Multivariable Zeta Functions

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Topics Covered

- Part I. Lerch Zeta Function and Lerch Transcendent
- Part II. Basic Properties
- Part III. Multi-valued Analytic Continuation
- Part IV. Other Properties: Functional Eqn, Differential Eqn
- Part V. Lerch Transcendent
- Part VI. Further Work: In preparation

References

- J. C. Lagarias and W.-C. Winnie Li ,
  The Lerch Zeta Function I. Zeta Integrals,
  Forum Math, **24** (2012) , 1–48.

  The Lerch Zeta Function II. Analytic Continuation,
  Forum Math **24** (2012), 49–84.

  The Lerch Zeta Function III. Polylogarithms and Special Values, arXiv:1506.06161, v1, 19 June 2015.

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Part I. Lerch Zeta Function: History and Objectives

- The Lerch zeta function is:

$$\zeta(s, a, c) := \sum_{n=0}^{\infty} \frac{e^{2\pi i n a}}{(n + c)^s}$$

- The Lerch transcendent is:

$$\Phi(s, z, c) = \sum_{n=0}^{\infty} \frac{z^n}{(n + c)^s}$$

- Thus

$$\zeta(s, a, c) = \Phi(s, e^{2\pi i a}, c).$$

Special Cases-1

- Hurwitz zeta function (1882)

$$\zeta(s,0,c) = \zeta(s,c) := \sum_{n=0}^{\infty} \frac{1}{(n+c)^s}.$$

- Periodic zeta function (Apostol (1951))

$$e^{2\pi i a}\zeta(s,a,1) = F(a,s) := \sum_{n=1}^{\infty} \frac{e^{2\pi i n a}}{n^s}.$$

Special Cases-2

- Fractional Polylogarithm

$$z\,\Phi(s,z,1) = Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$$

- Riemann zeta function

$$\zeta(s,0,1) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

History-1

- Lipschitz (1857) studied general Euler-type integrals including the Lerch zeta function

- Hurwitz (1882) studied Hurwitz zeta function, functional equation.

- Lerch (1883) derived a three-term functional equation. (Lerch's Transformation Formula)

$$\begin{aligned}\zeta(1-s,a,c) &= (2\pi)^{-s}\Gamma(s)\Big(e^{\frac{\pi i s}{2}}e^{-2\pi i a c}\zeta(s,1-c,a) \\ &+ e^{-\frac{\pi i s}{2}}e^{2\pi i c(1-a)}\zeta(s,c,1-a)\Big).\end{aligned}$$

History-2

- de Jonquiere (1889) studied the function

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{x^n}{n^s},$$

  sometimes called the fractional polylogarithm, giving integral representations and a functional equation.

- Barnes (1906) gave contour integral representations and method for analytic continuation of functions like the Lerch zeta function.

History-3

- Further work on functional equation: Apostol (1951), Berndt (1972), Weil 1976.

- Much work on value distribution: Garunkštis (1996), (1997), (1999), Laurinčikas (1997), (1998), (2000), Laurinčikas and Matsumoto (2000). Work up to 2002 summarized in L. & G. book on the Lerch zeta function.

Objective 1: Analytic Continuation

- Objective 1. Analytic continuation of Lerch zeta function and Lerch transcendent in three complex variables.

- Kanemitsu, Katsurada, Yoshimoto (2000) gave a single-valued analytic continuation of Lerch transcendent in three complex variables: they continued it to various large simply-connected domain(s) in $\mathbb{C}^3$.

- [L-L-Part-II] obtain a continuation to a multivalued function on a maximal domain of holomorphy in 3 complex variables. [L-L-Part-III] extends to Lerch transcendent.

Objective 2: Extra Structures

- Objective 2. Determine effect of analytic continuation on other structures: difference equations (non-local), linear PDE (local), and functional equations.

- Behavior at special values: $s \in \mathbb{Z}$.

- Behavior near singular values $a, c \in \mathbb{Z}$; these are "singularities" of the three-variable analytic continuation.

