Dürer:
Polyhedra and Shadows

Jeff Lagarias,
University of Michigan

University of Michigan
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- Conference: **500 Years of Melancholia in Mathematics**

- NYU Polytechnic, Brooklyn, NY

- Organizers: David and Gregory Chudnovsky

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Credits

Masud Hasan and Anna Lubiw, Equiprojective Polyhedra, Computational Geometry **40** (2008), 148–155.
+ (related talk slides)

M. Hasan, M. N. Hossain, A. Lopez-Ortiz, S. Nusrat, A. Quader, N. Rahman, Some new equiprojective polyhedra, arXiv:1009.2252

J. C. Lagarias and Yusheng Luo, Moser's shadow problem, arXiv:1310.4345

Table of Contents

Albrecht Dürer-1

- Albrecht Dürer (1471–1528) born in Imperial Free City of Nürnberg
- Hungarian: Ajtós= “door”. Germanized to: Thürer. Father was a jeweler.
- Learned trade of goldsmith and jeweler, apprenticed 1486 to Michel Wolgemut, painter and woodcut designer.
- Four years travel starting 1490 to Ulm, Konstanz, Basel, Colmar, Strasbourg, returned home 1494. Married Agnes Frey.

Dürer-2

- Second trip abroad 1494–1495 to Augsburg, Trento, Venice. Dürer's first trip to Italy. Met painter and printmaker Jacopo de' Barberi in Venice.

- Studied mathematics books 1495 in Nürnberg:
  Euclid's *Elements* ,
  Vitruvius on architecture,
  Leone Battista Alberti (1404–1472) on laws of perspective,
  Luca Pacioli (1445–1517) on theory of proportion.

- Second trip to Italy 1505-1507, Venice. Lent money for trip by lifelong friend Willibald Pirckheimer (1470-1530). Dürer was now famous. Possible side trip to meet Luca Pacioli.

Jacopo de' Barbari (1460/1470-before 1516)

- Jacopo de' Barbari, Italian painter, engraver and printmaker. A portrait of mathematician Fra Luca Pacioli, dated 1495, attributed to him.

- Famous for large woodcut view of Venice viewed from the air, taking three years to cut in late 1490's, publisher patent dated 1500.

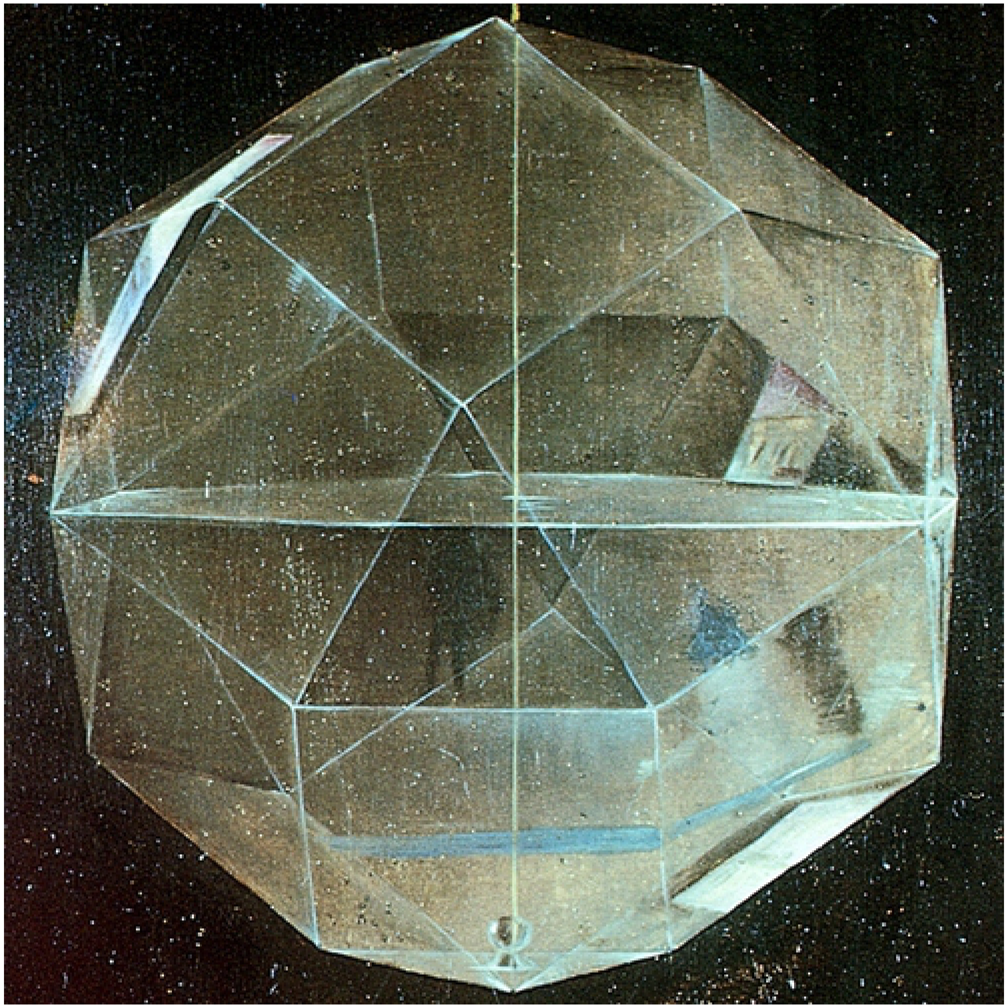
- He emigrated to Germany in 1500. Was in Nürnberg 1500–1501, met Dürer there. Later worked in Netherlands. Pension from Archduchess Margaret of Brussels 1611.

