Erdős, Klarner and the $3x + 1$ Problem

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Connections in Discrete Mathematics
A celebration of the work of Ron Graham
Simon Fraser University

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Credits

- Ron Graham created and maintained an ideal environment for mathematics research at Bell Labs for many years. He continued this at AT&T Labs-Research. He was an inspiring mentor for me.

- Work reported in this talk was supported by NSF Grant DMS-1401224.

Paul Erdős and $3x+1$ Problem

- *Q.* Why did Paul Erdős never study problems like the
  $3x+1$ problem?

- *A.* He came very close.

- It took place at the University of Reading, UK around 1972.

1. Contents of Talk

1. Hilton and Crampin: Orthogonal Latin squares (ca 1971)

2. Klarner and Rado: Integer Affine Semigroups (1971-1972)

3. Erdős work: problem and solution (1972)

4. Complement Covering Problem (1975)

5. Klarner: Free Affine Semigroups (1982)

6. Affine Semigroups and the 3x+1 Problem

1. Orthogonal Latin Squares-1

- Joan Crampin and Anthony J. W. Hilton at the University of Reading around 1971 studied the problem: For which n do self-orthogonal latin squares (SOLS) exist?

- A Latin square M of order n is an $n \times n$ matrix has integers 1 to n in each row and column.

- Two Latin squares (M, N) are orthogonal if their entries combined in each square give all n^2 pairs (i, j), $1 \leq i, j \leq n$ in some order.

1. Orthogonal Latin Squares-1

- L. Euler (1782) wrote an 60 page paper [E530] on magic squares. He found there were no orthogonal Latin squares of order 2 and 6, and tried to prove there were none of order $4k + 2$.

- This conjecture was disproved by:
  S. K. Bose, S.S. Shrikande and E. T. Parker (1959, 1960):
  They showed orthogonal Latin squares exist for *all* orders $4k + 2 \geq 10$.

- The used various constructions building new size orthogonal pairs from old, with some extra structure.

Self-Orthogonal Latin Squares-1

A Latin square is self-orthogonal if the pair (M, M^T) is orthogonal. An example for $n = 4$.

$$M = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \end{bmatrix}, \quad M^T = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 3 & 2 & 4 & 1 \\ 4 & 1 & 3 & 2 \\ 2 & 3 & 1 & 4 \end{bmatrix},$$

Then

$$(M, M^T) = \begin{bmatrix} 11 & 34 & 42 & 23 \\ 43 & 22 & 14 & 31 \\ 24 & 41 & 33 & 12 \\ 32 & 13 & 21 & 44 \end{bmatrix}.$$

Self-Orthogonal Latin Squares-2

• *Question:* For which orders n do there exist self-orthogonal Latin squares?

• Anthony J. W. Hilton (circa 1971) approached problem of construction using old ideas of A. Sade (1953, 1960).

• Method: Given an SOLS Q of order $p+q$ that has an upper left corner that is an SOLS of order p, plus an orthogonal pair (N_1, N_2) of order $q-p$, there exists for each SOLS of order x a construction of another one of order $f(x) = (q-p)x + p$.

• $f(x)$ is an integer affine function. If one has a lot of initial constructions x_i then by iterating the function get a lot of orders n that work. Solicited aid of Crampin to test on computer.

Self-Orthogonal Latin Squares-3

- **Theorem.** (Crampin, Hilton + computer)

  (1) There exist SOLS of every order $n > 482$*.*

  (2) There exist SOLS of every order $n \geq 36372$*, having another SOLS of order 22 in its upper left corner.*

- Result announced at British Math. Colloquium 1972. Authors slow to write up...

Self-Orthogonal Latin Squares-4

- Then came the announcement:

- **Theorem.** (Brayton, Coppersmith, A. Hoffman (1974))
  There exist SOLS of every order n except $2, 3$ and 6.

- Authors at IBM, bigger computers... (Done independently)

- Crampin and Hilton published in 1975, JCTA. Long version of Brayton, Coppersmith, Hoffman published 1976.

2. Klarner and Rado-1

- David A. Klarner (1940–1999) was a noted combinatorialist. He wrote to Martin Gardner about polyominoes in high school, contributed to Martin Gardner's column, and later edited "*The Mathematical Gardner*", to which RLG contributed.