Objectives: Singular Strata

- The values $a, c \in \mathbb{Z}$ give (non-isolated) singularities of this function of three complex variables. There is analytic continuation in the s-variable on the singular strata (in many cases, perhaps all cases).

- The Hurwitz zeta function and periodic zeta function lie on "singular strata" of real codimension 2. The Riemann zeta function lies on a "singular stratum" of real codimension 4.

- What is the behavior of the function approaching the singular strata?

Objectives: Automorphic Interpretation

- Is there a representation-theoretic or automorphic interpretation of the Lerch zeta function and its relatives?

- **Answer:** There appears to be at least one. This function has both a *real-analytic* and a *complex-analytic* version in the variables (a, c), so there may be two distinct interpretations.

Part II. Basic Structures

Important structures of the Lerch zeta function include:

1. Integral Representations

2. Functional Equation(s).

3. Differential-Difference Equations

4. Linear Partial Differential Equation

Integral Representations

• The Lerch zeta function has two different integral representations, generalizing integral representations in Riemann's original paper.

• Riemann's two integral representations are Mellin transforms:

$$(1) \qquad \int_0^\infty \frac{e^{-t}}{1-e^{-t}} t^{s-1} dt \quad = \quad \Gamma(s)\zeta(s)$$

$$(2) \qquad \int_0^\infty \vartheta(0; it^2) t^{s-1} dt \quad \text{``}=\text{''} \quad \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s),$$

where $\vartheta(0;\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}$ is a (Jacobi) theta function.

Integral Representations

- The generalizations to Lerch zeta function are

$$(1') \qquad \int_0^\infty \frac{e^{-ct}}{1-e^{2\pi i a}e^{-t}} t^{s-1} dt \;=\; \Gamma(s)\zeta(s,a,c)$$

$$(2') \qquad \int_0^\infty e^{\pi c^2 t^2} \vartheta(a+ict^2, it^2) t^{s-1} dt \;=\; \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s,a,c).$$

using the Jacobi theta function

$$\vartheta(z,\tau) = \vartheta_3(z,\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau} e^{2\pi i n z}.$$

Four Term Functional Equation-1

- Defn. Let a and c be real with $0 < a < 1$ and $0 < c < 1$. Set

$$L^{\pm}(s,a,c) := \zeta(s,a,c) \pm e^{-2\pi i a}\zeta(s,1-a,1-c).$$

Formally:

$$L^{+}(s,a,c) = \sum_{-\infty}^{\infty} \frac{e^{2\pi i n a}}{|n+c|^{s}}.$$

- Defn. The completed function

$$\hat{L}^{+}(s,a,c) := \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\, L^{+}(s,a,c)$$

and the completed function

$$\hat{L}^{-}(s,a,c) := \pi^{-\frac{s+1}{2}}\Gamma(\frac{s+1}{2})\, L^{-}(s,a,c).$$

Four Term Functional Equation-2

- Theorem (Weil (1976))

  Let $0 < a, c < 1$ be real. The completed functions $\hat{L}^{+}(s,a,c)$ and $\hat{L}^{-}(s,a,c)$ extend to entire functions of s and satisfy the functional equations

  $$\hat{L}^{+}(s,a,c) = e^{-2\pi i a c}\hat{L}^{+}(1-s, 1-c, a)$$

  and

  $$\hat{L}^{-}(s,a,c) = i\, e^{-2\pi i a c}\hat{L}^{-}(1-s, 1-c, a).$$

- **Remark.** These results "extend" to boundary $a = 0, 1$ and/or $c = 0, 1$. If $a = 0, 1$ then $\hat{L}^{+}(s,a,c)$ is a meromorphic function of s, with simple poles at $s = 0, 1$.

Functional Equation Zeta Integrals

- [L-L-Part-I] obtained a generalized functional equation for Lerch-like zeta integrals containing a test function. (This is in the spirit of Tate's thesis.)

- These equations relate a integral with test function $f(x)$ at point s to integral with Fourier transform $\hat{f}(\xi)$ of test function at point $1-s$.

