Hilbert Spaces of Entire Functions, Operator Theory and the Riemann Zeta Function

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Credits and References

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- J. C. Lagarias, Zero spacing distributions for differenced L-functions, Acta. Arithmetica **120** (2005), no. 2, 159–184.
- J. C. Lagarias, Hilbert spaces of entire functions and Dirichlet L-functions. in: *Frontiers in number theory, physics and geometry I*, 365–377, Springer, Berlin 2006.
- J. C. Lagarias, The Schrödinger operator with Morse potential on the right half-line, Commun. Number Theory Phys. **3** (2009), No. 2, 323–361.
- Disclaimer: Some results stated as facts in this talk are indicated by an asterisk (*). These represent results not yet written up for publication.

0. Overview: de Branges operator theory

- This talk concerns the applicability of the de Branges theory of Hilbert spaces of entire functions to functions arising in the theory of automorphic representations.
- These include: automorphic L-functions, Fourier-Whittaker coefficients of Eisenstein series.
- Phenomenology: The class of de Branges structure functions (in the Polya class) is a narrow class, and one interesting thing is that various "automorphic" objects seem to fall in this class. What you get is various (new) operators, whose properties might be investigated.

Contents of talk-1

- The talk first reviews "structure functions" $E(z)$ of de Branges theory. Various structure functions can be concocted out of functions in number theory such as L-function srising from the theory of automorphic representations.
- The de Branges theory attaches to a structure function $E(z)$ a Hilbert space of entire functions, with a multiplication operator on the space, which has a one-parameter family of self-adjoint extensions.
- It also attaches a "Fourier-like" transform which converts this operator to a 2×2 matrix ODE system ("canonical system") thought of over a finite interval $[0, a]$ (two regular endpoints) or a half-line $[0, \infty)$(one singular endpoint).

Contents of talk-2

- Assuming Riemann Hypothesis (RH), for L-functions associated to automorphic forms, one may hope in this fashion to produce a "Hilbert-Polya" operator that is a "canonical system."
- Some canonical systems can be nonlinearly transformed to a pair of Schrödinger operators on an interval or half-line.
- One can examine what such a "Hilbert-Polya" Schrödinger operator might look like.
- Toy models exist where all these steps can be carried out: Schrödinger operator with Morse potential on the half-line $[0, \infty)$.

Analytic functions and operator theory

There are several related kinds of analytic functions attached to non-self-adjoint operators which contain spectral information.

For one-dimensional Schrödinger operators an analytic spectral invariant is the *Weyl-Titchmarsh m-function*, introduced before 1920. The *characteristic operator function* of Livsic, developed in the 1940's, was developed by the Krein school. Another related theory is the Foias-Nagy theory of *unitary dilations.*

The de Branges theory was developed in the period 1959-1968. Its analytic data is a structure function $E(z)$. His theory has an extra feature: a uniqueness theorem, which proved a conjecture of Krein.

Structure Function-1

A *(de Branges) structure function* $E(z)$ is an entire function such that

$$E^\sharp(z) := \overline{E(\bar{z})}$$

has

$$|\frac{E^\sharp(z)}{E(z)}| \leq 1, \quad \text{when} \quad Im(z) > 0.$$

Remark. These functions called also *Hermite-Biehler functions*. Here $E(z)$ is bigger in upper half-plane than lower half-plane, so:

$$\Longrightarrow \text{ All zeros of } \; E(z) \text{ are in closed lower half-plane.}$$

Structure Function-2

One can write any entire function uniquely as

$$E(z) = A(z) - iB(z),$$

with $A(z), B(z)$ entire functions that are real on the real axis. Namely

$$A(z) := \frac{1}{2}\left(E(z) + E^\sharp(z)\right)$$

$$B(z) := \frac{-1}{2i}\left(E(z) - E^\sharp(z)\right)$$

Structure Function-3

Example 1. $E(z) = e^{-iaz}, a > 0$, is a structure function. Then: $e^{-iaz} = \cos az - i \sin az$ gives:

$$A(z) = \cos az, \qquad B(z) = \sin az.$$

Example 2. $E(z) = (z + i)^2 = (z^2 + 2iz - 1)$ is a structure function. Then:

$$A(z) = z^2 - 1 \qquad B(z) = -2z.$$

More generally $E_\theta(z) = e^{i\theta}(z + i)^2$ is a structure function for $0 \leq \theta < 2\pi$. Here

$$\begin{aligned} A_\theta(z) &= \cos\theta\,(z^2 - 1) + \sin\theta\,(-2z), \\ B_\theta(z) &= \quad \cos\theta\,(-2z) - \sin\theta\,(z^2 - 1). \end{aligned}$$

de Branges Structure Function-Key Property

A structure function $E(z)$ is *normalized* if it has no real zeros. (all zeros in open lower half-plane).

