Math 775, Homework 2-part 1

Large Sieve Problems

0. (Warm-up Brain Teaser) Let $||\alpha|| = \min\{|\alpha - n| : n \in \mathbb{Z}\}$ be the distance to the nearest integer. Show that for all real α,

$$|\sin \pi\alpha| \geq 2||\alpha||.$$

(Find a proof by picture.)

1. (*Brun-Titchmarsh Theorem*)

Take $\mathcal{A}$ to be the set of primes in $[M, M+N]$.

(i) Deduce from the large sieve inequality that

$$\pi(M+N) - \pi(M) \leq \frac{N + O(Q^2 \log Q)}{\sum_{q \leq Q} \frac{\mu(q)^2}{\varphi(q)}}$$

where $\varphi(q)$ is Euler's totient function.

(ii) Prove that

$$\sum_{q \leq Q} \frac{\mu(q)^2}{\varphi(q)} = \log Q + O(1).$$

(iii) Taking $Q = \frac{\sqrt{N}}{\log N}$, conclude that

$$\pi(M+N) - \pi(M) \leq \Big(2 + o(1)\Big) \frac{N}{\log N}.$$

[This result says that primes in short intervals can contain at most twice the number of primes that occur in the initial interval $[1, N]$. What would be the constant that replaces $2 + o(1)$, in the corresponding lower bound?]

2. (*Brun-Titchmarsh Theorem for Arithmetic Progressions*) Construct a similar argument to prove the a similar result for primes in arithmetic progressions:

$$\pi(x+y; q, a) - \pi(x; q, a) \leq \Big(2 + o(1)\Big) \frac{y}{\varphi(q) \log(y/q)}$$

3. (*) (*Improved Brun-Titchmarsh Theorem- A consequence*) Suppose that one can prove a version of the Brun-Titchmarsh theorem in which the constant 2 (on the right side in the inequality of problem 2) is replaced with 1.99.

Deduce from this improved theorem that the Dirichlet L-functions of real characters have no exceptional zeros! That is, deduce a good lower bound on how close the real zero can be to $s = 1$.

4. (*Large Sieve and Quadratic Residues*) Suppose that $\mathcal{A}$ is a set of integers in $[1, N]$ such that each $a \in \mathcal{A}$ is for each prime p, either divisible by p, or is a quadratic residue $\pmod p$. Prove that

$$|\mathcal{A}| \ll \sqrt{N \log N}.$$

[This is a "large sieve" inequality because we are sieving out a large number of residue classes–namely, half the residue classes for each prime p. A more detailed argument can show the stronger result $|\mathcal{A}| \ll \sqrt{N}$, which is optimal (up to a multiplicative constant) by taking $\mathcal{A} = \{k^2 : \quad k \leq \sqrt{N}\}$.]

Interlude. For the next two problems, let Φ be a vector in an inner product space V and suppose that $\varphi_1, ..., \varphi_N$ are orthonormal vectors in this space. Note that for any $u_i \in \mathbb{C}$,

$$||\Phi - \sum_i u_i \varphi_i||^2 \geq 0.$$

from which it follows that

$$||\Phi||^2 - 2Re(\sum_{i=1}^{N} \langle \Phi, u_i \varphi_i \rangle + \sum_{i=1}^{N} |u_i|^2 \geq 0.$$

Choosing $u_i = \langle \Phi, \varphi_i \rangle$ we deduce *Bessel's inequality*

$$||\Phi||^2 \geq \sum_{i=1}^{N} |\langle \Phi, \varphi_i \rangle|^2$$

If the φ_i span the space, then we deduce *Parseval's equality.*

5. (*Selberg's inequality*) Now suppose that the φ_i in V are not necessarily orthogonal. Then using in the above argument the bound

$$\begin{aligned} \sum_{i=1}^{N} \sum_{j=1}^{N} u_i \bar{u}_j \langle \varphi_i, \varphi_j \rangle &\leq \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (|u_i|^2 + |u_j|^2) |\langle \varphi_i, \varphi_j \rangle| \\ &= \sum_{i=1}^{n} |u_i|^2 \Big(\sum_{j=1}^{N} |\langle \varphi_i, \varphi_j \rangle| \Big), \end{aligned}$$

deduce Selberg's inequality: For all $\Phi \in V$ there holds

$$\sum_{i=1}^{N} \frac{|\langle \Phi, \varphi_i \rangle|}{\sum_{j=1}^{N} |\langle \varphi_i, \varphi_j \rangle|} \leq ||\Phi||^2.$$

Note. This reviews an inequality proved in class.

6. (*Large Sieve-Additive Form*) Use Selberg's inequality to derive an alternative proof of the large sieve bound in a weak additive form:

$$|\sum_{r=1}^{R} | \sum_{M+1}^{M+N} a_n e(n\alpha_r)|^2 \leq (N + O(\frac{1}{\delta} \log R)(\sum_{n=M+1}^{M+N} |a_n|^2).$$

Here the α_r are assumed δ-well spaced modulo one.