- He received his PhD in 1967 at Univ. of Alberta, with advisor John W. Moon.

- In 1970-1971 Klarner spent a year at University of Reading visiting Richard Rado.

Klarner and Rado-2

- Motivated by discussions with A. J. W. Hilton on SOLS problem, Klarner and Rado begin investigating integer orbits of affine maps. They considered a finite collection of maps

$$f(x_1, ..., , x_k) = m_1 x_1 + m_2 x_2 + \cdots + m_k x_x + b$$

with integer coefficients $m_i \geq 2$, all $b \geq 0$.

- Given a initial set of integers $A = \{a_i\}$, they formed the smallest set $T = \langle R : A \rangle \subset \mathbb{N}$ closed under iteration, substituting any member of T for each of the variables x_i.

- For two or more variables, they noted T often contained infinite arithmetic progressions, had positive density.

Klarner and Rado-3

- For one variable,more complicated...

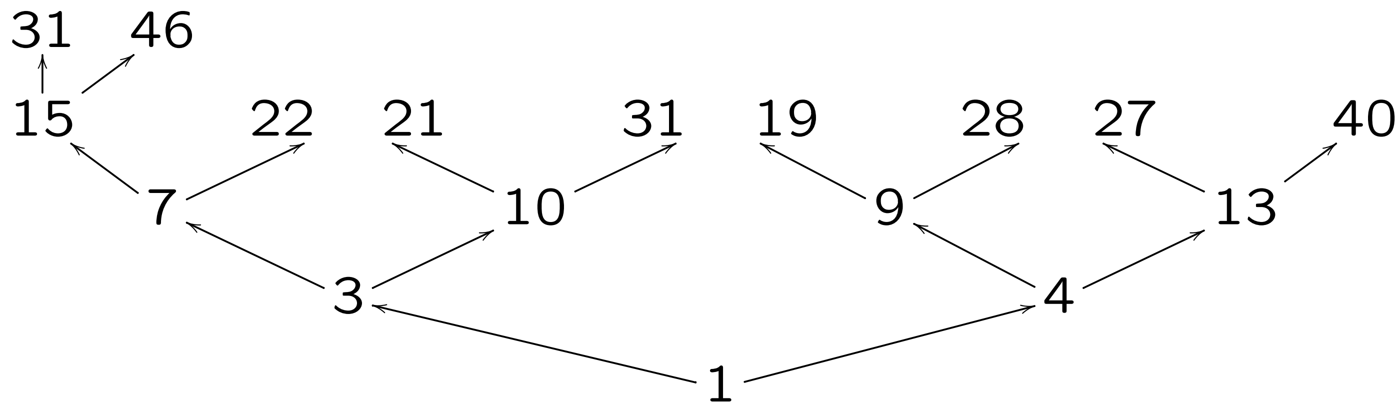
- Test problem: $f_1(x) = 2x + 1$, $f_2(x) = 3x + 1$.
  Will call it here: Klarner-Rado semigroup.

- *Question.* Does the orbit $\langle 2x + 1, 3x + 1 : 1 \rangle$ contain an infinite arithmetic progression?

- Orbit:

$$1, 3, 4, 7, 9, 10, 13, 15, 19, 21, 22, 27, 28, 31, 39, 40, 43...$$

Klarner-Rado Sequence



3. Erdős's Result -1

- Erdős was a long time collaborator with Rado, starting 1934. Eighteen joint papers, Erdős-Ko-Rado theorem.

- Erdős answered Klarner-Rado question: "No."

- He proved the orbit has density 0, giving a quantitative estimate of its size.

Erdős's Result -2

- **Theorem.** (Erdős(1972)) *Let* $\mathcal{S} = \langle f_1, f_2, ..., f_k, ... \rangle$ *with* $f_i(x) = m_x + b_i$ *with* $m_i \geq 2$ *and* $b_i \geq 0$. *Let exponent* $\sigma > 0$ *be such that*
$$\alpha := \sum_k \frac{1}{m_i^\sigma} < 1.$$
Then for any $N \geq 1$, *and any* $A = \{a_i\}$ *with* $a_i \to \infty$,
$$|\langle R : A \rangle^\sharp \cap [0, N]| \leq \frac{1}{1 - \alpha} T^\sigma.$$

- Application. For Klarner-Rado semigroup, we find τ with $\frac{1}{2^\tau} + \frac{1}{3^\tau} = 1$, which is $\tau = 0.78788...$ It suffices to take $\sigma = \tau + \epsilon$ for some $\epsilon > 0$. Since $\sigma < 1$ we get density 0.