Lemma 1

(1) If $E(z) = A(z) - iB(z)$ is a structure function, then $A(z)$ and $B(z)$ have only real zeros. Furthermore the zeros of $A(z)$ and $B(z)$ interlace (count zeros with multiplicity).

(2) If $E(z)$ is normalized, then all zeros of $A(z)$ and $B(z)$ are simple zeros.

Remark. In general given two entire functions, $A(z), B(z)$, real on real axis, with only real zeros, which interlace, the function $E(z) := A(z) - iB(z)$ need not be a structure function.

de Branges Structure Function-Two Invariants

- The function

$$S(z) := \frac{E^\sharp(z)}{E(z)}$$

  is a meromorphic inner function. (An *inner function* is a function holomorphic on the upper half plane $\mathbb{C}^+$, with $|F(z)| \leq 1$, which has boundary values on real axis of absolute value 1 a. e.) We call $S(z)$ the *scattering matrix*; it is a 1×1 matrix.

- The *de Branges m-function* is

$$m(z) := -\frac{B(z)}{A(z)}.$$

  It is a meromorphic *Herglotz function*, i.e. $m(z)$ has positive imaginary part in the upper half-plane $\mathbb{C}^+$.

Structure Function-Polya Class

- A de Branges structure function is in the *Polya class* if it is the uniform limit of (Hermite-Biehler) polynomials.
- The Polya class is characterized as set of functions whose modulus grows monotonically on vertical lines in $\mathbb{C}^+$

$$|E(x+iy_1)| \geq |E(x+iy_2)| \quad \text{when} \quad y_1 > y_2 \geq 0.$$

- Such functions are entire functions of order at most 2. The closer their order is to 2, the closer their zeros must be to the real axis (in the lower half-plane). It includes all structure functions that are entire functions of exponential type.
- The various functions in this talk are all in the Polya class.

2. de Branges theory

The de Branges theory of Hilbert spaces of entire functions provides:

- A normal form for a special class of non-self-adjoint operators: A subclass of symmetric operators, having deficiency indices (1,1).
- The operator is represented as a (generally unbounded) multiplication operator on a Hilbert space of entire functions.
- (1) Operators in the class are in 1-to-1 correspondence with normalized structure functions.
- (2) There is a "Fourier transform" transforming this operator to a special kind of 2×2 linear system of ordinary differential operators, a "canonical system". The "Fourier transform" (de Branges transform) is *"unique"*.

de Branges Hilbert Space-1

Given a structure function $E(z)$, the associated de Branges Hilbert space $\mathcal{H}(E)$ consists of all entire functions $f(z)$ such that

- Norm
$$||f(z)||^2 := \int_{-\infty}^{\infty} |\frac{f(x)}{E(x)}|^2 dx < \infty.$$
- The meromorphic functions $\frac{f(z)}{E(z)}$ and $\frac{f^\sharp(z)}{E(z)}$ have controlled size in the upper-half plane: They are in $H^2(\mathbb{C}^+)$.

The Hilbert space $\mathcal{H}(E)$ may be finite-dimensional, in which case it is spanned by the polynomials $1, z, z^2, ...$ up to the dimension of the space. In general it is infinite-dimensional.

de Branges Hilbert Space-2

- The Hilbert space $\mathcal{H}(E)$ scalar product is:

$$\langle f(z), g(z)\rangle_E := \int_{-\infty}^{\infty} \frac{f(t)\overline{g(t)}}{|E(t)|^2} dt.$$