Luca Pacioli (1445–1517)

- Fra Luca Bartolomeo de' Pacioli, a Franciscan friar, wrote 600 page book, *Summa de arithmetica, geometrica, proportioni et proportionlitá*, Venice 1494. First written treatment of double entry bookkeeping. The book incorporated work of Piero della Francisca (ca 1415–1492).

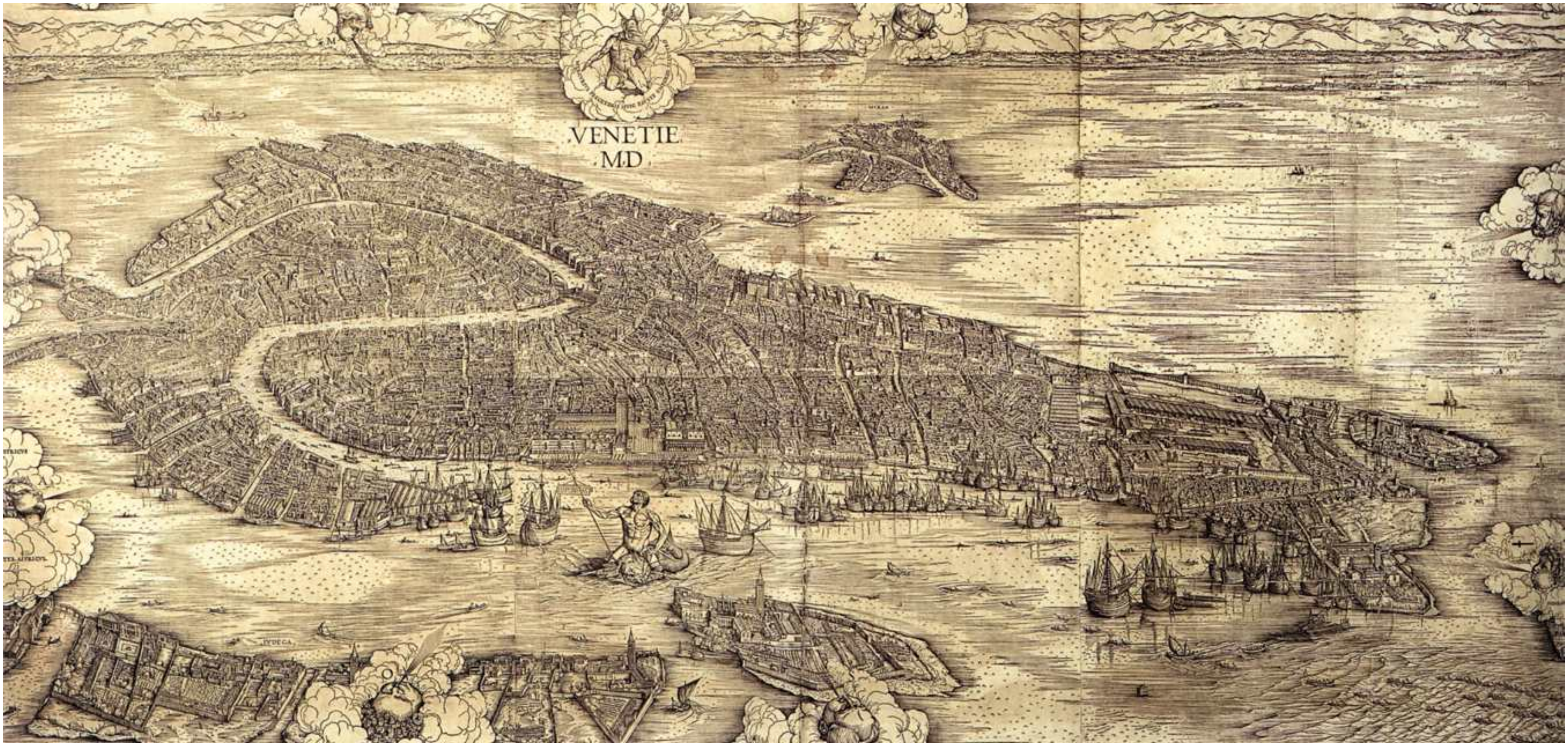
- Pacioli knew of archimedean polyhdedra, had models constructed in wood and glass.

- He was in Milan, 1497–1499, Florence 1499-1506, instructing Leonardo da Vinci on mathematics. Book: *De Divina Proportione*, written 1490's, published 1509 Venice. Illustrated by Leonardo da Vinci, many polyhedra.



Dürer self portrait(1500)

de' Barbari Map of Venice



Dürer-3

- In 1508 Dürer started collecting materials for mathematics and applications to arts.
  Viewpoint: art must be based upon science-in particular, upon mathematics, as the most exact logical and graphically constructive of the sciences

- Work for Holy Roman emperor Maximilian I, starting 1512. Made emperor's portrait in 1518.

