Math 775, Homework 2-part 2

7. (*An Integral Inequality*) Let $f(x)$ be a complex-valued function in $C^1([0,1])$.

(a) Using

$$f(x) = \int_0^1 f(u)du + \int_0^x uf'(u)du + \int_x^1 (u-1)f'(u)du$$

deduce that

$$|f(\frac{1}{2})| = \int_0^1 (|f(u)| + \frac{1}{2}|f'(u)|)du$$

and, for $0 \le x \le 1$,

$$|f(x)| \le \int_0^1 (|f(u)| + |f'(u)|)du$$

(b) Deduce from (a) that for fixed positive $\delta \le \frac{1}{2}$ and any α with $\delta \le \alpha \le 1-\delta$ there holds

$$|f(x)| \le \frac{1}{\delta}\int_{\alpha-\frac{1}{2}\delta}^{\alpha+\frac{1}{2}\delta} |f(u)|du + \frac{1}{2}\int_{\alpha-\frac{1}{2}\delta}^{\alpha+\frac{1}{2}\delta} |f'(u)|du$$

Note. Compare (b) with the fact that if $f(x)$ were a convex (real-valued) function then

$$f(x) \le \frac{1}{\delta}\int_{\alpha-\frac{1}{2}\delta}^{\alpha+\frac{1}{2}\delta} f(u)du.$$

These inequalities give pointwise estimates in terms of integral estimates over an interval.

8. (*Gallagher Version of Large Sieve*) Let $S(\alpha) = \sum_{n=M+1}^{M+N} a_n e(n\alpha)$. Let $\{\alpha_r : 1 \le r \le R$ be a set of δ-well-spaced points in the interval $[0,1]$ viewed as a torus mod 1.

(a) Apply Problem 7 with $f(\alpha) = S(\alpha)^2$ to deduce

$$\sum_{r=1}^{R} |S(\alpha_r)|^2 \le \sum_{r=1}^{R} \int_{\alpha_r-\frac{1}{2}\delta}^{\alpha_r+\frac{1}{2}\delta} \frac{1}{\delta}|S(\alpha)|^2 + |S(\alpha)S'(\alpha)|^2 d\alpha.$$

(b) Observe that the integration intervals in (a) do not overlap (why?), and deduce

$$\sum_{r=1}^{R} |S(\alpha_r)|^2 \le \frac{1}{\delta}\int_0^1 |S(\alpha)|^2 d\alpha + \int_0^1 |S(\alpha)S'(\alpha)|d\alpha.$$

(c) Bound the first integral on the right side of (b) by Parseval formula as $\sum_{n=M+1}^{M+N} |a_n|^2$. Then bound the second integral on the right side of (b) using Cauchy's inequality by

$$\le \Big(\sum_{n=M+1}^{M+N} |S(\alpha)|^2\Big)^{1/2}\Big(\sum_{n=M+1}^{M+N} |S'(\alpha)|^2\Big)^{1/2}.$$

Applying Parseval identity to both terms, deduce

$$\leq \Big(\sum_{n=M+1}^{M+N} |a_n|^2 \Big)^{1/2} \Big(\sum_{n=M+1}^{M+N} |2\pi i n a_n|^2 \Big)^{1/2}.$$

Next shift interval $M = -\frac{1}{2}(N+1)]$ so $|n| \leq \frac{1}{2}N$ (without changing left side of (b)) and deduce the second integral on the right side of (b) is

$$\leq \pi N \Big(\sum_{n=M+1}^{M+N} |a_n|^2 \Big)$$

(e) Conclude from (c) the Gallagher large sieve inequality

$$\sum_{r=1}^{R} |S(\alpha_r)|^2 \leq (\pi N + \frac{1}{\delta}) \sum_{n=M+1}^{M+N} |a_n|^2.$$

Note. The Gallagher method can be used with many functions $f(\alpha)$ so is flexible.

9. (*An inverse Mellin transform*)

(a) Let $c > 0$. Show that for each integer $k \geq 1$, and for real $y > 0$ that

$$I_k(c, y) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s \frac{ds}{s^{k+1}} = \begin{cases} \frac{1}{k!} (\log y)^k & \text{if } \ y > 1, \\ 0 & \text{if } \ 0 < y < 1. \end{cases}$$

Note that the integral $I_k(c, y)$ converges absolutely, unlike the conditionally convergent case $k = 0$.

(b) What is the value of this integral when $y = 1$?

(c) What happens if we allow the line of integration to be a value $c < 0$? What can you say about the value of $I_k(c, y)$, in the different cases above?

10. (*Sum of inverse Euler totient values*) Prove or disprove that there is a positive constant C such that for all $X \geq 2$,

$$\sum_{n=1}^{X} \frac{1}{\varphi(n)} \leq C \log X.$$

11. (*Minimal order of Euler totient function*) Prove that

$$\liminf_{n \to \infty} \varphi(n) \frac{\log \log n}{n} = e^{-\gamma},$$

where γ is Euler's constant.

Note. Compare this problem with Math 675, Problem 37.