- It is a *reproducing kernel Hilbert space* (RKHS) with kernel

$$K(w,z) := \frac{\overline{A(w)}B(z) - A(z)\overline{B(w)}}{\pi(z-\bar{w})}.$$

  That is, for each $w \in \mathbb{C}$, $K(w,\cdot) \in \mathcal{H}(E)$ and

$$f(w) = \langle f(z), K(w,z)\rangle_E.$$

- de Branges spaces can be characterized axiomatically by:
  (1) the RKHS property, (2) symmetry under a real involution:
  If $f(z) \in \mathcal{H}(E)$ then $f^\sharp(z) \in \mathcal{H}(E)$, and $||f(z)||_E^2 = ||f^\sharp(z)||_E^2$.
  (3) Zero reflection property: If $f(z) \in \mathcal{H}(E)$ with $f(z_0) = 0$, then $f^*(z) := f(z)(\frac{z-\bar{z}_0}{z-z_0}) \in \mathcal{H}(E)$ and $||f(z)||_E^2 = ||f^*(z)||_E^2$.

De Branges Multiplication Operator-1

- For $f(z) \in \mathcal{H}(E)$, define the *multiplication operator*

$$M_z : f(z) \to zf(z), \text{ whenever } f(z) \in \mathcal{D}_z \subset \mathcal{H}(E).$$

- The domain $\mathcal{D}_z$ consists of all functions $f(z) \in \mathcal{H}(E)$ such that $zf(z) \in \mathcal{H}(E)$. It is either dense in $\mathcal{H}(E)$ or has closure of codimension 1 in $\mathcal{H}(E)$.
- The operator $(M_z, \mathcal{D}_z)$ is symmetric and has deficiency indices $(1, 1)$. Its point spectrum is *empty*.
- (Von Neumann Theory) The operator $(M_z, \mathcal{D}_z)$ has a one-parameter family of self-adjoint extensions $(M_z, \mathcal{D}_z(\theta))$, where $w = e^{i\theta}$, $0 \leq \theta < 2\pi$.

de Branges Multiplication Operator-2

- The self-adjoint extensions of $(M_z, \mathcal{D}_z)$ are described by a parameter $e^{i\theta} \in U(1)$. Associate to it the structure function

  $$e^{i\theta} E(z) = A_\theta(z) - iB_\theta(z),$$

  and a self-adjoint extension corresponds to $A_\theta(z)$.
- The spectrum is real, *discrete* and *simple*, with an eigenvalue at each zero ρ of $A_\theta(z)$. It can be unbounded above and below. *Simple spectrum* is a *feature* of de Branges theory.
- One picks a single eigenvalue ρ and adjoins as eigenfunction

  $$f_\rho(z) = \frac{A_\theta(z)}{z - \rho} \in \mathcal{H}(E)$$

  to the domain $(M_z, \mathcal{D}_z)$, obtaining a bigger domain $\mathcal{D}_z[\theta]$, for extension.
- Heuristically one may think of zeros of $A(z)$ as like "Dirichlet boundary condition" spectrum , $B(z)$ as like "Neumann boundary conditions" spectrum.

3. Canonical System-1

A *canonical system* is a 2×2 system of linear ordinary differential equations on an interval I, for each $z \in \mathbb{C}$, with a boundary condition. For each fixed $z \in \mathbb{C}$,

- $$\frac{d}{dt}\begin{bmatrix} A(t,z) \\ B(t,z) \end{bmatrix} = z\, J\, M(t) \begin{bmatrix} A(t,z) \\ B(t,z) \end{bmatrix},$$

  with

  $$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

- The matrix function

  $$M(t) = \begin{bmatrix} \dot{\alpha}(t) & \dot{\beta}(t) \\ \dot{\beta}(t) & \dot{\gamma}(t) \end{bmatrix}$$

  is positive semidefinite symmetric, all $t \in I$.

- The "boundary conditions" (at regular endpoint) are:

  $$\lim_{t \to 0^+} A(t,z) = 1 \quad \text{and} \quad \lim_{t \to 0^+} B(t,z) = 0.$$

Canonical System-2

- A canonical system for *fixed* $z \in \mathbb{C}$ is a *Hamiltonian dynamical system with time-dependent Hamiltonian*.
  The Hamiltonian function is the quadratic form

  $$H(p,q,t) = \frac{z}{2}\left(\dot{\alpha}(t)p^2(t,z) + 2\dot{\beta}(t)p(t,z)q(t,z) + \dot{\gamma}(t)q^2(t,z)\right)$$

  with $p(t,z) := A(t,z)$ and $q(t,z) := B(t,z)$.
- $det(M(t)) = 0$ is permitted, for all t in an interval. This allows difference equations and Schrödinger equations to be encoded in this framework, also allows finite-dimensional de Branges spaces.
- *Krein normal form* for $M(t)$: Require $Trace(M(t)) \equiv 1$.