- In 1513–1514 produced three famous engravings, including "Melencolia I"

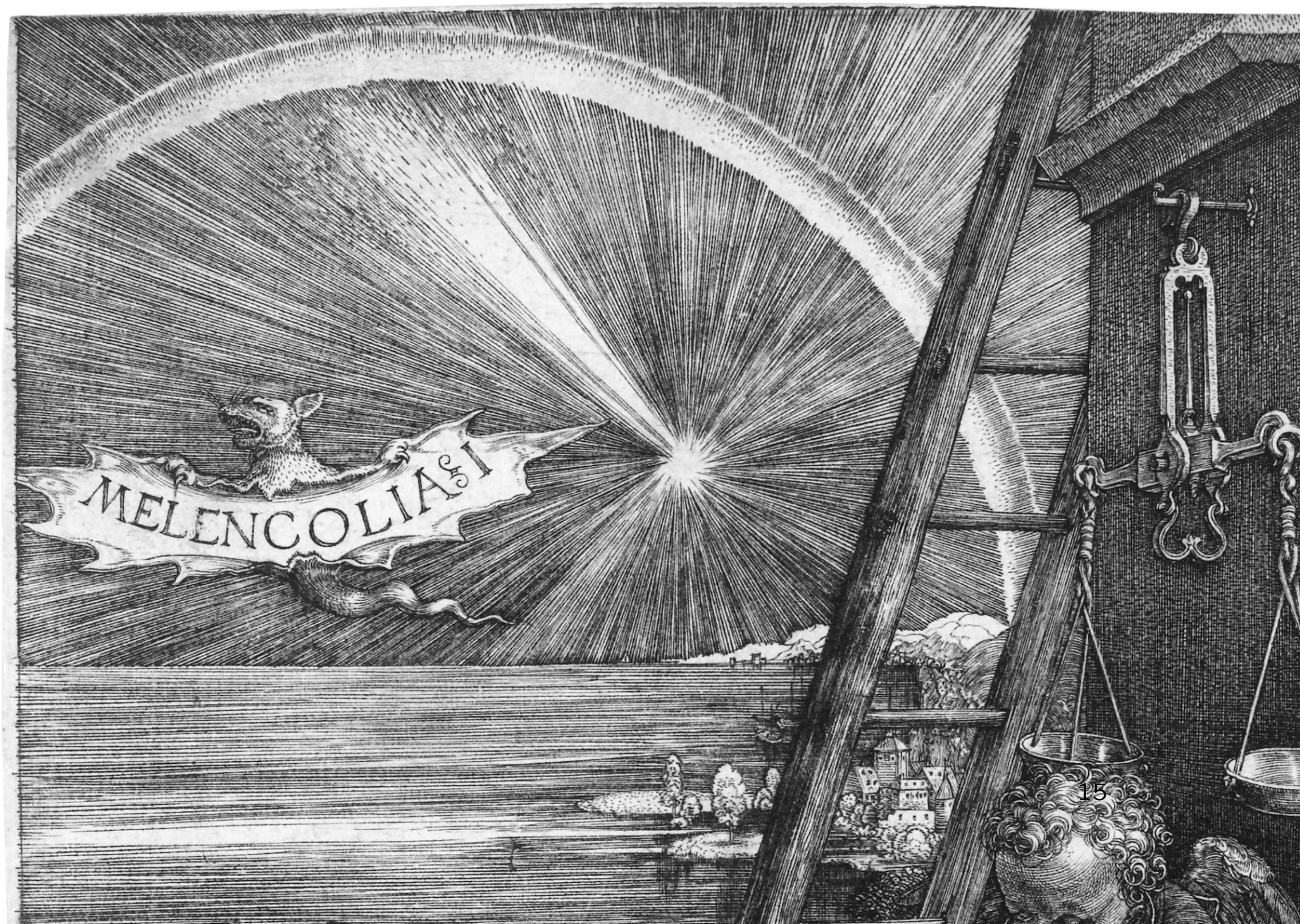
Dürer-4: Melencolia I

- The engraving is packed full of images of significance. There is an angel– contemplative or brooding? There is a putto: a secular symbol, often of a passion.

- There is a bat with a slogan, and an emaciated dog. A comet and a rainbow.

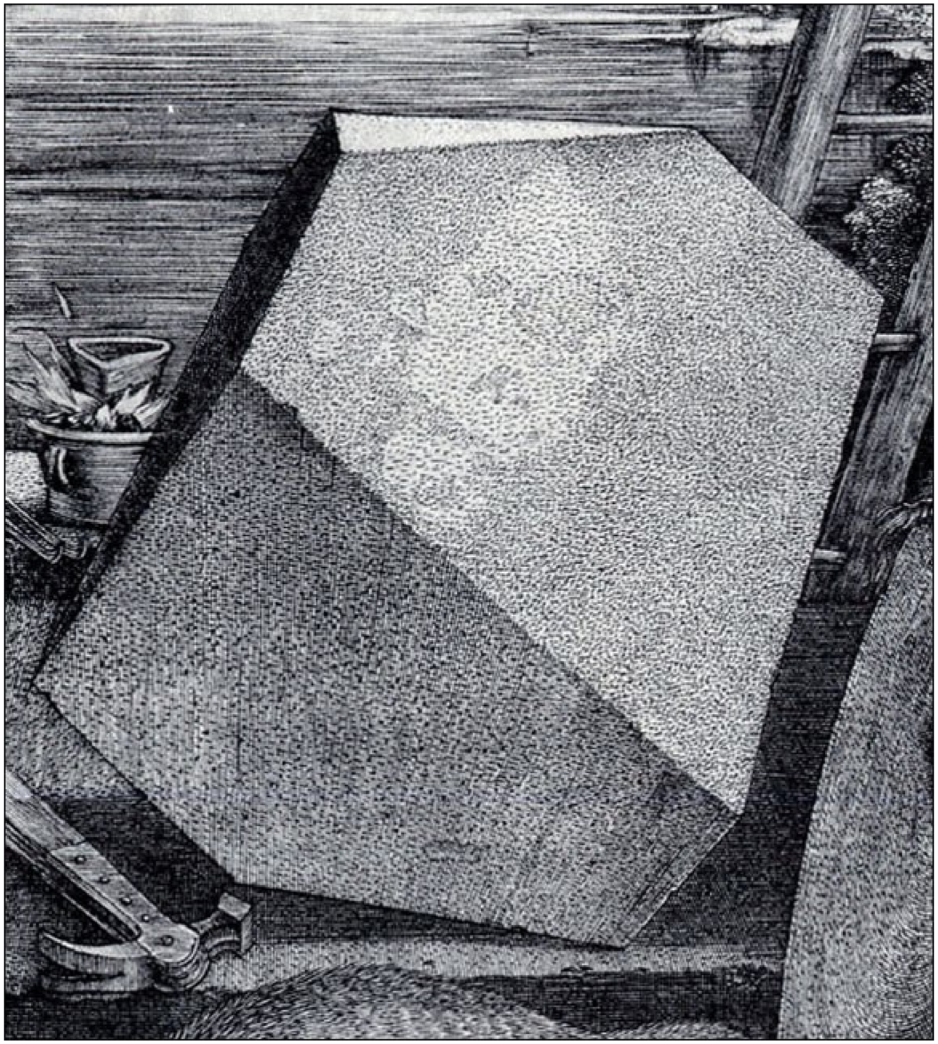
- There is a large polyhedron. It appears approximately a cube with two opposite corners cut off; two triangular faces, six pentagonal faces. (But not exactly so.)

