Intersections of Multiplicative Translates of 3-Adic Cantor Sets

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JMM, San Diego
AMS-SIAM Special Session on Mathematics of Computation:
Algebra and Number Theory
January 11, 2013

Topics Covered

- Part I. Ternary expansions of powers of 2 and a 3-Adic generalization
- Part II. Intersections of translates of 3-adic Cantor sets

References

- Part III reports:

  W. Abram and J. C. Lagarias , *Path sets and their symbolic dynamics*, `arXiv:1207.5004`

  W. Abram and J. C. Lagarias, *p-adic path set fractals and arithmetic*, `arXiv:1210.2478`

  W. Abram and J. C. Lagarias, *Intersections of Multiplicative Translates of* 3*-adic Cantor sets,* in preparation.

- Work of J.C.Lagarias supported by NSF grant DMS-1101373. Work by W. Abram supported by an NSF Graduate Fellowship.

Part I. Erdős Ternary Digit Problem and 3-adic generalization

- Problem. Let $(M)_3$ denote the integer M written in ternary (base 3). How many powers 2^n of 2 omit the digit 2 in their ternary expansion?

Examples	Non-examples
$(2^0)_3 = 1$	$(2^3)_3 = 22$
$(2^2)_3 = 11$	$(2^4)_3 = 121$
$(2^8)_3 = 100111$	$(2^6)_3 = 2101$

- Conjecture. (Erdős 1979) There are no solutions for $n \geq 9$.

3-Adic Dynamical System-1

- Approach: View the set $\{1, 2, 4, ...\}$ as a forward orbit of the discrete dynamical system $T : x \mapsto 2x$.

- The forward orbit $\mathcal{O}(x_0)$ of x_0 is

$$\mathcal{O}(x_0) := \{x_0, T(x_0), T^{(2)}(x_0) = T(T(x_0), \cdots\}$$

Thus: $\mathcal{O}(1) = \{1, 2, 4, 8, \cdots\}.$

- Generalized Problem. Study the forward orbit $\mathcal{O}(\lambda)$ of an arbitrary initial starting value λ. For how many λ can it have infinite intersection with the "Cantor set" (omit the digit 2)? View orbit inside the 3-adic integers.

3-adic Integer Dynamical System-2

- The integers $\mathbb{Z}$ are contained in the set of 3-adic integers $\mathbb{Z}_3$ (and are dense in it.)

- The 3-adic integers $\mathbb{Z}_3$ are the set of all formal expansions

$$\beta = d_0 + d_1 \cdot 3 + d_2 \cdot 3^2 + ...$$

  where $d_i \in \{0, 1, 2\}$. Call this the 3-adic expansion of β.

- Now view $\{1, 2, 4, 8, ...\}$ as a subset of the 3-adic integers, still a forward orbit of $x \mapsto 2x$.

3-adic Integer Dynamical System-3

- The 3-adic Cantor set Σ is the set of all 3-adic integers whose 3-adic expansion omits the digit 2. The Hausdorff dimension of Σ is $\log_3 2 \approx 0.63092$.

- *Generalization:* Consider the set of all $\lambda \in \mathbb{Z}_3$ for which the forward orbit

$$\mathcal{O}(\lambda) = \{\lambda, 2\lambda, 4\lambda, \cdots, 2^n\lambda, \cdots\}$$

  intersects Σ infinitely many times. Call this the 3-adic exceptional set and denote it $\mathcal{E}^*_\infty(\mathbb{Z}_3)$.

3-adic Integer Dynamical System-4

- The Erdös Conjecture asserts that $\lambda = 1$ is not in the exceptional set.

- This problem seems hopelessly hard. Instead will consider question:

- The 3-adic exceptional set $\mathcal{E}_{\infty}^{*}(\mathbb{Z}_3)$ ought to be very small. Conceivably it is just one point $\{0\}$. Can one show it is "small"?

3-adic Integer Dynamical System-5

- Exceptional Set Conjecture.
  The 3-adic exceptional set $\mathcal{E}^*_\infty(\mathbb{Z}_3)$ has
  Hausdorff dimension 0.

- This conjecture may be approachable, due to nice symbolic dynamics!