Canonical System-3

de Branges Direct Theorem [Theorem 41]

Any canonical differential system with the property that the 2×2 *matrix function* $M(t)$ *has properties:*

1. *Measurable over the interval* $(0, b]$
2. *Entries are integrable over the interval* $(0, b]$*, with*

$$\int_0^\delta \alpha(t)\dot{\gamma}(t)dt < \infty$$

   where $\alpha(t) = \int_0^t \dot{\alpha}(u)du$*.*

3. *M(t) is symmetric, positive semi-definite on interval* $(0, b]$*.*

Then its solutions $E(t,z) = A(t,z) - iB(t,z)$ *with* t *constant and* $z \in \mathbb{C}$ *are strict, normalized de Branges structure functions, in the Polya class.*

Canonical System-4

de Branges Inverse Theorem [Theorem 40]

Given any structure function $E(z)$ in the Polya class, there exists a canonical system on an interval or half-line which produces $E(z) = A(t, z) - iB(t, z)$ at a "regular value" $t = c$. The canonical system is unique if it is "trace-normalized" $Tr(M(t)) \equiv 1$.

This theorem encodes: there is a *unique* totally ordered chain of closed subspaces of $\mathcal{H}(E)$ that are themselves de Branges spaces $\mathcal{H}(E_t)$ for some (normalized) structure function $E_t(z)$.

Canonical System-5

- The inverse theorem asserts the existence of a chain of very nice invariant subspaces of $\mathcal{H}(E)$ ordered by inclusion. ("Triangular decomposition".)
- Inverse problem seems *extremely hard*: Find $M(t)$ given $E(z)$. (de Branges's proof is non-constructive.)
- de Branges found explicit form of $M(t)$ in various special cases where a large symmetry group is acting: often these involve special functions in mathematical physics.

4. Automorphic L-Functions-1

- The theory of automorphic representations is a "theory of everything." Main point of this talk: this theory naturally produces a large collection of entire functions which are (unconditionally or conditionally) structure functions to which the de Branges theory applies.
- These include: zeta and L-functions, Fourier coefficients of Eisenstein series, (inverses of) local factors in Euler products, and Hecke operator eigenvalues.
- Two main conjectures in this theory make sense in the de Branges theory: Ramanujan conjecture for local L-factors, and (Grand) Riemann hypothesis for global L-functions.

Automorphic L-functions-2

- Manifesto: To determine the attached canonical systems guaranteed to exist by de Branges inverse theorem.
- There are large group actions present, leaving some hope for explicit constructions. [Or characterization by "extra properties": an approach of de Branges.]
- Good News: de Branges theory includes difference operators.
- Bad News: Quotienting by discrete subgroups has (so far) been an obstacle to explicit constructions.
- Important Point: Many interesting structure functions fall in the *singular endpoint case.*

Riemann zeta function

- The Riemann xi-function is:

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$$

- It is an *entire function*. It satisfies the functional equation

$$\xi(s) = \xi(1-s).$$

- It is real-valued on the critical line and has a reflection symmetry around the critical line

$$\xi(\frac{1}{2}+h+it) = \overline{\xi(\frac{1}{2}+h+it)}$$

- This can be rewritten

$$\xi(s) = \overline{\xi(1-\bar{s})}$$

  This symmetry is the de Branges "real symmetry".

One-Parameter Family of Structure Functions

Theorem [L. '05]

Let h be real and set $E_h(s) = A_h(s) - iB_h(s)$ with

$$A_h(s) := \quad \frac{1}{2}\left(\xi(s+h) + \xi(s-h)\right)$$

$$B_h(s) := -\frac{1}{2ih}\left(\xi(s+h) - \xi(s-h)\right).$$

1. *Then $\tilde{E}_h(z) := E_h(\frac{1}{2} - iz)$ is a structure function whenever $|h| \geq 1/2$.*
2. *Assuming Riemann Hypothesis, $\tilde{E}_h(\frac{1}{2} - iz)$ is also a structure function for $0 < |h| < \frac{1}{2}$.*

