The Lerch Zeta Function and the Heisenberg Group

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Topics Covered

- Part I. Lerch Zeta Function : History
- Part II. Basic Properties
- Part III. Two-Variable Hecke Operators
- Part IV. LZ and the Heisenberg Group

Summary

This talk reports on work on the Lerch zeta function extending over many years. Much of it is joint work with Winnie Li.

This talk focuses on:

• **Two-variable Hecke operators** and their action on function spaces related to Lerch zeta function. (with Winnie Li).

• **Heisenberg group** representation theory interpretation of (generalized) Lerch zeta functions.

Credits

- J. C. Lagarias and W.-C. Winnie Li , The Lerch Zeta Function I. Zeta Integrals, Forum Math, 2012.
- J. C. Lagarias and W.-C. Winnie Li , The Lerch Zeta Function II. Analytic Continuation, Forum Math, 2012
- J. C. Lagarias and W.-C. Winnie Li , The Lerch Zeta Function III. Polylogarithms and Special Values, Research in Mathematical Sciences, 2016.
- J. C. Lagarias and W.-C. Winnie Li , The Lerch Zeta Function IV. Hecke Operators Research in Mathematical Sciences, submitted.
- J. C. Lagarias, The Lerch zeta function and the Heisenberg Group `arXiv:1511.08157`

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Part I. Lerch Zeta Function:

- The Lerch zeta function is:

$$\zeta(s,a,c) := \sum_{n=0}^{\infty} \frac{e^{2\pi i n a}}{(n+c)^s}$$

- The Lerch transcendent is:

$$\Phi(s,z,c) = \sum_{n=0}^{\infty} \frac{z^n}{(n+c)^s}$$

- Thus

$$\zeta(s,a,c) = \Phi(s, e^{2\pi i a}, c).$$

Special Cases-1

- Hurwitz zeta function (1882)

$$\zeta(s, 0, c) = \zeta(s, c) := \sum_{n=0}^{\infty} \frac{1}{(n+c)^s}.$$

- Periodic zeta function (Apostol (1951))

$$e^{2\pi ia}\zeta(s, a, 1) = F(a, s) := \sum_{n=1}^{\infty} \frac{e^{2\pi ina}}{n^s}.$$

Special Cases-2

- Fractional Polylogarithm

$$z\,\Phi(s,z,1) = Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$$

- Riemann zeta function

$$\zeta(s,0,1) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

History-1

- Lipschitz (1857) studies general Euler integrals including the Lerch zeta function
- Hurwitz (1882) studied Hurwitz zeta function.
- Lerch (1883) derived a three-term functional equation. (Lerch's Transformation Formula)

$$\begin{aligned}\zeta(1-s,a,c) &= (2\pi)^{-s}\Gamma(s)\Big(e^{\frac{\pi is}{2}}e^{-2\pi iac}\zeta(s,1-c,a) \\ &+ e^{-\frac{\pi is}{2}}e^{2\pi ic(1-a)}\zeta(s,c,1-a)\Big).\end{aligned}$$

History-2

- de Jonquiere (1889) studied the function

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{x^n}{n^s},$$

  sometimes called the fractional polylogarithm, getting integral representations and a functional equation.

- Barnes (1906) gave contour integral representations and method for analytic continuation of functions like the Lerch zeta function.

History-3

- Further work on functional equation: Apostol (1951), Berndt (1972), Weil 1976.

- Much work on value distribution of Lerch zeta function by Lithuanian school: Garunkštis (1996), (1997), (1999), Laurinčikas (1997), (1998), (2000), Laurinčikas and Matsumoto (2000).

- This work up to 2002 summarized in book of Laurinčikas and Garunkštis on the Lerch zeta function.

Lerch Zeta Function and Elliptic Curves

• The Lerch zeta function is a Mellin transform of a Jacobi theta function containing its (complex) elliptic curve variable z, viewed as two real variables (a, c). The Mellin transform averages the elliptic curve data over a particular set of moduli.

• **Paradox.** The Lerch zeta function "elliptic curve variables" give it some "additive structure". Yet the variables specialize to a "multiplicative object", the *Riemann zeta function*.

• Is the Lerch zeta function "modular"? This talk asserts that it can be viewed as an automorphic form ("Eisenstein series") on a solvable Lie group. This group falls outside the Langlands program.

Part II. Basic Structures

1. Functional Equation(s).

2. Differential-Difference Equations

3. Linear Partial Differential Equation

4. Integral Representations

5. Three-variable Analytic Continuation

2.1 Four Term Functional Equation-1

- Defn. For real variables $0 < a < 1$ and $0 < c < 1$, set

$$L^{+}(s,a,c) = \sum_{-\infty}^{\infty} \frac{e^{2\pi i n a}}{|n+c|^{s}}, \quad L^{-}(s,a,c) = \sum_{-\infty}^{\infty} \mathrm{sgn}(n+\frac{1}{2}) \frac{e^{2\pi i n a}}{|n+c|^{s}}$$

More precisely,

$$L^{\pm}(s,a,c) := \zeta(\mathbf{s},\mathbf{a},\mathbf{c}) \pm \mathbf{e}^{-2\pi \mathbf{i}\mathbf{a}} \zeta(\mathbf{s}, \mathbf{1}-\mathbf{a}, \mathbf{1}-\mathbf{c}).$$