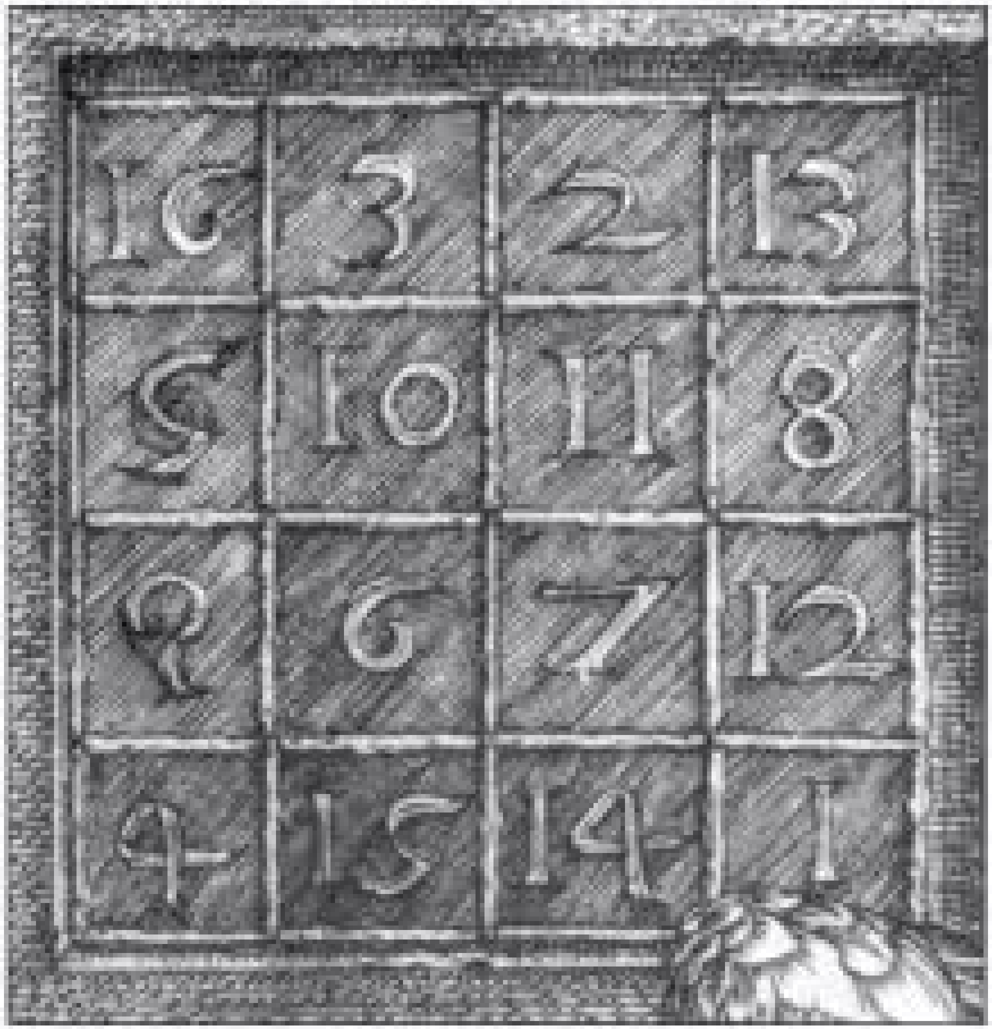
- There is a magic square.


MELENCOLIA § I

Putto 


courtesy of www.albrecht-durer.org

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Melancholy of Cornelius Agrippa-1

- Cornelius Agrippa (Heinrich Cornelius Agrippa von Netteshelm)(1486–1535) a German occult writer, astrologer and alchemist.

  [“occult” = hidden from view]

- Agrippa studied in 1510 with Johannes Trithemius (1462–1516), Benedictine abbot and cryptographer.