Remarks on this One-Parameter Family

- Zeros of $A_h(s)$ on critical line match ordinates of zeros of $Re(\xi(s))$ on line
$$Re(s) = 1/2 + h,$$
those of $B_h(s)$ match the zeros of $Im(\xi(s))$ there.
- The unconditional case comes from showing that, when $|h| \geq 1/2$, then for $Re(s) > 1/2$,
$$|\xi(h+s)| > |\xi(h+1-\bar{s})|.$$
- The GUE normalized spacing distribution (conjectured to hold for the zeta zeros) fails to hold when $h \neq 0$. Instead, the normalized zeros are asymptotically evenly spaced: normalized spacing distribution is a delta function at spacing 1.

Parameter Value $h = 0$ (obtained as a limit)

Theorem [L.-06]

Let h be real and set

$$\tilde{E}_0(z) := \xi(\frac{1}{2} - iz) + \xi^{'}(\frac{1}{2} - iz).$$

1. *Assuming the Riemann hypothesis, $\tilde{E}_0(z)$ is a structure function.*
2. *Assuming RH and also simple zeros, $\tilde{E}_0(z)$ is a normalized structure function, up to a scaling.*

Parameter Value $h = 0$-Continued.

- Assume Riemann Hypothesis is true. Then the associated de Branges space $\mathcal{H}(\tilde{E}_0(z))$ encodes the "explicit formula" of prime number theory in the Hilbert space scalar product. The elements of the Hilbert space serve as test functions.
- (*) The Hilbert space scalar product agrees with the Weil scalar product, and its positive definiteness encodes the RH.
- ("Hilbert-Polya" operator) Assuming RH, one particular self-adjoint extension of the associated canonical system to $\tilde{E}_0(z)$ functions as a Hilbert-Polya operator: call it E_{HP}. (Important: it will fall in the "singular" endpoint case.)

General automorphic L-functions

- The theorems above generalize to all Dirichlet L-functions (GL(1)) case. (*) The theorems above also generalize to all principal automorphic L-functions attached to $GL(N)$ over $\mathbb{Q}$. A key property is the symmetry

$$\xi(s,\pi) = \overline{\xi(1-\bar{s},\pi)}$$

- For each automorphic L-function there is a one-parameter family $E_h(z,\pi)$ with real parameter h. The functions $A_h(z,\pi)$, $B_h(z,\pi)$ involve differences of the corresponding ξ-function, $\xi(s,\pi)$, which is the automorphic L-function with archimedean factors added. For $|h| \geq 1/2$, these are unconditionally structure functions. Inside the critical strip one must assume RH. The value $h=0$ involves a derivative and is the interesting value, where one gets a "Hilbert-Polya" operator.

Quotation from Atle Selberg

Atle Selberg on "Hilbert-Polya" Operator
[Bulletin Amer. Math. Soc. 45 (October 2008), p. 632]:

"In fact there have been some people that have been able to construct such a space, if they assume the Riemann hypothesis is correct, and where they can define an operator that is relevant. Well and good, but it gives us basically nothing, of course. It does not help much if one has to postulate the results beforehand–there is not much worth in that."

Approach to RH?

- If one could "guess" the correct canonical system and integrate it to show it produces the function

$$E_{HP}(z) = \xi(\frac{1}{2} - iz) + \xi^{'}(\frac{1}{2} - iz),$$

  this would certify that $E_{HP}(z)$ is a structure function and so prove the Riemann hypothesis.
- Certification means: (1) Finding the functions $\dot{\alpha}(t), \dot{\beta}(t), \dot{\gamma}(t)$; (2) Checking that $M(t)$ is positive semidefinite on the appropriate interval; and (3) Checking that integration produces the appropriate $(A(z), B(z))$.
- If found, this would exhibit a (new) integral representation of $\xi(s)$ and $\xi^{'}(s)$.
- No one knows how to approach this!

5. Toy Model-1

- de Branges and earlier Krein found many cases where the transform and canonical system can be explicitly computed, in the regular endpoint case.
- Singular endpoint case. These are cases where the structure function is NOT entire of exponential type, it grows faster. We present a "toy model" example that involves Whittaker functions. Earlier work in singular endpoint case: de Branges constructed examples. J.-F. Burnol's work especially significant.
- A point of "toy model" is that the number of zeros to distance T on the real axis (corresponds to critical line) has

$$N(T) = C_1 T \log T + C_2 T + o(T),$$

so that the asymptotics of zeros behaves like that of the Riemann zeta function.