- Defn. The **completed functions** with *gamma-factors* are:

$$\hat{L}^{+}(s,a,c) := \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})\, L^{+}(s,a,c)$$

and

$$\hat{L}^{-}(s,a,c) := \pi^{-\frac{s+1}{2}} \Gamma(\frac{s+1}{2})\, L^{-}(s,a,c).$$

2.1 Four Term Functional Equation-2

- Theorem (Weil (1976)) Let $0 < a, c < 1$ be real. Then:

  (1) The completed functions $\hat{L}^{+}(s,a,c)$ and $\hat{L}^{-}(s,a,c)$ extend to **entire functions** of s. They satisfy the functional equations

$$\hat{L}^{+}(s,a,c) = e^{-2\pi i a c}\,\hat{L}^{+}(1-s, 1-c, a)$$

  and

$$\hat{L}^{-}(s,a,c) = i\,e^{-2\pi i a c}\,\hat{L}^{-}(1-s, 1-c, a).$$

  (2) These results extend to $a = 0, 1$ and/or $c = 0, 1$. For $a = 0, 1$ then $\hat{L}^{+}(s,a,c)$ is a **meromorphic function** of s, with simple poles at $s = 0, 1$. In all other cases these functions remain **entire functions** of s.

2.1 Functional Equation- Zeta Integrals

- Part I paper obtains a generalized functional equation for Lerch-like zeta integrals depending on a **test function**. (This work is in the spirit of Tate's thesis.)

- These equations relate a integral with test function $f(x)$ at point s to integral with Fourier transform $\widehat{f}(\xi)$ of test function at point $1 - s$.

- The self-dual test function $f_0(x) = e^{-\pi x^2}$ yields the function $\hat{L}^{+}(s, a, c)$. The test function $f_1(x) = xe^{-\pi x^2}$ yields $\frac{1}{\sqrt{2\pi}} \hat{L}^{-}(s, a, c)$. More generally, eigenfuctions $f_n(x)$ of the oscillator representation yield functional equations with **Zeta Polynomials** (local RH of Bump and Ng(1986)).

Functional Equation- Zeta Integrals-2

- An adelic generalization of the Lerch functional equation, also with test functions, was found by my student Hieu T. Ngo in 2014. He uses ideas from Tate's thesis, but his results fall outside that framework.

- His results include generalizations to number fields, to function fields over finite fields, and new zeta integrals for local fields.

2.2 Differential-Difference Equations

- The Lerch zeta function satisfies two differential-difference equations.

- (Raising operator)

$$\frac{\partial}{\partial c}\zeta(s,a,c) = -s\zeta(s+1,a,c).$$

- Lowering operator)

$$\left(\frac{1}{2\pi i}\frac{\partial}{\partial a} + c\right)\zeta(s,a,c) = \zeta(s-1,a,c)$$

- These operators are non-local in the s-variable.

2.3 Linear Partial Differential Equation

- The Lerch zeta function satisfies a linear PDE:

$$\left(\frac{1}{2\pi i}\frac{\partial}{\partial a} + c\right)\frac{\partial}{\partial c}\zeta(s,a,c) = -s\,\zeta(s,a,c).$$

  Set

$$D_L := \frac{1}{2\pi i}\frac{\partial}{\partial a}\frac{\partial}{\partial c} + c\frac{\partial}{\partial c}.$$

- The (formally) skew-adjoint operator

$$\Delta_L := \frac{1}{2\pi i}\frac{\partial}{\partial a}\frac{\partial}{\partial c} + c\frac{\partial}{\partial c} + \frac{1}{2}I$$

  has

$$\Delta_L \zeta(s,a,c) = -(s - \frac{1}{2})\zeta(s,a,c).$$

2.4 Integral Representations-1

• The Lerch zeta function has two different integral representations, generalizing two of the integral representations in Riemann's 1859 paper.

• Riemann's first formula is:

$$\int_0^\infty \frac{e^{-t}}{1-e^{-t}} t^{s-1} dt = \Gamma(s)\zeta(s)$$

• Generalization to **Lerch zeta function** is:

$$\int_0^\infty \frac{e^{-ct}}{1-e^{2\pi i a}e^{-t}} t^{s-1} dt = \Gamma(s)\zeta(s,a,c)$$

2.4 Integral Representations-2

- Riemann's second formula is: (formally)

$$\int_0^\infty \vartheta(0; it^2) t^{s-1} dt \text{ ``}=\text{''}\ \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s),$$

where

$$\vartheta(0;\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}.$$

- Generalization to **Lerch zeta function** is:

$$\int_0^\infty e^{\pi c^2 t^2} \vartheta(a + ict^2, it^2) t^{s-1} dt = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s, a, c).$$

where the Jacobi theta function is

$$\vartheta(z;\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau} e^{2\pi i n z}.$$

2.5 Analytic Continuation

• Paper II (with Winnie Li) showed the Lerch zeta function has an analytic continuation in **three complex variables** (s, a, c). It is an entire function of s but is then **multi-valued** analytic function in the (a, c)-variables.

• Analytic continuation becomes single-valued on the maximal abelian covering of the complex surface $(a, c) \in \mathbb{C} \times \mathbb{C}$ punctured at all integer values of a and c. We explicltly computed the monodromy describing the multivaluedness.