- Agrippa’s manuscript of magic and alchemy “De occulta philosophia libri tres”, copies circulated in 1510. Rewitten many times, published 1531.

Melancholy of Cornelius Agrippa-2

- Several kinds of Melancholy: *inspired melancholy* comes in three grades: *of imagination*, *of reason*, and *of intellect*. "Melencolia I" may signify the first of these.

- Erwin Panofsky: engraving expresses the frustration of creation, suffering at being unable to express its visions, melancholy inactivity

- Dame Frances Yates: The angel in Melencolia I is in an intense visionary trance of creativity.

Dürer-5

- Trip to Netherlands 1520, to collect a pension, and search for de' Barbari manuscript. In ill health on return to Nürnberg 1520. Later work: students, writing, publishing.

- Book: *Unterweisung der Messung mit dem Zirkel und Richtscheit,* 1525

  [First German book on mathematics in vernacular.]

- Book: On fortifications 1527.

- Book: Four books on human proportion, in proofsheets at his death in 1528.

Dürer-6: Book "Unterweisung" 1525

- Book 1. Construction of curves: Archimedes spiral, logarithmic spiral, conchoid

- Book 2. Construction of regular polygons, with cutout models, trisects the angle (approximately)

- Book 3. Pyramids, cylinders, solid bodies, sundials

- Book 4. Five Platonic solids, semi-regular polyhedra. Theory of shadows and of perspective.

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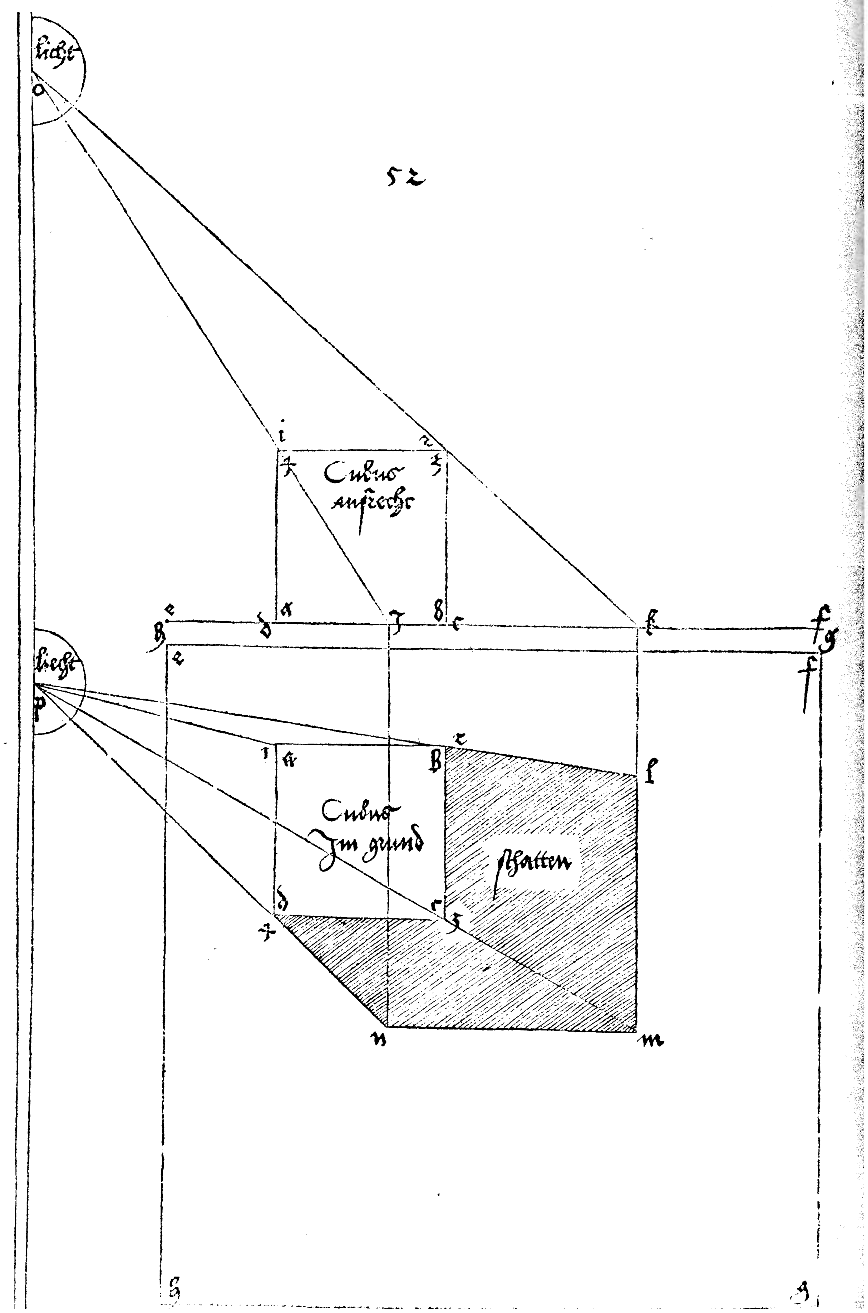