- The self-dual test function $f_0(x) = e^{-\pi x^2}$ yields the function $\hat{L}^{+}(s,a,c)$. The eigenfuctions $f_n(x)$ of the oscillator representation yield similar functional equations: Here $f_1(x) = xe^{-\pi x^2}$ yields $\frac{1}{\sqrt{2\pi}}\hat{L}^{-}(s,a,c)$.

Differential-Difference Equations

- The Lerch zeta function satisfies two differential-difference equations.

- (Raising operator) $\partial_L^+ := \frac{\partial}{\partial c}$

$$\frac{\partial}{\partial c}\zeta(s,a,c) = -s\zeta(s+1,a,c).$$

- Lowering operator) $\partial_L^- := \left(\frac{1}{2\pi i}\frac{\partial}{\partial a} + c\right)$

$$\left(\frac{1}{2\pi i}\frac{\partial}{\partial a} + c\right)\zeta(s,a,c) = \zeta(s-1,a,c)$$

Linear Partial Differential Equation

- Canonical commutation relations

$$\partial_L^{+}\partial_L^{-} - \partial_L^{-}\partial_L^{+} = I.$$

- The Lerch zeta function satisfies a linear PDE: The (formally) skew-adjoint operator

$$\Delta_L = \frac{1}{2}(\partial_L^{+}\partial_L^{-} + \partial_L^{-}\partial_L^{+}) = \frac{1}{2\pi i}\frac{\partial}{\partial a}\frac{\partial}{\partial c} + c\frac{\partial}{\partial c} + \frac{1}{2}I$$

has

$$\Delta_L \zeta(s,a,c) = -(s - \frac{1}{2})\zeta(s,a,c).$$

Part III. Analytic Continuation for Lerch Zeta Function

- Theorem. [L-L-Part-II] $\zeta(s,a,c)$ *analytically continues to a multivalued function over the domain*

$$\mathcal{M} = (s \in \mathbb{C}) \times (a \in \mathbb{C} \smallsetminus \mathbb{Z}) \times (c \in \mathbb{C} \smallsetminus \mathbb{Z}).$$

  It becomes single-valued on the maximal abelian cover of $\mathcal{M}$.

- The monodromy functions giving the multivaluedness are computable. For fixed s, they are built out of the functions

$$\phi_n(s,a,c) \ := \ e^{2\pi i n a}(c-n)^{-s}, \qquad n \in \mathbb{Z}.$$

$$\psi_{n'}(s,a,c) \ := \ e^{2\pi c(a-n')}(a-n')^{s-1} \qquad n' \in \mathbb{Z}.$$

Analytic Continuation-Features

- Fact. The manifold $\mathcal{M}$ is invariant under the symmetries of the functional equation: $(s, a, c) \mapsto (1-s, 1-c, a)$.

- Fact. The four term functional equation extends to the maximal abelian cover $\mathcal{M}^{ab}$ by analytic continuation. It expresses a non-local symmetry of the function.

Lerch Analytic Continuation: Proof

- Step 1. The first integral representation defines $\zeta(s, a, c)$ on the simply connected region

$$\{0 < Re(a) < 1\} \times \{0 < Re(c) < 1\} \times \{0 < Re(s) < 1\}.$$

Call it the fundamental polycylinder.

- Step 2a. Weil's four term functional equation extends to fundamental polycylinder by analytic continuation. It leaves this polycylinder invariant.

- Step 2b. Extend to entire function of s on fundamental polycylinder in (a, c)-variables, together with the four-term functional equation.

Lerch Analytic Continuation: Proof -2

- Step 3. Integrate single loops around $a = n, c = n'$ integers, using contour integral version of first integral representation to get initial monodromy functions

  Here monodromy functions are difference (functions) between a function and the same function traversed around a closed path. They are labelled by elements of $\pi_1(\mathcal{M})$.

- Step 4. The monodromy functions themselves are multivalued, but in a simple way: Each is multivalued around a single value $c = n$ (resp. $a = n'$). They can therefore be labelled with the place they are multivalued. (This gives functions $\phi_n, \psi_{n'}$)

Lerch Analytic Continuation: Proof -3

- Step. 5. Iterate to the full homotopy group in (a, c)-variables by induction on generators; use fact that a-loop homotopy commutes with c-loop homotopy.