• Paper III (with Winnie Li) extended the analysis to **Lerch transcendent** and polylogarithms. More monodromy occurs.

• In the remainder of this talk we will stick to a, c being **real variables**.

Part III. Two-Variable Hecke Operators

• Recall the role of **Hecke operators** in modular forms on a homogeneous space $\Gamma\backslash\mathbb{H}$, say $\Gamma = PSL(2, \mathbb{Z})$.

• Without defining them exactly, *Hecke correspondences* form an infinite commuting family of discrete "arithmetic" symmetries on such a manifold.

• They correspond to a family of *Hecke operators* acting on functions, which commute with a Laplacian operator, and that can be simultaneously diagonalized to give a basis of simulteous eigenfunctions on spaces of modular forms.

• There are associated L-**functions** that go with these diagonalizations, having **Euler products**. The prime power coefficients are "Hecke eigenvalues."

Two-Variable Hecke Operators

- Paper IV (with Winnie Li) intoduces the two-variable "Hecke operators"

$$T_m(F)(a,c) := \frac{1}{m}\sum_{k=0}^{m-1} F(\frac{a+k}{m}, mc)$$

- These operators dilate in the c-direction while contract and shift in the a-direction.

- Domain of definition: Seems to require at least a half-line in c variable due to dilations. We will view it on domain $\mathbb{R} \times \mathbb{R}$.

Two-Variable Hecke Operators-2

• Consider the restriction to functions constant in the a-direction: then $F(c) = F(a, c)$. Then the operator becomes the *one-variable operator*

$$\mathsf{T}_m(F)(c) = F(mc)$$

• This is the "*dilation operator*". It corresponds to the operator V_m in modular forms which acts on q-expansions as

$$V_m(\sum_n a_n q^n) = \sum_n a_n q^{mn}.$$

Take $q = e^{2\pi i \tau}$ so that $f = \sum_n a_n q^n = \sum_n a_n e^{e\pi i n \tau}$. Then indeed:

$$V_m(f)(\tau) = f(m\tau).$$

Two-Variable Hecke Operators-3

• Consider the restriction to functions constant in the c-direction: $F(a) = F(a,c)$. It becomes one-variable operator

$$\mathsf{T}_m(F)(a) = \frac{1}{m}\sum_{k=0}^{m-1} F(\frac{a+k}{m}).$$

These operators studied under many different names.

• Atkin (1969) called them " *Hecke operators*". Also called "*Atkin operator*" U_m, in modular forms, acts on q-expansions as

$$U_m(\sum_n a_n q^n) = \sum_n a_{mn} q^n.$$

Take $q = e^{2\pi i\tau}$ and $f = \sum_n a_n q^n = \sum_n a_n e^{2\pi i n\tau}$. Then indeed:

$$U_m(f)(\tau) = \frac{1}{m}\sum_{k=0}^{m-1} F(\frac{\tau+k}{m}).$$

Milnor's Theorem for Kubert Functions-1

• In 1983 Milnor proved a result characterizing the Hurwitz zeta function

$$\zeta(s,z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^s}$$

as a simultaneous eigenfunction of "Kubert operators":

$$\mathsf{T}_m(F)(z) = \frac{1}{m} \sum_{j=0}^{m-1} F(\frac{z+k}{m})$$

Here, for $Re(s) > 1$, plus analytic continuation (except $s = 1$)

$$\mathsf{T}_m(\zeta(s,z)) = m^{s-1}\zeta(s,z).$$

Milnor's Theorem for Kubert Functions-2

Theorem. (Milnor (1983)) Let $\mathcal{K}_s$ denote the set of continuous functions $f : (0,1) \to \mathbb{C}$ which satisfy for $m \geq 1$,

$$\mathsf{T}_m f(x) = m^{-s} f(x) \quad \text{for all} \quad x \in (0,1)\ .$$

(1) $\mathcal{K}_s$ is a **two-dimensional** complex vector space and consists of *real-analytic* functions.

(2) $\mathcal{K}_s$ is an invariant subspace for the involution

$$\mathsf{J}_0 f(x) := f(1-x)$$

and decomposes into one-dimensional eigenspaces $\mathcal{K}_s = \mathcal{K}_s^+ \oplus \mathcal{K}_s^-$ spanned by an *even eigenfunction* $f_s^+(x)$ and an *odd eigenfunction* $f_s^-(x)$, respectively, which satisfy

$$\mathsf{J}_0 f_s^\pm(x) = \pm f_s^\pm(x)\ .$$

Milnor's Theorem for Kubert Functions-3

• Milnor gave an explicit basis for $\mathcal{K}_s$, which for $s \neq 0, -1, -2, \ldots$ is given in terms of the (analytic continuation in s of the) Hurwitz zeta function

$$\zeta_s(x) := \zeta(s, 0, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s},$$

namely $\mathcal{K}_s =< \zeta_{1-s}(x), \zeta_{1-s}(1-x) >$. He also gave very interesting basis functions at the exceptional values $s = 0, -1, -2, \ldots$, related to polylogarithms. These s values are "trivial zeros" of the Dedekind zeta function $\zeta_{\mathbb{Q}(i)}(s)$.

• Differentiation $\frac{\partial}{\partial x}$ maps $\mathcal{K}_s$ to $\mathcal{K}_{s-1}$, acting as a "lowering operator". Because Kubert operators are contracting, this fact suffices for Milnor's proof.