3-adic Integer Dynamical System-6

Can approach the Exceptional Set Conjecture by nested intervals.

- Define Level k exceptional set $\mathcal{E}_k^*(\mathbb{Z}_3)$ to be all λ with at least k distinct powers of 2 with $\lambda 2^k$ in the Cantor set.

- Level k exceptional sets are nested by increasing k:

$$\mathcal{E}_\infty^*(\mathbb{Z}_3) \subset \cdots \subset \mathcal{E}_3^*(\mathbb{Z}_3) \subset \mathcal{E}_2^*(\mathbb{Z}_3) \subset \mathcal{E}_1^*(\mathbb{Z}_3)$$

- Goal: Study the Hausdorff dimension of $\mathcal{E}_k^*(\mathbb{Z}_3)$; it gives an upper bound on $dim_H(\mathcal{E}^*(\mathbb{Z}_3))$.

3-adic Integer Dynamical System-7

In 2009, one author (J. L.) showed:

- Theorem. (Upper Bounds on Hausdorff Dimension)

$$(1).\qquad dim_H(\mathcal{E}_1^*(\mathbb{Z}_3)) = \alpha_0 \approx 0.63092.$$

$$(2).\qquad dim_H(\mathcal{E}_2^*(\mathbb{Z}_3)) \leq 0.5.$$

- Remark. There is also a lower bound:

$$dim_H(\mathcal{E}_2^*(\mathbb{Z}_3)) \geq \log_3(\frac{1+\sqrt{5}}{2}) \approx 0.438$$

3-adic Integer Dynamical System-8

- Upper Bound Theorem: Proof Idea:
  The set $\mathcal{E}_k^*(\mathbb{Z}_3)$ is a countable union of closed sets

$$\mathcal{E}_k^*(\mathbb{Z}_3) = \bigcup_{0 \le r_1 < r_2 < \ldots < r_k} \mathcal{C}(2^{r_1}, 2^{r_2}, \ldots, 2^{r_k}),$$

  with: $\mathcal{C}(2^{r_1}, 2^{r_2}, \ldots, 2^{r_k}) := \{\lambda : \ (2^{r_i}\lambda)_3 \ \text{ omits digit 2}\}.$

- We have

$$dim_H(\mathcal{E}_k^*(\mathbb{Z}_3)) = \sup\{dim_H\left(\mathcal{C}(2^{r_1}, 2^{r_2}, \ldots, 2^{r_k})\right)\}$$

- Proof for $k = 1, 2$: obtain upper bounds on Hausdorff dimension of all the sets $\mathcal{C}(2^{r_1}, 2^{r_2}, \ldots, 2^{r_k})$.

3-adic Integer Dynamical System-9

- Question. Could it be true that

$$\lim_{k \to \infty} dim_H(\mathcal{E}_k^*(\mathbb{Z}_3)) = 0?$$

- If so, this would imply that the complete exceptional set $\mathcal{E}^*(\mathbb{Z}_3)$ has Hausdorff dimension 0.

Part III. Intersections of Translates of 3-adic Cantor sets

- New Problem. For positive integers $r_1 < r_2 < \cdots < r_k$ set

$$\mathcal{C}(2^{r_1}, 2^{r_2}, ..., 2^{r_k}) := \{\lambda : (2^{r_i}\lambda)_3 \text{ omits the digit 2}\}$$

  Determine the Hausdorff dimension of $\mathcal{C}(2^{r_1}, 2^{r_2}, ..., 2^{r_k})$.

- More generally, allow arbitrary positive integers $N_1, N_2, ..., N_k$. Determine the Hausdorff dimension of:

$$\begin{aligned}\mathcal{C}(N_1, N_2, \cdots, N_k) &:= \{\lambda : \text{all } (N_i\lambda)_3 \text{ omit the digit 2}\} \\ &= N_1\Sigma \cap N_2\Sigma \cap \cdots \cap N_k\Sigma.\end{aligned}$$

Discovery and Experimentation

- The Hausdorff dimension of sets $\mathcal{C}(N_1, N_2, ..., N_k)$ can in principle be determined exactly. (Structure of these sets describable by finite automata.)