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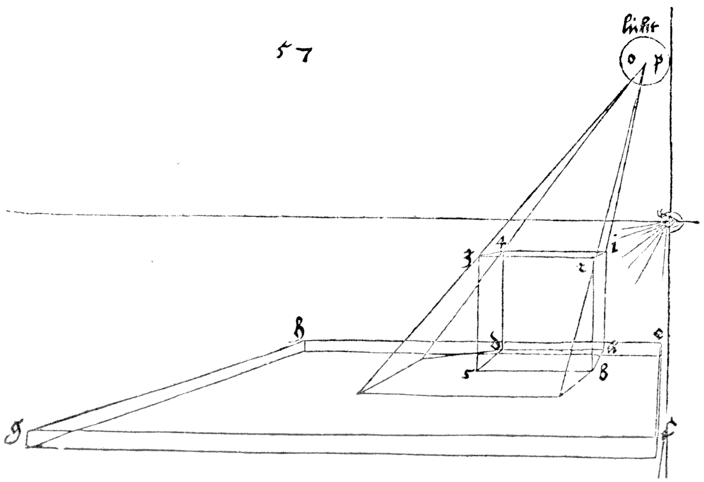

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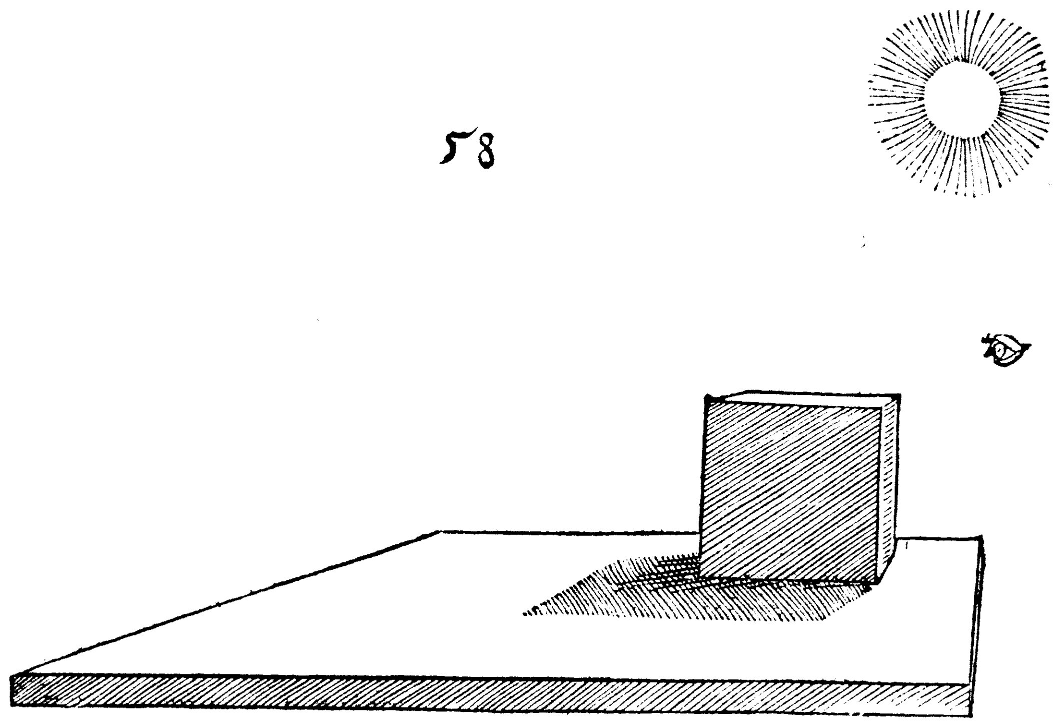

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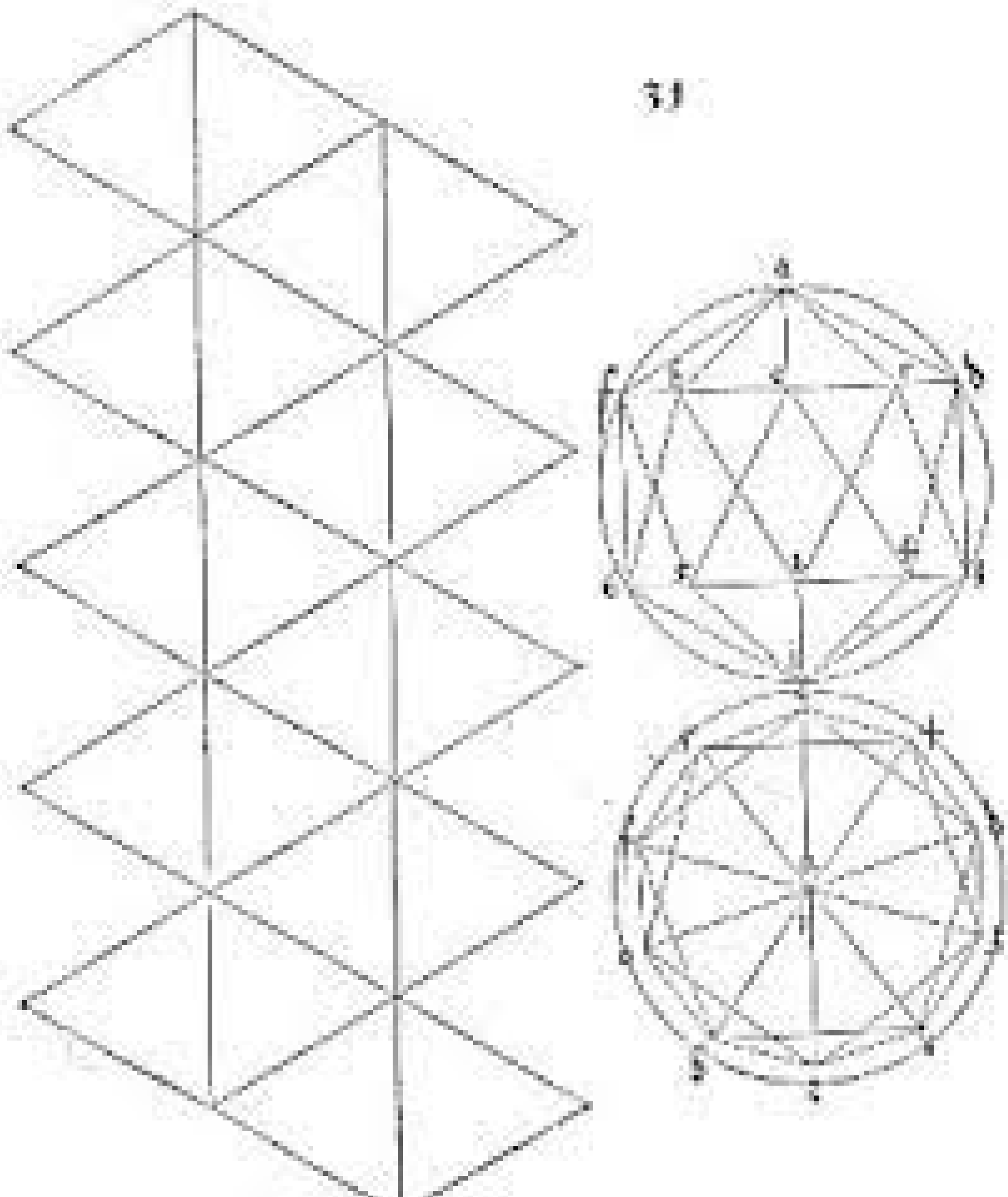
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2. Polyhedra and Shadows