Two-Variable Hecke Operators-4

• We establish a generalization to the Lerch zeta function. (A variable change requires replacing m^{-s} by m^{s-1}.)

• We consider a special class of functions $F(a,c)$ on $\mathbb{R} \times \mathbb{R}$: those satisfying **Twisted Periodicity** for the lattice $\mathbb{Z} \times \mathbb{Z}$.

$$F(a+1,c) \;=\; F(a,c)$$

$$F(a,c+1) = e^{-2\pi i a} F(a,c).$$

• For **twisted periodic** functions: Given any values $F(a,c)$ on the open unit square $\square^{\circ} = \{(a,c):\; 0 < a < 1, 0 < c < 1\}$, twisted periodicity extends it uniquely to $\mathbb{R} \times \mathbb{R}$ (off a set of measure zero), so two-variable Hecke operators become well defined.

Two-Variable Hecke Operators-5

• **Proposition A.** (1) *If $F(a,c)$ is twisted-periodic on $\square^{\circ}$, then $\mathsf{T}_m(F)(a,c)$ is twisted-periodic for all $m \geq 1$.*

(2) Acting on the space of twisted-periodic functions (allowing linear discontinuities) the two-variable Hecke operators $\{\mathsf{T}_m : m \geq 1\}$ form a commuting family of operators.

• **Proposition B.** (1) *For fixed s with $s \in \mathbb{C}$ the Lerch zeta function on $\square^{\circ}$ is "naturally" extendable to be* **twisted-periodic.**

(2) **Lerch zeta function** $\zeta(s,a,c)$ *is then a simultaneous eigenfunction of two-variable Hecke operators,*

$$\mathsf{T}_m(\zeta)(s,a,c) = m^{-s}\zeta(s,a,c)$$

(There will be discontinuities at integer a, c when $Re(s) < 1$.)

R-operator and J-operator-1

- Throw in some additional operators on functions on $\square^{\circ}$, use them to define additional two-variable Hecke operators.

- The R-**operator** is defined on functions with domain $\square^{\circ}$ by:

$$\mathsf{R}(F)(a,c) := e^{-2\pi i a c} F(1-c,a).$$

It is an operator of order 4, i.e $\mathsf{R}^4 = I$.

- The J-**operator** is $\mathsf{J} = \mathsf{R}^2$. It is given by

$$\mathsf{J}(F)(a,c) = e^{-2\pi i a} F(1-a,1-c).$$

R-operator -2

• **Observation.** The Lerch functional equation(s) can be put in a nice form using the R-*operator*, as

$$\hat{L}^{+}(s,a,c) = \mathsf{R}(\hat{L}^{+})(1-s,a,c).$$

and

$$\hat{L}^{-}(s,a,c) = i\mathsf{R}(\hat{L}^{-})(1-s,a,c).$$

(This happens because the R-operator intertwines with the Fourier transform on a suitable space.)

• **Lemma.** *The* R-*operator acting on functions in the Hilbert space* $L^2(\Box^\circ)$ *is a unitary operator.*

R-operator and Two-variable Hecke ops.

The operator R does **not commute** with the two-variable Hecke operators T_m. It generates three new families of two-variable Hecke operators under conjugation:

$$\mathsf{S}_m := \mathsf{R}\mathsf{T}_m\mathsf{R}^{-1}\ ,\ \mathsf{T}_m^{\vee} := \mathsf{R}^2\mathsf{T}_m\mathsf{R}^{-2},\quad \text{and}\quad \mathsf{S}_m^{\vee} := \mathsf{R}^3\mathsf{T}_m\mathsf{R}^{-3},$$

These are:

$$\begin{aligned}
\mathsf{S}_m f(a,c) &= \frac{1}{m}\sum_{k=0}^{m-1} e^{2\pi i k a} f\left(ma, \frac{c+k}{m}\right), \\
\mathsf{T}_m^{\vee} f(a,c) &= \frac{1}{m}\sum_{k=0}^{m-1} e^{2\pi i (\frac{(1-m)a+k}{m})} f\left(\frac{a+k}{m}, 1+m(c-1)\right), \\
\mathsf{S}_m^{\vee} f(a,c) &= \frac{1}{m}\sum_{k=0}^{m-1} e^{2\pi i (m-(k+1))a} f\left(1+m(a-1), \frac{c+m-(k+1)}{m}\right).
\end{aligned}$$

Commuting Two-variable Hecke operators-1

Theorem 1. (Commuting Operator Families-1)
(1) The four sets of two variable Hecke operators $\{\mathsf{T}_m, \mathsf{S}_m, \mathsf{T}_m^{\vee}, \mathsf{S}_m^{\vee} : m \geq 1\}$ *continuously extend to bounded operators on each Banach space* $L^p(\square, da\, dc)$ *for* $1 \leq p \leq \infty$*.*

(Here we view functions on $\square$ as extended to $\mathbb{R} \times \mathbb{R}$ via **twisted-periodicity**.)