Erdős's Result -3

- Asymptotics of non-linear recurrences analyzed by D. Knuth PhD student Mike Fredman (PhD. 1972) Some results published in paper Fredman and Knuth (1974).

- One special case of Fredman's thesis improves the upper bound to CN^{τ}, with

$$\sum_i \frac{1}{m_i^{\tau}} = 1.$$

  He applies result to Klarner-Rado sequence.

Erdős's Problem (1972)

- **Problem** (Erdős £10) Consider the semigroup $\mathcal{S}$ generated by

$$R = \{f_2(x) = 2x + 1, \quad f_3(x) = 3x + 1, \quad f_6(x) = 6x + 1\}$$

with "seed" $A = \{1\}$. Does the orbit $S := \langle R : A \rangle$ have a positive density? More precisely, does S have a positive lower asymptotic density $\underline{d}(S) > 0$?

- These parameters are "extremal": $\tau = 1$,

$$\sum_{i=1}^{3} \frac{1}{m_i} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1.$$

Erdős's Problem-2

- This Erdős problem was solved by Crampin and Hilton with the answer "No". The orbit has zero density.

- *Key Idea.* The semigroup $\mathcal{S}$ is not free, it has a nontrivial relation:

$$f_{232}(x) = f_2 \circ f_3 \circ f_2(x) = f_6 \circ f_2(x) = f_{62}(x) = 12x + 7.$$

- One can show the orbit density up to N is less than $cN^{5/6}$, as $N \to \infty$. [The proof uses Erdős's theorem.]

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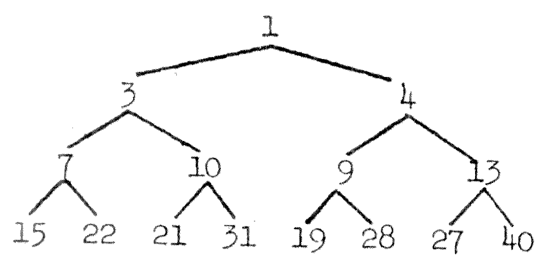
4. Klarner-Rado Complement Covering Problem

• In 1972 Chvatal, Klarner and Knuth wrote a Stanford Technical Report giving a problem list in Combinatorial Topics.

• One problem considered the complement $\mathbb{N} \smallsetminus S$ of the Klarner-Rado set S, which has density one, and asked if it can be covered by infinite arithmetic progressions.

Problem 1.

Consider the set $\langle 2x+1, 3x+1 : 1\rangle$ defined to be the smallest set of natural numbers which contains 1 and is closed under the operations $x \to 2x+1$ or $3x+1$. The set can be constructed by iterating these operations as indicated in the following tree.



...

Michael Fredman showed in his thesis that this set has density 0 in the set of all natural numbers; hence, $S = \langle 2x+1, 3x+1 : 1\rangle$ does not contain an infinite arithmetic progression. Let N denote the set of all natural numbers. Is it true that $N \backslash S$ may be expressed as a disjoint union of infinite arithmetic progressions?

Klarner-Rado Complement Covering Problem-3

• In 1975 Don Coppersmith wrote a paper that answered the question. The answer is "Yes".

• Coppersmith gave two results.

• First result gave a sufficient condition for a semigroup generated by relations R to have all sets $S = \langle R : A \rangle$ for $A = \{a\}$, $a \geq 1$ with $\mathbb{N} \smallsetminus S$ covered by infinite arithmetic progressions. He showed the covering can always be done by *disjoint* arithmetic progressions.

• Second result gave a (very complicated) sufficient condition for there to exist some a where such S cannot be covered by infinite arithmetic progressions.

Klarner-Rado Complement Covering Problem-4

• For the set $S = \langle 2x+1, 3x+1 : 1\rangle$ examined $\pmod 6$ the residues on the third level all occur at lower levels. Thus there at most $6-(1+2)=3$ classes $\pmod 6$ completely in $\mathbb{N} \setminus S$.