- One may consider the shadows of a polyhedron cast onto a flat screen behind it. The shadow will be a polygon, having a finite number of corners k.

- *First model.* View shadows coming from a finite point source of light. (Projective geometry)

- *Second model.* View shadows from a source at infinity (parallel light rays) projected onto a screen perpendicular to rays. This is orthogonal projection. (Euclidean geometry)

Mathematical Polyhedra-1

- A *bounded polyhedron* or *polytope* P is the smallest convex set containing a given finite set of points S in n-space (The minimal such set S = vertices of P.)

- Smallest convex set P containing $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is the set of all points $\mathbf{x} := \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n$ with nonnegative weights λ_i with $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$.

- This set P is called the *convex hull* of S.

Mathematical Polyhedra-2

- (Bounded) polygons are the 2-dimensional version of the convex hull construction,

- (Bounded) polyhedra are the 3-dimensional version of convex hull construction.

Mathematical Polyhedra-3

- A *(possibly unbounded polyhedron P* is the set cut out as the intersection of a finite number of half-spaces. Half spaces are specified by linear inequalities

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n \leq b.$$

- This construction may give unbounded objects. If they are bounded they coincide with convex hull construction.

- Use of coordinates here is anachronistic:
due to *René Descartes* (1596–1650)

Uses of Polyhedra and Shadows

- Mathematical Optimization: Linear Programming
  (Shadow Vertex Algorithm)

- Robotics: Motion Planning

- Fraud Detection: Funny shadows

3. Equiprojective Polyhedra

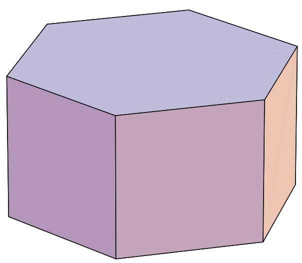
- **Definition** A polyhedron P is *equiprojective* if its orthogonal projection ("shadow") to a plane is a k-gon in every direction not parallel to a face of P.

- That is, all shadows cast by parallel light rays (from infinity) avoiding a finite set of directions have the same number of edges k. (The finite set of "bad" directions have fewer edges.)

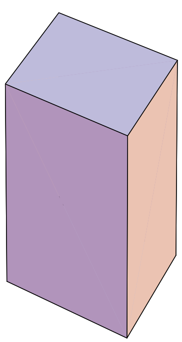
- Concept introduced by G. Shephard (1968)

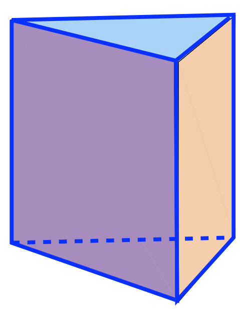
Definition

Convex polyhedron P is *k*-equiprojective if all orthogonal projections, except those parallel to faces of P, are k-gons.

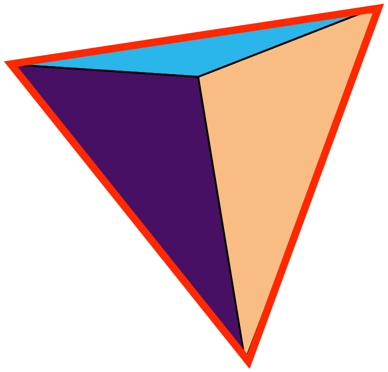


Any k-gonal prism is $(k+2)$-equiprojective

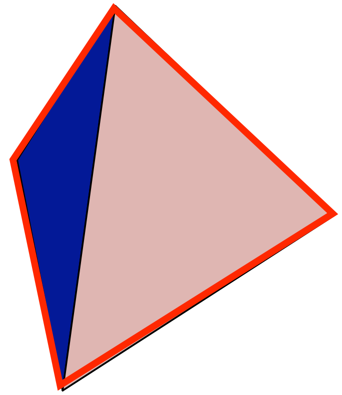




The Tetrahedron is Not Equiprojective



3-sided



4-sided

Equiprojective Polyhedra-2

- **Theorem.** Any k-gonal prism is $(k+2)$-equiprojective, for $k \geq 3$.