(2) These operators satisfy $\mathsf{T}_m = \mathsf{T}_m^{\vee}$*,* $\mathsf{S}_m = \mathsf{S}_m^{\vee}$ *and* $\mathsf{S}_m = \frac{1}{m}(\mathsf{T}_m)^{-1}$ *for all* $m \geq 1$*.*

Commuting Two-variable Hecke operators-2

Theorem 1. (continued)

(3) The $\mathbb{C}$-algebra $\mathcal{A}_0^p$ of operators on $L^p(\Box, da\, dc)$ generated by **all four sets of operators** $\{\mathsf{T}_m, \mathsf{S}_m, \mathsf{T}_m^\vee, \mathsf{S}_m^\vee : m \geq 1\}$ under addition and operator multiplication is **commutative.**

(4) On the Hilbert space $L^2(\Box, da\, dc)$ the **adjoint Hecke operator** $(\mathsf{T}_m)^* = \mathsf{S}_m$, and $(\mathsf{S}_m)^* = \mathsf{T}_m$. In particular the $\mathbb{C}$-algebra $\mathcal{A}_0^2$ is a $\star$-algebra.

(5) On the Hilbert space $L^2(\Box, da\, dc)$ each rescaled operator $\sqrt{m}\mathsf{T}_m, \sqrt{m}\mathsf{S}_m$ is a **unitary operator** on $L^2(\Box, da\, dc)$.

Lerch eigenspace

• For fixed $s \in \mathbb{C}$ the **Lerch eigenspace** $\mathcal{E}_s$ is the vector space over $\mathbb{C}$ spanned by the four functions

$$\begin{aligned}\mathcal{E}_s \;:=\; & < L^+(s,a,c), \quad L^-(s,a,c), \\ & \quad e^{-2\pi iac}L^+(1-s,1-c,a), \quad e^{-2\pi iac}L^-(1-s,1-c,a) >,\end{aligned}$$

viewing the (a,c)-variables on $\square^\circ$.

• The gamma factors are omitted from functions in this definition, since s is constant. These functions satisfy linear dependencies by virtue of the two functional equations that $L^{\pm}(s,a,c)$ satisfy. The resulting space is **two-dimensional**.

Operators on Lerch eigenspaces-1

Theorem 2. (Operators on Lerch eigenspaces-1)

(1) For each $s \in \mathbb{C}$ the space $\mathcal{E}_s$ is a **two-dimensional** *vector space.*

(2) All functions in $\mathcal{E}_s$ have the following four properties.

(i) (Lerch differential operator eigenfunctions) *Each $f \in \mathcal{E}_s$ is an eigenfunction of the Lerch differential operator $\mathsf{D}_L = \frac{1}{2\pi i}\frac{\partial}{\partial a}\frac{\partial}{\partial c} + c\frac{\partial}{\partial c}$ with eigenvalue $-s$, namely*

$$(\mathsf{D}_L f)(s,a,c) = -sf(s,a,c)$$

holds at all $(a,c) \in \mathbb{R} \times \mathbb{R}$, with both a and c non-integers.

Operators on Lerch eigenspaces-2

Theorem 2. (Operators on Lerch eigenspaces-2)

(ii) *(Simultaneous Hecke operator eigenfunctions)* Each $f \in \mathcal{E}_s$ is a simultaneous eigenfunction with eigenvalue m^{-s} of all two-variable Hecke operators

$$\mathsf{T}_m(f)(a,c) = \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{a+k}{m}, mc\right)$$

in the sense that, for each $m \geq 1$,

$$\mathsf{T}_m f = m^{-s} f$$

holds on the domain $(\mathbb{R} \smallsetminus \frac{1}{m}\mathbb{Z}) \times (\mathbb{R} \smallsetminus \mathbb{Z})$.

Operators on Lerch eigenspaces-3

Theorem 2 (Operators on Lerch eigenspaces-3)

(iii) (J-*operator eigenfunctions*) The space $\mathcal{E}_s$ admits the involution

$$\mathsf{J}f(a,c) := e^{-2\pi i a} f(1-a, 1-c),$$

under which it decomposes into one-dimensional eigenspaces $\mathcal{E}_s = \mathcal{E}_s^{+} \oplus \mathcal{E}_s^{-}$ with eigenvalues ± 1, that is, $\mathcal{E}_s^{\pm} =< F_s^{\pm} >$ and

$$\mathsf{J}(F_s^{\pm}) = \pm F_s^{\pm}.$$

(iv) *(R-operator action)* The R-operator acts: $\mathsf{R}(\mathcal{E}_s) = \mathcal{E}_{1-s}$

$$\mathsf{R}(L^{\pm}(s,a,c)) = w_{\pm}^{-1} \gamma^{\pm}(1-s) L^{\pm}(1-s,a,c),$$

where $w_{+} = 1$, $w_{-} = i$, $\gamma^{+}(s) = \Gamma_{\mathbb{R}}(s)/\Gamma_{\mathbb{R}}(1-s)$, and $\gamma^{-}(s) = \gamma^{+}(s+1)$, and $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$.