- Key Fact. Multiplication by integer N of 3-adic set X described by a finite automaton gives set NX describable by another finite automaton.

- It turns out that even the special cases $\mathcal{C}(1, N)$ already have a complicated and intricate structure!

Basic Structure of the answer-1

- The 3-adic expansions of allowed members λ of sets $\mathcal{C}(N_1, N_2, ..., N_k)$ are describable dynamically as having the symbolic dynamics of a sofic shift, given as the set of allowable infinite paths in a suitable labelled graph (finite automaton). Actually we need a slight generalization of sofic shift, which we call path set.

- The sequence of allowable paths is characterized by the topological entropy of the dynamical system. This is the growth rate ρ of the number of allowed label sequences of length n. It is the maximal (Perron-Frobenius) eigenvalue ρ of the weight matrix of the labelled graph, a non-negative integer matrix. (Adler-Konheim-McAndrew (1965))

Basic Structure of the answer-2

- The Hausdorff dimension of the associated "fractal set" $\mathcal{C}(N_1, ..., N_k)$ is given as the base 3 logarithm of the topological entropy of the dynamical system.

- This is $\log_3 \rho$ where ρ is the Perron-Frobenius eigenvalue of the symbol weight matrix of the labelled graph.

- Remark. These sets $\mathcal{C}(N_1, ..., N_k)$ are 3-adic analogs of "self-similar fractals" in sense of Hutchinson (1981), as extended in Mauldin-Williams (1985). Such a set is a fixed point of a system of set-valued functional equations.

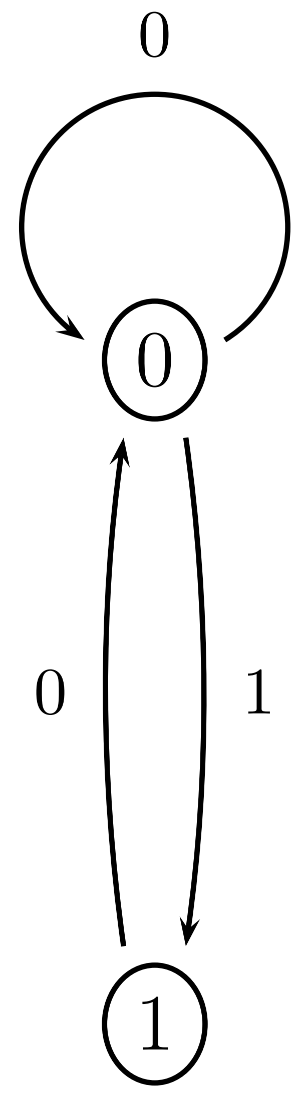
Basic Structure of the answer-3

Some reductions to the problem:

- If some $N_j \equiv 2 \pmod 3$ occurs, then Hausdorff dimension $\mathcal{C}(N_1, N_2, ..., N_k)$ will be 0.

- If one replaces N_j with $3^k N_j$ then the Hausdorff dimension does not change.

- Can therefore reduce to case: All $N_j \equiv 1 \pmod 3$.



Graph: $\mathcal{C}(1, N)$, $N = 2^2 = 4$

Associated Matrix $N = 4$

- Weight matrix is:

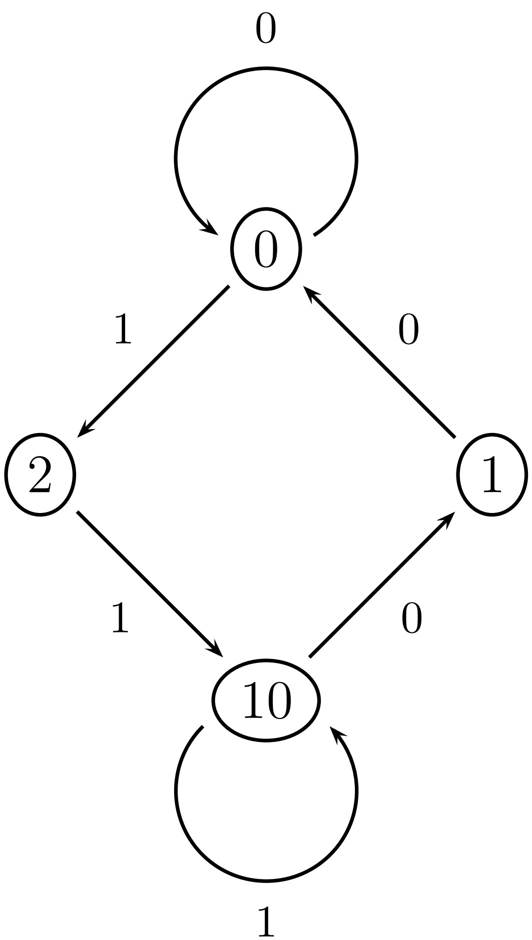
	state 0	state 1
state 0	[1	1]
state 1	[0	1]