Toy Model-2

Consider the canonical system with matrix

$$M(u) := \begin{bmatrix} e^{-2ru}e^{-e^u} & 0 \\ 0 & e^{2ru}e^{e^u} \end{bmatrix}.$$

on the half-line $(0, \infty)$. Then it has solutions

$$\begin{aligned} A(u,z) &= e^{(r-\frac{1}{2})u}e^{\frac{1}{2}e^u}W_{-r+\frac{1}{2},\pm i\sqrt{z^2-r^2}}(e^u) \\ B(u,z) &= -z\,e^{(-r-\frac{1}{2})u}e^{-\frac{1}{2}e^u}W_{-r-\frac{1}{2},\pm i\sqrt{z^2-r^2}}(e^u). \end{aligned}$$

where $W_{k,m}(x)$ is the Whittaker function with parameters (k, m) and $x = e^u$ lies on positive real axis.

Toy Model-3

- The Whittaker function $W_{k,m}(x)$ is a confluent hypergeometric function. It is an entire function of two complex variables (k, m) but is multivalued in the x-variable, with singular points $x = 0$ and $x = \infty$ on the Riemann sphere.
- Since we obtained structure functions via the canonical system, we can deduce information on the complex zeros of Whittaker functions $W_{k,m}(x)$:

  Theorem. *If $x > 0$ is positive real, and k is real, then, as an entire function of the remaining variable m, all zeros lie on the line $Re(m) = 0$ and are simple zeros, except possibly for $m = 0$.*
- The asymptotics of zeros can be deduced from the Schrödinger equation connection given next.

6. Conversion to Schrödinger Operator-1

We suppose we have a canonical system $M(t)$ such that:

- $M(t)$ is invertible everywhere in $(0, b]$, and time is rescaled so that
$$\det M(t) \equiv 1.$$
- $M(t)$ is diagonal, so that
$$M(t) = \begin{bmatrix} \alpha(u) & 0 \\ 0 & \frac{1}{\alpha(u)} \end{bmatrix}.$$
- A necessary condition for a structure function $E(z)$ to give a diagonal canonical system is that $A(z)$ be an even function and $B(z)$ be an odd function. It is sufficient condition under a small additional condition. (de Branges [Theorem 47] and following problems).

Conversion to Schröodinger Operator-2

- Now set
$$\phi^+(u,z^2) := \sqrt{\alpha(u)}A(u,z)$$
$$\phi^-(u,z^2) := \sqrt{\alpha(u)}^{-1}B(u,z).$$
- These functions satisfy the Schrödinger equation
$$[-\frac{d^2}{du^2} + V^{\pm}(u)]\phi^{\pm}(u) = z^2\phi^{\pm}(u)$$
with eigenvalue $E = z^2$, where the potentials are:
$$V^{\pm}(u) := W(u)^2 \pm W'(u)$$
- The "superpotential" $W(u)$ is:
$$W(u) = \frac{d}{du}\log\sqrt{\alpha(u)} = \frac{1}{2}\frac{\alpha'(u)}{\alpha(u)}.$$
- This is related to Schrödinger's "factorization method".

Schrödinger Operator for Morse Potential

- The *Morse potential* is:

$$V_k(u) = \frac{1}{4}e^{2u} + ke^u$$

  where k is a real parameter.

- Consider the Schrödinger operator

$$-\frac{d^2}{du^2} + V_k(u)$$

  on the half-line $[u_0, \infty)$ for a constant u_0.

- The toy model canonical system above leads to such Schrödinger operators for special cases of the parameter r.

Schrödinger Operator for Morse Potential-2

- The boundary value problem on $[u_0, \infty)$ is *singular* and in the *limit point case* at $+\infty$. Morse potential eigenfunctions are given in terms of Whittaker functions.
- One can find the spectrum for standard boundary conditions at the left endpoint u_0. The spectrum is pure discrete. For Dirichlet boundary conditions they correspond to zeros of the Whittaker functions as above.
- One can compute the density of the spectrum by (rigorous) semiclassical estimates. One obtains

$$|\{n : E_n \leq T\}| = c_1\sqrt{T}\log T + c_2\sqrt{T} + O(1).$$

  The square root occurs here because $E = z^2$. (No GUE!)
- *Reference:* [L] Commun. Number Theory and Physics, 3 (2009), 329-361.