Analytic Properties of Lerch eigenspaces-1

Theorem 3. *(1) For $s \in \mathbb{C}$ the functions in the Lerch eigenspace $\mathcal{E}_s$ are* **real-analytic** *functions of $(a, c) \in (\mathbb{R} \smallsetminus \mathbb{Z}) \times (\mathbb{R} \smallsetminus \mathbb{Z})$, which may be discontinuous at values $a, c \in \mathbb{Z}$.*

(2) In addition they have properties:

(i) (Twisted-Periodicity Property) All functions $F(a, c)$ in $\mathcal{E}_s$ satisfy the twisted-periodicity functional equations

$$\begin{aligned} F(a+1,\ c\ ) &= F(a,c), \\ F(\ a\ ,c+1) &= e^{-2\pi i a} F(a,c). \end{aligned}$$

Analytic Properties of Lerch eigenspaces-2

Theorem 3. (continued) (ii) *(Integrability Properties)* (a)-(b)
(a) If $\Re(s) > 0$, then for each $0 < c < 1$ all functions in $\mathcal{E}_s$ have $f_c(a) := F(a, c) \in L^1[(0, 1), da]$, and all their Fourier coefficients

$$f_n(c) := \int_0^1 F(a, c) e^{-2\pi i n a} da, \qquad n \in \mathbb{Z},$$

are continuous functions of c on $0 < c < 1$.

(b) If $\Re(s) < 1$, then for each $0 < a < 1$ all functions in $\mathcal{E}_s$ have $g_a(c) := e^{2\pi i a c} F(a, c) \in L^1[(0, 1), dc]$, and all Fourier coefficients

$$g_n(a) := \int_0^1 e^{2\pi i a c} F(a, c) e^{-2\pi i n c} dc, \qquad n \in \mathbb{Z},$$

are continuous functions of a on $0 < a < 1$.

Generalized Milnor Converse Theorem

Theorem 4. (Lerch Eigenspace Converse Theorem) *Let* $s \in \mathbb{C}$ *be fixed. Suppose that* $F(a, c) : (\mathbb{R} \smallsetminus \mathbb{Z}) \times (\mathbb{R} \smallsetminus \mathbb{Z}) \to \mathbb{C}$ *is a continuous function that satisfies the following three conditions.*

- (Twisted-Periodicity Condition) $F(a, c)$ *is twisted periodic.*

- (Integrability Condition) L^1*-condition on* a *or on* c *depending on* $Re(s) < 1$ *or* $Re(s) > 0$*, as in Theorem 3.*

- (Hecke Eigenfunction Condition) *For all* $m \geq 1$,

$$\mathsf{T}_m(F)(a, c) = m^{-s} F(a, c)$$

Then $F(a, c)$ *is the restriction to noninteger* (a, c)*-values of a function in the Lerch eigenspace* $\mathcal{E}_s$.

V. Lerch Zeta Function and Heisenberg Group

• The **Heisenberg group** $Heis(\mathbb{R})$ is a 3-dimensional real Lie group. It has a one-dimensional center given by z-variable.

• Abstract group law (parameter $\lambda \in \mathbb{R}$)

$$[x_1, y_1, z_1]_\lambda \circ [x_2, y_2, z_2]_\lambda :=$$

$$[x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2 + \lambda x_1 y_2 + (1 - \lambda) y_1 x_2]_\lambda .$$

• 3×3 Matrix group Representations (for $\lambda = 0, 1$ only.)

$$[x, y, z]_0 = \begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [x, y, z]_1 = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Heisenberg Group-2

- 4×4 Matrix group representation (general case)

$$[x, y, z]_\lambda = \begin{bmatrix} 1 & x & y & z \\ 0 & 1 & 0 & \lambda y \\ 0 & 0 & 1 & -(1-\lambda)x \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- Maximally symmetric case ($\lambda = \frac{1}{2}$)

$$[p, q, z]_{1/2} =: \begin{bmatrix} 1 & p & q & 2z \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & -p \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Sub-Jacobi Group

- The actual group H^J involved with the Lerch zeta function is a semidirect product of $GL(1)$ with the real Heisenberg group, which we call the sub-Jacobi group. The Lerch integral representation is a Mellin transform integrating over the characters of the $GL(1)$-action, so the three Heisenberg variables (a, c, z) remain as parameters in the resulting integral. The dependence on z is simple, so can be omitted.

- $GL(1)$ is **not** in the center of the sub-Jacobi group.

- One particular (asymmetric) matrix representation of H^J is:

$$[c, a, z, t] = \begin{bmatrix} 1 & c & a & z \\ 0 & t & 0 & ta \\ 0 & 0 & \frac{1}{t} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Representation Theory of the Heisenberg Group

• The *Heisenberg group* $\mathcal{N} = Heis(\mathbb{R})$ is one of the eight three-dimensional geometries of Thurston, called Nil.

• For each real $\lambda \neq 0$, there exists a unique infinite dimensional irreducible representation $\mathbf{P}_\lambda$ on which the central character takes value $\chi([0, 0, z]) = e^{2\pi i \lambda z}$. For the parameter $\lambda = 0$, the central character is trivial, and there are an uncountable number of 1-dimensional representations $\mathbf{J}_{\mu_1,\mu_2}$, parametrized by $(\mu_1, \mu_2) \in \mathbb{R}^2$, with $\chi([a, c, 0]) = e^{2\pi i(\mu_1 a+\mu_2 c)}$.

• All the infinite-dimensional irreducible representations $\mathbf{P}_\lambda$ are the "same" in that is an automorphism of $Heis(\mathbb{R})$ taking the representation $\mathbf{P}_\lambda$ to $\mathbf{P}_1$. The value of λ is "Planck's constant."