- Step. 6. Explicitly calculate that the monodromy functions all vanish on the commutator subgroup $[\pi_1(\mathcal{M}), \pi_1(\mathcal{M})]$ of $\pi_1(\mathcal{M})$. This gives single-valuedness on the maximal abelian covering of $\mathcal{M}$.

Exact Form of Monodromy Functions-1

- At points $c = m \in \mathbb{Z}$,

$$M_{[Y_m]}(Z) = c_1(s) e^{2\pi i m a} (c-m)^{-s}$$

in which

$$c_1(s) = 0 \qquad \text{for} \quad m \geq 1,$$

$$c_1(s) = e^{2\pi i s} - 1 \quad \text{for} \quad m \leq 0.$$

Also

$$M_{[Y_m]^{-1}}(Z) = -e^{2\pi i s} M_{[Y_m]}(Z).$$

$$M_{[Y_m]^{\pm k}}(Z) = \frac{e^{\pm 2\pi i k s} - 1}{e^{\pm 2\pi i s} - 1} M_{[Y_m]^{\pm 1}}(Z).$$

Exact Form of Monodromy Functions-2

- At points $a = m \in \mathbb{Z}$,

$$M_{[X_m]}(Z) = c_2(s) e^{2\pi i c(a-m)} (a-m)^{s-1}$$

where

$$c_2(s) = -\frac{(2\pi)^s e^{\frac{\pi i s}{2}}}{\Gamma(s)}.$$

Also

$$M_{[X_m]^{-1}}(Z) = -e^{2\pi i s} M_{[X_m]}(Z)$$

$$M_{[X_m]^{\pm k}}(Z) = \frac{e^{\mp 2\pi i k s} - 1}{e^{\mp 2\pi i s} - 1} M_{[X_m]^{\pm 1}}(Z).$$

“Toy Model”: interpretation?

- The original Lerch zeta function is the “ground state”.

- Each homotopy class of loops encoded by integer “charge” at each $[X_n]$ and at each $[Y_n]$. The “charge” can be positive or negative. There are finitely many nonzero “charges”.

- The “charge” at $[X_n]$ is localized near $[X = n]$, sitting on a one-dimensional lattice. Same for $[Y_n]$ sitting on a second copy of the lattice.

- This model is a memory aid to keep track of the monodromy structure. But does it have a physics interpretation?

Extended Lerch Analytic Continuation

- Theorem. [L-L-Part-II] $\zeta(s,a,c)$ *analytically continues to a multivalued function over the (larger) domain*

$$\mathcal{M}^\sharp = (s \in \mathbb{C}) \times (a \in \mathbb{C} \smallsetminus \mathbb{Z}) \times (c \in \mathbb{C} \smallsetminus \mathbb{Z}_{\leq 0}).$$

Here the extra points $c = 1, 2, 3, ...$ *are glued into* $\mathcal{M}$. *The extended function is single-valued on the maximal abelian cover of* $\mathcal{M}^\sharp$.

- The manifold $\mathcal{M}^\sharp$ is *not invariant* under the four term Lerch functional equation. There is a *broken symmetry* between a and c variables.

Part IV. Consequences: Other Properties

We determine the effect of analytic continuation on the other properties

(1) Functional Equation. This is inherited by analytic continuation on $\mathcal{M}$ but not on $\mathcal{M}^{\sharp}$.

(2) Differential-Difference Equations. These equations lift to the maximal abelian cover of $\mathcal{M}$. However they are not inherited individually by the monodromy functions.

Consequences: Other Properties

(3) Linear PDE. This lifts to the maximal abelian cover. That is, this PDE is *equivariant* with respect to the covering map. The monodromy functions are all solutions to the PDE.

For fixed s the monodromy functions give an infinite dimensional vector space of solutions to this PDE.
(View this vector space as a direct sum.)