• Examined $\pmod{36}$ all nodes at the fifth level take residue classes that occur at lower levels. Thus at least $36-(1+2+4+8)=21$ residue classes $\pmod{36}$ fall completely in $\mathbb{N} \setminus S$.

• Assuming this pattern continues, we find $6^n-(1+2+4+\cdots+2^{2n-1})$ residue classes $\pmod{6^n}$ fall completely in $\mathbb{N} \setminus S$. In this limit these a.p's cover a set of natural density one.

5. Klarner Free Semigroup Criterion-1982

• In 1982 Klarner gave a sufficient condition for a set of affine maps to generate a free semigroup.

• **Claim.** The affine maps $f_i(x) = m_i x + b_i$, with $m_i \geq 2$, $b_i \geq 0$ can always be ordered so that

$$0 \leq \frac{b_1}{m_1 - 1} < \frac{b_2}{m_2 - 1} < ... < \frac{b_k}{m_k - 1}.$$

Set $p_i := \frac{b_i}{m_i - 1}$, then: $0 \leq p_1 < p_2 < ... < p_k$.

• Why? If equality holds, two generators commute, not free semigroup.

Klarner Free Semigroup Criterion-2

- **Theorem** (Klarner 1982) *Given affine maps* $f_i(x) = m_i x + b_i$ *satisfying the claim,* $0 \leq p_1 < p_2 < ... < p_k$*, and if in addition*

$$\frac{p_k + a_i}{m_i} \leq \frac{p_1 + a_{i+1}}{m_{i+1}} \quad \textit{holds for} \quad 1 \leq i \leq k-1,$$

then this semigroup on k *generators is free.*

- *Proof idea:* Show product order matches auxiliary lexicographic order, on elements of same level $2^j 3^k x + c_\ell$, puts a total order on semigroup.

Klarner Free Semigroup Criterion-3

- Klarner gave six examples of three generator semigroups with parameters $m_i = 2, 3, 6$ that are free.

- $R = \langle 2x,\ 3x + 2,\ 6x + 3 \rangle$

- $R^* = \langle 3x,\ 6x + 2,\ 2x + 1 \rangle$, and four more.

- Klarner raised the problem: Do any of these semigroups have orbits of positive density, starting from the single integer seed 1? (Unsolved Problem!)
  (Listed in various Richard Guy problem lists and books.)

6. Affine Semigroups and the $3x+1$ Problem

Recall the $3x+1$ Problem:

- Iterate $T(x) = x/2$ if x even, $T(x) = \frac{3x+1}{2}$ if x odd.

- $3x+1$ Conjecture. Every positive integer $x \geq 1$ iterates to 1 using T.

- Observation. Running the iteration backwards from $x = 1$ produces a tree of inverse iterates, generated by two affine maps. $f_1(x) = 2x$, $f_2(x) = \frac{2x-1}{3}$.

$3x + 1$ Problem-2

- $3x + 1$ Conjecture first appeared in print 1971, record of lecture of Coxeter on frieze patterns in Australia (1970).

- Coxeter offered 50 dollars for a proof, 100 dollars for a counterexample.

- J. H. Conway proved undecidability result in 1972.

- These results done about same time as Klarner-Rado work: such problems were "in the air."