Schrödinger Representationof the Heisenberg Group

• This is a model of the infinite dimensional irreducible unitary representation $\mathbf{P}_1$ of $Heis(\mathbb{R})$. It acts on $L^2(\mathbb{R}, dx)$. (Use the model $[a, c, z]_0$ for $Heis(\mathbb{R})$.)

• Translation $[0, c, 0]$: $f(x) \mapsto f(x + c)$.

• Modulation: $[a, 0, 0]$ $f(x) \mapsto e^{2\pi i a x} f(x)$.

• Center $[0, 0, z]$: $f(x) \mapsto e^{2\pi i z} f(x)$

• Translation and Modulation satisfy **canonical commutation relations** (up to a scaling).

• **Stone-von Neumann theorem.** Repesentation is unique unitary irreducible representation with given central character.

Representation Theory of the Heisenberg Nilmanifold

• The *Heisenberg nilmanifold* $\mathcal{N} = Heis(\mathbb{Z})\backslash Heis(\mathbb{R})$. It is a compact manifold of volume 1 with respect to Haar measure $dadcdz$ on $Heis(\mathbb{R})$. The manifold $\mathcal{N}$ is a homogeneous space for the action of the Heisenberg group.

• The space $L^2(Heis(\mathbb{Z})\backslash Heis(\mathbb{R}))$ can be decomposed under the (right) Heisenberg action as

$$L^2(Heis(\mathbb{Z})\backslash Heis(\mathbb{R})) = \oplus_{N \in \mathbb{Z}} \mathcal{H}_N,$$

in which:

(1) For $N \neq 0$ the space $\mathcal{H}_N$ consists of $|N|$ copies of the infinite-diml. repn. $\mathbf{P}_N$ having central character $e^{2\pi i N z}$.

(2) For $N = 0$ the space $\mathcal{H}_0 = \oplus_{(m_1,m_2)\in\mathbb{Z}^2} \mathbf{J}_{m_1,m_2}$.

Representation Theory of the Heisenberg Nilmanifold-2

• **Fact.** All the infinite-dimensional spaces $\mathcal{H}_N$ with $N \neq 0$ carry a nontrivial action of the sub-Jacobi group.

• The space $\mathcal{H}_0$ does not carry such an action, but we show carries a discrete remnant of this action from analogues of **two-variable Hecke operators** (defined on next slide)

• All the spaces $\mathcal{H}_N$ also carry an action of a *"Laplacian" operator*, which is left-invariant but not two-sided invariant. It is not in the center of the universal enveloping algebra of $Heis(\mathbb{R})$. This operator is

$$\Delta := \frac{1}{2\pi i}\left(\frac{\partial}{\partial a}\frac{\partial}{\partial c} + c\frac{\partial}{\partial c}\frac{\partial}{\partial z}\right)$$

Two-Variable Hecke Operators

• There are two-variable Hecke operators given by

$$\mathsf{T}_m(F)(a,c,z) := \frac{1}{|m|}\sum_{j=0}^{|m|-1} F\left(\frac{a+j}{m}, mc, z\right) \ ,$$

• Here $\mathsf{T}_m : C^0_{bdd}(Heis(\mathbb{R})) \to C^0_{bdd}(Heis(\mathbb{R}))$, and the two variables in the operator name refer to variables (a,c), noting that the action on the z-variable is rather simple.

Symmetrized Lerch Zeta Functions as Eisenstein Series

Theorem. *(1)The two symmetrized Lerch-zeta functions*

$$L^{\pm}(s,a,c) = \zeta(s,a,c) \pm e^{2\pi i a}\zeta(s,1-a,1-c)$$

are "Eisenstein series" for the real Heisenberg group $H(\mathbb{R})$ with respect to the discrete subgroup given by the integer Heisenberg group $H(\mathbb{Z})$.

• **Eisenstein series** (in the theory of reductive Lie groups) are generalized eigenfunctions a "Laplacian" operator having pure continuous spectrum, which in arithmetic (adelic) contexts are also simultaneous eigenfunctions of a family of "Hecke operators".

Symmetrized Lerch Zeta Functions as Eisenstein Series-2

Theorem. (continued)

(2) The two functions $L^{\pm}(s, a, c)$ *form a family of eigenfunctions in the* s*-parameter with eigenvalue* $s - \frac{1}{2}$ *with respect to a "Laplacian operator"* $\Delta_L = \frac{1}{2\pi i}\frac{\partial}{\partial a}\frac{\partial}{\partial c} + c\frac{\partial}{\partial c} + \frac{1}{2}$*. The operator* Δ_L *defines a left-invariant vector field on* $H(\mathbb{R})$*, and acts on the Hilbert space* $\mathcal{H}_1$ *of the Schrödinger representation of* $Heis(\mathbb{R})$*. It is specified with a dense domain* $\mathcal{W}(\mathcal{D}_{1,1})$ *with respect to which it is skew-adjoint. It has pure continuous spectum for* Δ_L *with spectral measure =Lebesgue.*

Lerch L-Functions

• The continuous spectrum result extends to all the spaces $\mathcal{H}_N$ for all $N \geq 0$ as follows:

• Let χ be a (primitive or imprimitive) Dirichlet character $(\text{mod}\, d)$ with d dividing N, the *level* of the Heisenberg representation. The associated **Lerch L-function** is:

$$L^{\pm}_{N,d}(\chi, s, a, c) := \sum_{n \in \mathbb{Z}} \chi(\frac{nd}{N})(\text{sgn}(n + Nc))^k e^{2\pi i n a}|n + Nc|^{-s},$$

in which $(-1)^k - = \pm$ with $k = 0$ or 1.