Consequences: Special Values

- Theorem. [L-L-Part-II]
  The monodromy functions vanish identically when $s = 0, -1, -2, -3, \ldots$. *That is: for these values of* s *the value of the Lerch zeta function is well-defined on the manifold* $\mathcal{M}$*, without lifting to the maximal abelian cover* $\mathcal{M}^{ab}$.

- It is well known that at the special values $s = 0, -1, -2, \ldots$ the Lerch zeta function simplifies to a rational function of c and $e^{2\pi i a}$.

- At nonnegative integer values of $s = 1, 2, \ldots$ monodromy partially degenerates: the monodromy functions satisfy *extra linear dependencies*.

Approaching Singular Strata

- [L-L-Part-I] There are (sometimes!) **discontinuities** in the Lerch zeta function's behavior approaching a singular stratum: these depend on the value of the s-variable.

  Observation. The location of discontinuities depends only on the *real part* of the s-variable. Three regimes:

$$Re(s) < 0; \quad 0 \le Re(s) \le 1; \qquad Re(s) > 1.$$

Part V. Lerch Transcendent

[L-L-Part III] determines the effect of analytic continuation on the Lerch transcendent

$$\Phi(s, z, c) := \sum_{n=0}^{\infty} \frac{z^n}{(n+c)^s}.$$

We make the change of variable $z = e^{2\pi i a}$ so that

$$a = \frac{1}{2\pi i} \log z.$$

This introduces extra multivaluedness: a is a multivalued function of z.

Polylogarithm

- The Lerch transcendent (essentially) specializes to the m-th order polylogarithm at $c = 1$, $s = k \in \mathbb{Z}_{>0}$.

$$Li_m(z) := \sum_{m=1}^{\infty} \frac{z^m}{m^k} = z\Phi(k, z, 1).$$

- The m-th order polylogarithm satisifes an $(m+1)$-st order linear ODE in the complex domain. This equation is Fuchsian on the Riemann sphere, i.e. it has regular singular points. These are located at $\{0, 1, \infty\}$.

- The point $c = 1$ is on a regular stratum. This uses the extended analytic continuation, which is not invariant under the functional equation.

Analytic Continuation for Lerch Transcendent

- Theorem. [L-L-Part III] $\Phi(s,z,c)$ *analytically continues to a multivalued function over the domain*

$$\mathcal{N} = (s \in \mathbb{C}) \times (z \in \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}) \times (c \in \mathbb{C} \smallsetminus \mathbb{Z}).$$

It becomes single-valued on a two-fold solvable cover of $\mathcal{N}$.

- The monodromy functions giving the multivaluedness are explicitly computable, but complicated.

Monodromy Functions for Lerch Transcendent

For fixed s, the monodromy functions are built out of the functions

$$\phi_n(s, z, c) \; := \; z^n (c-n)^{-s}, \qquad n \in \mathbb{Z}.$$

and

$$f_n(s, z, c) := e^{\pi i (s-1)} e^{2\pi i n c} z^{-c} (n - \frac{1}{2\pi i} Log\, z)^{s-1} \quad \text{if} \quad n \geq 1.$$

$$f_n(s, z, c) := e^{2\pi i n c} z^{-c} (\frac{1}{2\pi i} Log\, z - n)^{s-1} \qquad \text{if} \quad n \leq 0.$$

taking $z^{-c} = e^{-cLogz}$. where $Log\, z$ denotes a branch of the logarithm cut along the positive real axis.

Functional Equations: Lerch Transcendent

- Fact. The Lerch transcendent satisfies four term functional equations inherited from the Lerch zeta function. They are *multivalued*, relate different sheets of covering. They "break down" at the integer points $c \in \mathbb{Z}$, including all the polylogarithm values.

- Fact. Polylogarithms satisfy various "new" functional equations, of a completely different kind, some related to physics.

Hilbert's Problem List-after Problem 18

• Functions that satisfy algebraic differential equations are "significant functions."

