from when first $T \to \infty$, and then y is moved? Change variables $y = e^u$ and interpret this result in terms of a Fourier transform. Explain what is the role of the requirement $c > 0$, versus $c < 0$. Also, what would happen if for finite T fixed we let $y \to \infty$ (resp. let $y \to 0$). Compare the three answers you get for $I(y,T)$ in this case.

Note. A point of this problem is to get you to understand this lemma "from all sides".

7. (Half of the Poisson summation formula)

(*) (i) Find a version of the Poisson summation formula that sums on the half line from $n = 0$ through positive integers to $n = +\infty$. It will take the form

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2}\Big(\sum_{n=-\infty}^{\infty} \hat{f}(n)\Big) + \Big(\textit{an extra term}\Big).$$

Your job, should you wish to undertake it, is to *find that extra term.*

(ii) More or less equivalently, in terms of tempered distributions, find the Fourier transform of

$$\sum_{n=0}^{\infty} \delta_n,$$

where δ_n is the Dirac delta function at $x = n$, as a tempered distribution.

Note. Here the Fourier transform of a function is

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx.$$

[Here (ii) is optional. If you pursue it, you may go to Wikipedia to find out about tempered distributions and their Fourier transforms.]

8. (Formula for $b(\chi)$)

Let χ be a nonprincipal primitive Dirichlet character with $\chi(-1) = 1$.

Define $b(\chi) = b_0(\chi)$ by the Laurent expansion at $s = 0$,

$$\frac{L'}{L}(s, \chi) = \frac{1}{s} + b_0(\chi) + b_1(\chi)s + O(|s|^2).$$

Determine a formula relating $b(\chi)$ and the constant $B(\chi)$ appearing in the Hadamard product

$$\xi(s, \chi) = e^{A(\chi)+B(\chi)s} \prod_{\rho=\rho_{chi}} (1 - \frac{s}{\rho})e^{\frac{s}{\rho}}.$$

Here $B(\chi) = \frac{\xi(0,\chi)}{\xi'(0,\chi)}$.

9. (Upper Bound for $B(\chi)$)

Let χ be a non-principal (primitive) character (mod m). The following problem deduces that the upper bound (*) holds:

$$Re(B(\chi)) \leq -\log m + O(1).$$

(The O-constant is independent of m and χ.)

(i) Show that

$$\frac{1-\beta}{(1-\beta)^2 + \gamma^2} + \frac{\beta}{\beta^2 + \gamma^2} \geq \frac{1}{1+\gamma^2}$$

holds for $0 \leq \beta \leq 1$.

(ii) Deduce that

$$Re(B(\chi)_{\leq} - \frac{1}{2}\Big(\sum_{\rho=\beta+i\gamma} \frac{1}{1+\gamma^2}\Big)$$

(iii)) Show that

$$\frac{\xi'}{\xi}(2, \chi) = \frac{1}{2}\log m + O(1).$$

(iv) Show that

$$Re(\frac{\xi'}{\xi}(2, \chi)) = \sum_{\rho} Re(\frac{1}{2-\rho}).$$

(v) Show that

$$Re(\frac{\xi'}{\xi}(2, \chi)) = \frac{1}{2}\sum_{\rho} Re\Big(\frac{1}{2-\rho} + \frac{1}{1+\bar{\rho}}\Big).$$

(vi) Show that

$$\frac{2-\beta}{(2-\beta)^2+\gamma^2} + \frac{1+\beta}{(1+\beta)^2+\gamma^2} \leq \frac{3}{1+\gamma^2}.$$

(vi) Deduce that (*) holds for primitive characters. What happens to the bound for imprimitive characters? Does it still hold?

Note. There is no good lower bound for $Re(B(\chi))$ in some cases because of the exceptional zero.