• The Lerch L-functions for fixed parameter s on the critical line $Re(s) = \frac{1}{2}$ are (continuous spectrum) eigenfunctions of the operator Δ, for those parts of the space $\mathcal{H}_N$, when $N \neq 0$.

Weil-Brezin Transform-1

• Method of proof: Weil-Brezin transforms.

• The *Weil-Brezin map* $\mathcal{W} : L^2(\mathbb{R}, dx) \to \mathcal{H}_1$ is defined for Schwartz functions $f \in \mathcal{S}(\mathbb{R})$ by

$$\mathcal{W}(f)(a, c, z) := e^{2\pi iz} \left(\sum_{n \in \mathbb{Z}} f(n + c) e^{2\pi ina} \right) .$$

Under Hilbert space completion this map extends to an isometry of Hilbert spaces.

• The Weil-Brezin image of the scaled Gaussian function $f_t(x) = e^{-\pi tx^2}$ is closely related to a Jacobi theta function. Image is $\theta_t(a, c, z) := e^{2\pi iz} e^{-\pi tc^2} \vartheta_3(it, a + ict)$. For $|x|^s$ (not a Schwartz function) the image is Lerch L-function.

Weil-Brezin Transform-2

• The Weil-Brezin transform takes the Schrödinger representation on $L^2(\mathbb{R}, dx)$ to $\mathcal{H}_1 \subset L^2(Heis(\mathbb{R}), da\, dc\, dz)$.

• The Hilbert space $L^2(\mathbb{R}, dx)$ carries a dilation action $D_\lambda(f(x)) = |x|^{-1/2} f(\lambda x)$. This action together with the Heisenberg action on $L^2(\mathbb{R}, dx)$ given by the Schrödinger representation gives a sub-Jacobi group H^J representation on this space.

• The differential operator corresponding to the infinitesimal dilation action is $x\frac{d}{dx} + \frac{1}{2}$. Under the Weil-Brezin map this operator maps to $\Delta_L = \frac{1}{2\pi i}\frac{\partial}{\partial a}\frac{\partial}{\partial c} + c\frac{\partial}{\partial c} + \frac{1}{2}$ on $\mathcal{H}_1$.

Weil-Brezin Transform-3

• The Weil-Brezin map intertwines the *Fourier transform* on $L^2(\mathbb{R}, dx)$ with the Heisenberg analogue of the R-operator on the space $\mathcal{H}_1$.

• The analogue of the R-operator for all spaces $\mathcal{H}_N$ is

$$\mathsf{R}(F)([a,c,z]) := F([-c,a,z-Nac]).$$

Weil-Brezin Transform-4

• There are generalizations of the Weil-Brezin map to all infinite-dimensional irreducible representations of the Heisenberg nilmanifold.

• The *twisted Weil-Brezin map* $\mathcal{W}_{N,d}(\chi) : L^2(\mathbb{R}, dx) \to \mathcal{H}_{N,d}(\chi)$ is defined for Schwartz functions $f \in \mathcal{S}(\mathbb{R})$ by

$$\mathcal{W}_{N,d}(\chi)(f)(a,c,z) := \sqrt{C_{N,d}} e^{2\pi i N z} \sum_{n \in \mathbb{Z}} \chi\left(\frac{nd}{N}\right) f(n + Nc) e^{2\pi i n a}$$

in which we set $\chi(r) := 0$ if $r \notin \mathbb{Z}$, and

$$C_{N,d} := \frac{N}{\phi(d)}$$

is a normalizing factor. (Note also that $\chi(r) = 0$ for those $r \in \mathbb{Z}$ having $(r, d) > 1$.)

Weil-Brezin map-5

• **Proposition C.** *The twisted Weil-Brezin map* $\mathcal{W}_{N,d}(\chi) : \mathcal{S}(\mathbb{R}) \to C_\infty(\mathcal{H}_N)$ *extends to a Hilbert space isometry*

$$\mathcal{W}_{N,d}(\chi) : L^2(\mathbb{R}, dx) \longrightarrow \mathcal{H}_{N,d}(\chi) \subseteq \mathcal{H}_N$$

whose range $\mathcal{H}_{N,d}(\chi)$ *is a closed subspace of* $\mathcal{H}_N$*. The Hilbert space* $\mathcal{H}_{N,d}(\chi)$ *is invariant under the action of* $Heis(\mathbb{R})$*.*

Concluding Remarks

• There are many more details relating to the spectral decomposition of $L^2(Heis(\mathbb{Z})\backslash Heis(\mathbb{R}))$ away from $\mathcal{H}_0$.

(1) In the Heisenberg group interpretation, the special case $N = 1$ corresponds to results with Winnie Li on the Lerch zeta function.

(2) The "Laplacian" spectrum is pure continuous on all $\mathcal{H}_N$, $N \neq 0$. (But on $\mathcal{H}_0$ it is discrete.)

(3) All Dirichlet characters occur, primitive and imprimitive, in infinitely many levels N. (All characters of $GL(1, \mathbb{Q})$ occur.)

Thank you for your attention!