• "The function of two variables s and x defined by the infinite series

$$\zeta(s,x) = x + \frac{x^2}{2^s} + \frac{x^3}{3^s} + \frac{x^4}{4^s} + \cdots$$

which stands in close relation with the function $\zeta(s)$, probably satisfies no algebraic differential equation. In the investigation of this question the functional equation $\frac{d\zeta(s,x)}{dx} = \zeta(s-1,x)$ will have to be used."

• No algebraic differential equation proved by Ostrowski (1920).

Differential-Difference Equations: Lerch Transcendent

- The Lerch transcendent satisfies two differential-difference equations. These operators are non-local in the s-variable.

- (Raising operator) $D_L^+ = \frac{\partial}{\partial c}$

$$\frac{\partial}{\partial c}\Phi(s,z,c) = -s\Phi(s+1,z,c).$$

- Lowering operator) $D_L^- = \left(z\frac{\partial}{\partial z} + c\right)$

$$\left(z\frac{\partial}{\partial z} + c\right)\Phi(s,z,c) = \Phi(s-1,z,c)$$

Linear Partial Differential Equation: Lerch Transcendent

- As with Lerch zeta function, the Lerch transcendent satisfies a linear PDE:

$$\left(z\frac{\partial}{\partial z}+c\right)\ \frac{\partial}{\partial c}\Phi(s,z,c)=-s\,\Phi(s,a,c).$$

- The (formally) skew-adjoint operator

$$\tilde{\Delta}_L := \ \left(z\frac{\partial}{\partial z}+c\right)\frac{\partial}{\partial c}+c\frac{\partial}{\partial c}+\frac{1}{2}I$$

has

$$\tilde{\Delta}_L\Phi(s,z,c)=-(s-\frac{1}{2})\Phi(s,z,c).$$

Specialization to Polylogarithm-1

- For positive integer value $s = m$, and c as a parameter, the function $z\Phi(m, z, c)$ gives a deformation of the polylogarithm in c-variable:

$$Li_m(z, c) := \sum_{n=0}^{\infty} \frac{z^n}{(n+c)^m}.$$

- Viewing c as fixed, it satisfies the Fuchsian ODE $D_c \, Li_m(z, c) = 0$ where the differential operator is:

$$D_c := z^2 \frac{d}{dz} (\frac{1-z}{z}) (z \frac{d}{dz} + c - 1)^m.$$

- The singular stratum points are $c = 0, -1, -2, -3, ...$

Specialization to Polylogarithm-2

- A basis of solutions for each regular stratum point is

$$\{Li_m(z,c), z^{1-c}(\log z)^{m-1}, z^{1-c}(\log z)^{m-2}, \cdots, z^{1-c}\}.$$

- The monodromy of the loop $[Z_0]$ on this basis is:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & e^{-2\pi ic} & e^{-2\pi ic}\frac{2\pi i}{1!} & \cdots & e^{-2\pi ic}\frac{(2\pi i)^{m-2}}{(m-2)!} & e^{-2\pi ic}\frac{(2\pi i)^{m-1}}{(m-1)!} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{-2\pi ic} & e^{-2\pi ic}\frac{2\pi i}{1!} \\ 0 & 0 & 0 & \cdots & 0 & e^{-2\pi ic} \end{pmatrix}.$$

- The monodromy of the loop $[Z_1]$ is unipotent and is independent of c.

Specialization to Polylogarithm-3

- A basis of solutions for each singular stratum point is

$$\{Li_m^*(z,c), z^{1-c}(\log z)^{m-1}, z^{1-c}(\log z)^{m-2}, \cdots, z^{1-c}\}$$

- The monodromy of the loop $[Z_0]$ in this basis is unipotent:

$$\begin{pmatrix} 1 & \frac{2\pi i}{1!} & \frac{(2\pi i)^2}{2!} & \cdots & \frac{(2\pi i)^{m-1}}{(m-1)!} & \frac{(2\pi i)^m}{m!} \\ 0 & 1 & \frac{2\pi i}{1!} & \cdots & \frac{(2\pi i)^{m-2}}{(m-2)!} & \frac{(2\pi i)^{m-1}}{(m-1)!} \\ 0 & 0 & 1 & \cdots & \frac{(2\pi i)^{m-3}}{(m-3)!} & \frac{(2\pi i)^{m-2}}{(m-2)!} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{2\pi i}{1!} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

- The monodromy of the loop $[Z_1]$ is also unipotent and independent of c.