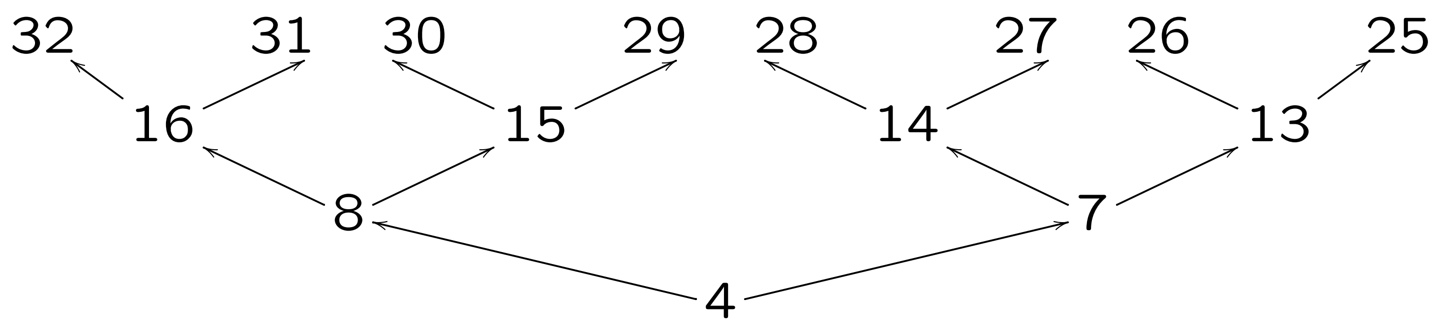
$x + 1$ Problem

- Treat a simpler case first: the $x + 1$ problem.
  Iterate $T(x) = x/2$ if x even, $T(x) = \frac{x+1}{2}$ if x odd.

- All forward orbits of $T(x)$ go "downhill" to 1, a periodic point.

- Inverse iterate maps: $f_1(x) = 2x, \quad f_2(x) = 2x - 1$.
  (These are affine maps of Klarner-Rado type.)

$x + 1$ Tree: Seed $a = 4$



$x+1$ Tree: Orbit Properties

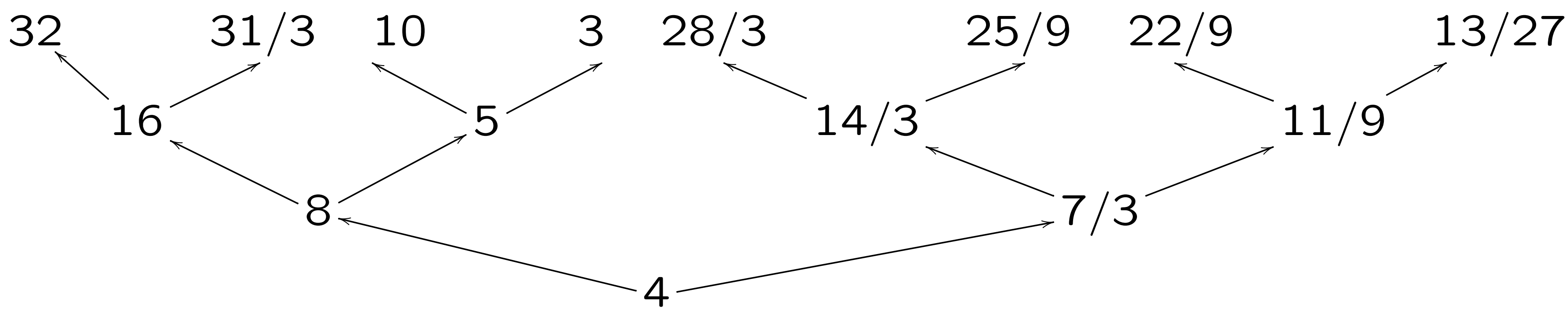
- Orbit $S = \langle 2x, 2x-1 : 4 \rangle$ has positive lower density. Density $\frac{1}{n}|S \cap [1,N]|$ oscillates as N increases.

- Complement $\mathbb{N} \smallsetminus S$ cannot be covered by complete arithmetic progressions!

- In 1972 Ron Graham gave Klarner and Rado a related bad example.

$3x+1$ Problem- Affine Semigroup Form

$3x+1$ Conjecture (Affine Semigroup Form) *The semigroup orbit* $S^* = \langle 2x, \frac{2x-1}{3} : 4 \rangle$ *contains every positive integer larger than 2.*



$3x + 1$ Problem vs. Klarner (1982) problem

• $3x + 1$ Semigroup problem is more complicated than Klarner-Rado semigroup problem.

(1) Maps can have negative entries, orbits can have negative numbers in them.

(2) Maps can have rational entries, orbits have rational numbers. (The $3x + 1$ problem only cares about the integer entries in backwards orbits.)

(3) Only a small fraction of orbit values are integers.

7. Summary

- In response to work of Klarner and Rado, in 1972 Erdős proved a theorem on iterates of affine semigroups and formulated a prize problem, solved the same year by Crampin and Hilton.

- In 1982 Klarner produced a "corrected" semigroup problem, currently unsolved.

- The 3x+1 Problem can be formulated in affine semigroup orbit framework. It has new features making it hard. (P. Erdős (1984) said: "*Hopeless. Absolutely hopeless!* ")

Thank You!