- The case $k = 6$ includes the cube, as a kind of 4-gonal prism with extra symmetries.

- **Theorem** (Hasan et al 2010) *Any equiprojective polyhedron has two faces that are parallel.*

- The k-gonal prism can be chosen to have exactly one set of parallel faces.

3. Equiprojective Polyhedra-Characterization-1

- *Idea:* Rotating an equiprojective polyhedron, at a point where one edge disappears a new parallel edge must appear.

- Such new edges can be on the same face or on a different face; they need not always have the same length.

- Call a "compensating pair" a collection of two such pairs, with opposite orientations:
(edge, face it belongs to) ["Before pair" and "After pair"]

3. Equiprojective Polyhedra-Characterization-2

- *Abstract Euclidean combinatorial graph:* Graph of vertices, edges, faces, edges oriented clockwise on outside of each face, parallel edges marked, parallel faces marked.

- **Theorem.** (Hasan- Lubiw 2008) A polyhedron P is equiprojective if and only if its abstract Euclidean combinatorial graph has a partition into compensating pairs.

- **Theorem.** (Hasan-Lubiw 2008) Given a polyhedron P with n edges, there is an $O(n)$ algorithm to determine if it is equiprojective, starting from its associated abstract Euclidean combinatorial type.

3. Equiprojective Polyhedra-Characterization-3

- **Corollary.** For each given n and k there are finitely many different (abstract Euclidean combinatorial) types of equiprojective polyhedra having n edges and of equiprojectivity k.

- Note: n edges on a polyhedron P implies that it has at most n vertices and at most n faces. In addition, either n vertices or n faces on P implies at most n^2 edges.

4. Moser Shadow Problem-1

- Leo Moser (1921–1970) born in Vienna, emigrated to Canada, received Ph.D. with Alfred Brauer, Univ. North Carolina in 1950.

- Moser formulated a well known problem list in Discrete and Combinatorial Geometry in the 1960's. It had 50 problems.

- **Moser Shadow Problem.** (1966) *Estimate the largest* $f(n)$ *such that every convex polyhedron of* n *vertices has an orthogonal projection onto the plane with* n *vertices on the 'outside'.*

Moser Shadow Problem-2

- **Definition.** The *Shadow number* $s(P)$ of polyhedron P is the maximal number of vertices on any shadow it casts, using parallel light rays from infinity. [A screen perpendicular to light rays displays orthogonal projection.]

- **Definition.** For a given number of vertices n on a polyhedron, the *shadow function* $f(n)$ is the minimal shadow number of any polyhedron having n vertices

- **Moser Shadow Problem.** Determine the behavior of $f(n)$ as $n \to \infty$. Is it unbounded?

Moser Shadow Problem-3

- For an equiprojective polyhedron P, its equiprojectively number k equals its shadow number $s(P)$.

- The shadow number $s(P)$ is an invariant of Euclidean geometry. It is unchanged by Euclidean motions of space. *But it can be changed by projective transformations.*

Moser Shadow Problem-4

- **Definition.** The *(projective) shadow number* $s^*(P)$ is the largest number of vertices on any shadow it casts from a point source (not at infinity).

- *Fact.* $s^*(P) \geq s(P)$.

- **Definition.** The *(projective) shadow function* $f^*(n)$ is the minimal projective shadow number over all bounded polyhedra having n vertices.

- *Fact.* $f^*(n) \geq f(n)$.

Moser Shadow Problem-5

- Moser construction: The shadow function $f(n)$ is no larger than $C \log n$. According to R. K. Guy, Moser believed this was the right answer. But then, 20 years later:

- **Theorem.** (Chazelle, Edelsbrunner, Guibas 1989)
  For $n \geq 5$ the *projective shadow function* $f^*(n)$ satisfies
  $$C_1 \frac{\log n}{\log \log n} \leq f^*(n) < C_2 \frac{\log n}{\log \log n}.$$
  for certain positive constants C_1, C_2.

- The construction for the upper bound is not obvious.

Moser Shadow Problem-6

- **Corollary.** (CEG, 1989) The shadow function satisifes

$$f(n) \leq C_2 \frac{\log n}{\log \log n}.$$

- Recently a full answer to the Moser problem was found. It differs for bounded and unbounded polyhedra!

Moser Shadow Problem-7: Bounded Polyhedron Case

- **Theorem.** For $n \geq 5$ the *shadow number* $f_b(n)$ restricted to bounded polyhdedra satisfies

$$C_1' \frac{\log n}{\log \log n} \leq f_b(n) < C_2' \frac{\log n}{\log \log n}.$$

for certain positive constants C_1', C_2'.

- This result is implicit in in the CEG paper, but is not explicitly stated there.

- In work with Yusheng Luo, we recently found a modified reduction method leading to this result, which also applies to the unbounded polyhedron case.

Moser Shadow Problem-8: Unbounded Polyhedron Case

- **Theorem.** (J. L., Yusheng Luo 2013) For $n \geq 5$ the *shadow number* $f(n)$ allowing unbounded polyhdedra satisfies

$$f(n) \leq 6.$$

- Unbounded polyhedra with arbitrarily many vertices do exist.

Zen Rock Garden: a hidden stone

The End

- This topic of shadows illustrates the aphorism:

  Ars Longa, Vita Brevis

- Thank you for your attention!