Specialization to Polylogarithm-4

Observations:

- The monodromy representation (of π_1 of the Riemann sphere minus $0, 1, \infty$) is upper triangular, and is unipotent exactly when c is a positive integer (regular strata) or c is a nonpositive integer (singular strata).

- The differential equation makes sense on the singular strata, and remains Fuchsian (i.e. regular singular points). The monodromy representation continues to be unipotent, paralleling the positive integer case (polylogarithms at $c = 1$). However it takes a discontinuous jump at these points.

Part VI. Further Work (in preparation)

• [L-L-Part IV] studies two-variable "Hecke operators"

$$T_m(F)(a,c) := \frac{1}{m}\sum_{j=0}^{m-1} F(\frac{a+k}{m}, mc).$$

• These operators mutually commute, and also commute with Δ_L. Operators dilate in the c-direction while contract and shift in the a-direction.

• For fixed s the LZ function is a simultaneous eigenfunction of these operators, with eigenvalue m^{-s} for T_m.

• Show generalization of Milnor's 1983 result (to LZ function) characterizing the Hurwitz zeta function $\zeta(s,z)$ as a simultaneous eigenfunction of "Kubert operators":

Automorphic Interpretation-1 ([L])

• Automorphic Representation. The Lerch zeta function, or rather the functions $L^{\pm}(s, a, c)$, [in *real-analytic* version] may be viewed as a (non-holomorphic) *"Eisenstein series"* attached to the four dimensional solvable Lie group $H^J = GL(1, \mathbb{R}) \ltimes Heis(\mathbb{R})$ acting on a space of functions on the Heisenberg group, with $GL(1, \mathbb{R})$-action $(a, c, b) \mapsto (ta, t^{-1}c, b)$.

• The representation corresponds to the standard infinite-dimensional Schrödinger representation on $Heis(\mathbb{Z})\backslash Heis(\mathbb{R})$ having Planck constant $\hbar = 1$.

Automorphic Interpretation-2

• The space $L^2(Heis(\mathbb{Z})\backslash Heis(\mathbb{R})) = \oplus_{n\in\mathbb{Z}}\mathcal{H}_n$ where $n \in \mathbb{Z}$ corresponds to the value of Planck constant. The discete group

$$\Gamma = GL(1, \mathbb{Z}) \ltimes Heis(\mathbb{Z}).$$

acts separately on all the spaces $\mathcal{H}_n$. There is a "Laplacian" Δ acting on $Heis(\mathbb{R})$ and two-variable Hecke operators acting on all $\mathcal{H}_n$ ($n \neq 0$).

• The action of Δ is **pure continuous** acting on all spaces $n \neq 0$. The continuous spectrum for $\mathcal{H}_1$ is parametrized by $L^{\pm}(\frac{1}{2} + it, a, c)$. Action for $\mathcal{H}_n$ parametrized by Lerch functions twisted by various Dirichlet characters. The action is *not* semisimple. Also $\mathcal{H}_n$ is not irreducible for $|n| \geq 2$, there are "superselection sectors".

Summary

- The Lerch zeta function carries many extra algebraic and analytic structures, in its real-analytic and complex-analytic versions. The former assigns it a role as an "Eisenstein series" attached to a solvable Lie group.

- Observation. The analytic continuation of Lerch zeta function fails at values a, c integers, which are the most interesting values: the Hurwitz and Riemann zeta functions appear. These are singular points. Understanding the behavior as the singular points are approached might shed interesting new light on these limit functions.

Summary-2

- **Question.** Does the *Lindelöf hypothesis* hold for the Lerch zeta function? (Possibility raised by Garunkštis-Steuding).

  Our results imply: If so, Lindelöf hypothesis will hold for all the multivalued branches as well, because the monodromy functions are all of slow growth in the t-direction.

Thank You!