7. Concluding Remarks-1

- The de Branges theory is completely general, it applies to any structure function. It is a transducer: takes a problem in one form, converts it to another.
- For Riemann Hypothesis: it appears to apply *only* if the RH is true. It provides only a "reverse-engineering" approach: it then constructs new objects which, one hopes, will turn up in some different context allowing progress to be made.
- Different perspective suggests various new questions:

Hilbert-Polya Schrödinger operator on half-line?

Can the "Hilbert-Polya" operator canonical system (existing if RH holds) be converted to (a pair of) Schrödinger operators on a half-line $[u_0, \infty)$?

- The associated canonical system will be diagonal, as required.
- Canonical system $M(t)$ must have nonzero determinant for all relevant t.
- The transform to Schrödinger operator is non-linear, and requires some smoothness in the functions $\dot{\alpha}(t)$ in the canonical system.
- Speculation: the HP potential, if it exists, is not a locally L^1-function, but to be some kind of generalized function.

Scattering Theory Interpretation: Lax-Phillips Viewpoint

- The de Branges theory has a scattering theory interpretation. The full structure function $E(z)$ describes scattering data, its zeros in lower half-plane correspond to scattering poles in $S(z)$.
- From scattering viewpoint, the de Branges theory is a *narrow special case* of the Lax-Phillips theory:

  (a) The deBranges scattering matrix is a 1×1 *scalar* matrix. (Lax-Phillips theory allows an $n \times n$ matrix, any $n \geq 1$);

  (b) The associated function is *meromorphic* inner function (Lax-Phillips theory allows any inner function)

Eisenstein Series-1

- Eisenstein series form the continuous spectrum of the Laplacian operator (acting on a suitable domain). Scattering matrix coming from constant term is the source of L-functions.
- Fadeev-Pavlov (1972) noted an formulation of RH in terms of scattering theory. Operator is $GL(2)$ Laplacian. This was reworked by Lax-Phillips (1978) in terms of their scattering theory.
- de Branges paper [(1986) RH for Hilbert Spaces of Entire Functions] considers case of particular congruence subgroup $\Gamma(N)$ of $SL(2,\mathbb{Z})$ where operators describable in his theory. An important point is that the space of Eisenstein series factorizes into 1-dimensional blocks, to each of which separately de Branges theory applies. (RH is encoded in terms of the zeros of $E(z)$ in the lower half-plane. This is different from the "Hilbert-Polya" interpretation.)

Eisenstein Series-2

- The condition that the Eisenstein series for a congruence subgroup splits into one-dimensional subspaces, on which the Eisenstein series has a standard functional equation, seems to be a necessary condition for applicability of the de Branges theory to apply to all L-functions.
- This splitting holds in de Branges's case, which has $\phi(N)$ cusps.
- *Fact.* For principal congruence subgroups $\Gamma_0(N)$, splitting of Eisenstein series into one-dimensional subspaces holds for squarefree level N.
- More complicated behavior for non-squarefree level.

Natural Occurrence of de Branges spaces

- de Branges spaces occur arise in nature in terms of the Mellin transform.
- The relevant operator is the GL(1) Laplacian.
- The Mellin transform gives an isometry $L^2([1,\infty), dt)$ to the Hardy space $H^2(\{Re(s) > \frac{1}{2}\})$. It also gives an isometry $L^2((0,1], dt)$ to $H^2(\{Re(s) < \frac{1}{2}\})$.
- de Branges spaces can be viewed as ("small") closed subspaces of these Hardy spaces.
- This viewpoint is compatible with the Beurling-Nyman type real variable reformulations of RH.

Understanding de Branges "Subordination"

Question 1. De Branges theory does not prescribe GUE or anything like it. But one can ask what GUE would mean in terms of this framework. How would one recognize it? If the structure function $E(z)$ has $A(z)$ satisfying a limiting GUE distribution, do the functions $A_h(z)$ in the associated de Branges chain inherit this property?

Question 2. Call a structure function $E_1(z)$ *subordinate* to $E_2(z)$ if it appears in the de Branges chain for $E_1(z)$? This relation defines a partial order on the set of all structure functions. What can one say about this partial order?

End

Thank you!