- This is Fibonacci shift. Perron-Frobenius eigenvalue is:

$$\rho = \frac{1+\sqrt{5}}{2} = 1.6180...$$

- Hausdorff Dimension $= \log_3 \rho \approx 0.438$.

Graph: $\mathcal{C}(1, N)$, $N = (21)_3 = 7$



Associated Matrix $N = 7$

- Weight matrix is:

		state 0	state 2	state 10	state 1	
state 0	[	1	1	0	0	]
state 2	[	0	0	1	0	]
state 10	[	0	0	1	1	]
state 1	[	1	0	0	0	]

- Perron-Frobenius eigenvalue is : $\rho = \frac{1+\sqrt{5}}{2} = 1.6180...$

- Hausdorff Dimension $= \log_3 \rho \approx 0.438$.

Graphs for $N = (10^k 1)_3$

- Theorem. ("Fibonacci Graphs" Infinite Family)
  For $N = (10^k 1)_3$, (i.e. $N = 3^{k+1} + 1$)

$$dim_H(\mathcal{C}(1, N)) := dim_H(\Sigma \cap \frac{1}{N}\Sigma) = \log_3(\frac{1+\sqrt{5}}{2}) \approx 0.438$$

- Remark. The finite graph associated to $N = 3^{k+1} + 1$ has 2^k states and is strongly connected.

- The eigenvector for the maximal eigenvalue (Perron-Frobenius eigenvalue) of the adjacency matrix of this graph has an explicit self-similar structure, and has all entries in $\mathbb{Q}(\sqrt{5})$. (Many other eigenvalues.)

Graphs for family $N = (20^k 1)_3$

- This family has more complicated graphs.

- Computations. Here $N = 2 \cdot 3^{k+1} + 1$.
  For $1 \leq k \leq 7$, the graphs have increasing numbers of strongly connected components, which are nested.

- There is an outer component with about k states, whose Hausdorff dimension goes rapidly to 0 as k increases.

- The Hausdorff dimension of the inner component(s) start small but eventually exceed that of the outer component.

A Bad Case: $N = 139 = (12011)_3$

- This value N=139 is a value of $N \equiv 1 \pmod{3}$ where the associated set has Hausdorff dimension 0.

- The corresponding graph has 5 strongly connected components; each one separately has Perron-Frobenius eigenvalue 1, giving Hausdorff dimension 0!

Lower Bound for Hausdorff Dimension

- Theorem. (**Lower Bound Theorem**) For any any $k \geq 1$ there exist

$$N_1 < N_2 < \cdots < N_k, \quad \text{all } N_i \equiv 1 \pmod 3$$

such that

$$dim_H(\mathcal{C}(N_1, N_2, ..., N_k)) := dim_H(\bigcap_{i=1}^{k} \frac{1}{N_i}\Sigma) \geq 0.35.$$

Thus: the maximal Hausdorff dimension of intersection of translates is uniformly bounded away from zero.

- Proof. Take suitable N_i of the form $3^{k_i} + 1$ for various large k_i. One can show the Hausdorff dimension of intersection remains large (There is large overlap of symbolic dynamics).

Conclusions: Part II

(1) The graphs for $\mathcal{C}(1, N)$ exhibit a complicated structure depending on an irregular way on the ternary digits of N.

(2) Approach to prove Hausdorff dimension 0 by nested sets cannot be done if one generalizes it from powers of 2 to all $N \equiv 1 \pmod 3$. (Lower Bound Theorem).

Thank you for your attention!